





# Twenty Years of GdR IFM, seen from GT ALEA

## 1 Introduction

Discrete structures play a central role in modern scientific research. They allow us to model particles, data, and their relationships, and more broadly to understand how complex systems emerge from simple components. Whether it is data structures in computer science, the organization of matter in statistical physics, or large biological molecules such as DNA and RNA, discrete models provide powerful tools for describing and analyzing real-world phenomena.

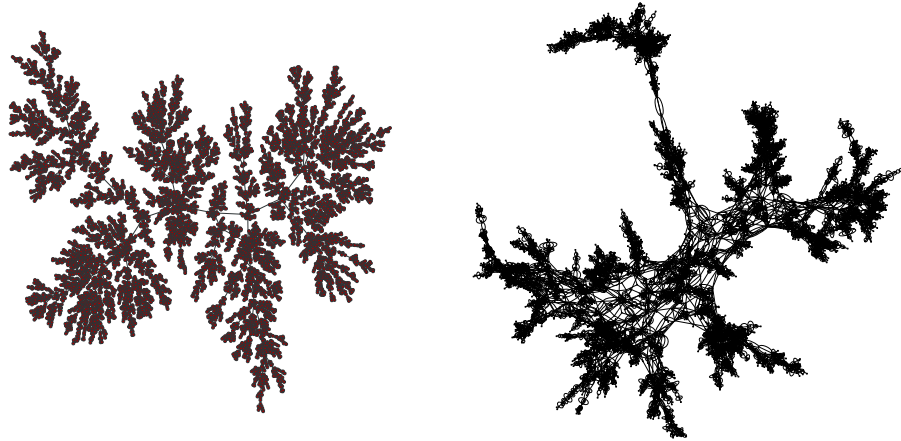
The GT ALEA aims to explore these models, to develop new theoretical approaches, and to promote their applications across disciplines. In fact, the *raison d'être* of the ALEA group is the study of discrete random structures that appear in all sciences, mainly fundamental computer science and algorithms, discrete mathematics and probabilities, statistical physics, and more marginally, bioinformatics. A distinctive feature of the group is the systematic use of combinatorial insights – particularly techniques from analytic combinatorics – to analyze and understand these structures.

More precisely, one fundamental line of inquiry concerns simulation methods – the design, programming, and analysis of algorithms that generate these structures efficiently. Alongside this, researchers investigate algorithms operating on these structures (or on more elementary structures such as lists of numbers to be sorted), characterizing their typical behavior. A second major area involves the study of parameters of random discrete objects: investigating quantities such as the average height of a large random tree or determining the limiting distribution of its height after appropriate normalization. Closely related is the study of scaling limits, which seeks to demonstrate that the normalized global object converges in distribution to a limiting random object, typically of a continuous nature. A third major direction is enumerative in nature, focusing on the computation and analysis of generating functions for combinatorial families. This is traditionally done via solving functional equations satisfied by these generating functions. Their study amplified over the past twenty years, with the emergence of a variety of new tools and techniques to solve them, at the interface of many subfields of mathematics. Beyond these core themes, the ALEA group studies numerous other random objects, each with their own characteristic questions (probabilistic cellular automata, random geometry, particle systems, . . .), driven by a keen interest in models from theoretical physics and the rich, often unexpected, phenomena they reveal.

This survey presents a selective panorama of recent developments addressing each of these questions, with a particular emphasis on contributions from the ALEA working group. Its aim is not only to describe major advances, but also to highlight the conceptual unification emerging from these diverse domains, and to illustrate how the combination of combinatorial insight, analytic techniques, and probabilistic methods continues to reshape the field.

## 2 Analysis of algorithms

The precise and systematic analysis of algorithm performance dates back to the work of Donald Knuth, who developed techniques for studying both worst-case and average-case scenarios. The methodology for average-case analysis relies on discrete probability and combinatorics, particularly through their central object: generating functions. For example, if the number of occurrences of a fixed pattern  $X$  in a random text of length  $n$  is required to



■ **Figure 1** Simulation of a uniform tree with 10,000 nodes and of a uniform quadrangulation with 10,000 faces.

analyze an algorithm, this approach consists of studying its *generating function*, the formal power series in two variables  $A_X(z, u) = \sum_{n,k \geq 0} p_{n,k} z^n u^k$ , where  $p_{n,k}$  is the probability that a word of length  $n$  contains  $k$  occurrences of  $X$ . This generating function captures all the necessary information about the distribution of the number of occurrences of the pattern and presents two fundamental advantages: first, it allows us to obtain an explicit expression directly without computing the  $p_{n,k}$ 's (or implicitly as a solution of a system of functional equations); second, the desired information (expectation, standard deviation, limit distribution, large deviations, and more) can be extracted using ready-made theorems from complex analysis, by viewing the generating function as a holomorphic function of the parameters  $z$  and  $u$ . This approach has been systematized by the ALEA community, notably under the impulsion of Philippe Flajolet, and is fully described in his seminal book *Analytic Combinatorics* [31], written with Robert Sedgewick.

Nowadays, a thorough examination of the algorithms implemented in the standard libraries of modern programming languages reveals significant deviations from traditional textbook algorithms. Even with classical data structures and algorithms, such as hash tables or sorting methods, a closer look at the source code shows that engineers often diverge from the expected paths, creating new algorithms or combining existing techniques in innovative ways.

An emblematic example is the sorting algorithm TIMSORT, designed by the Python's engineer Tim Peters in the early 2000's, which progressively gained popularity and which is now used in Java, Rust, etc. Its story has been eventful, and the final version of TIMSORT, its formal proof of correctness, and its complexity [5, 33] were established with the decisive help of academic researchers from the GT ALEA. Importantly, TIMSORT remained unnoticed by the academic community for more than 10 years despite being widely used in real life. Eventually, the techniques and metrics used for the theoretical study of TIMSORT led the academic researchers to propose new sorting algorithms, including the POWERSORT algorithm [38], which was recently adopted by Python in 2021 as their main sorting procedure.

Members of GT ALEA also began, about fifteen years ago, to incorporate architectural elements into computer models and to develop techniques for refining algorithm analyses in order to better describe their actual performance. For example, an internal processor mechanism, branch prediction, tries to anticipate what the next instruction will be during a

conditional jump (such as an `if` or `while`) in order to better take advantage of the speed-up provided by pipelining. Adding this feature to the computer model allows for a complexity analysis that is closer to what is observed in practice. This has been done by the community for the dual-pivot variant of QUICKSORT used in Java [37], as well as for standard pattern-matching algorithms [39]. In an orthogonal direction, fundamental algorithms were revisited and modified while taking this mechanism into account; this led to the discovery of surprising variants (such as a binary search that does not split in the middle to make prediction easier), which were proved theoretically and validated experimentally [6].

### 3 Random generation

A (uniform) random generation algorithm aims at efficiently sampling an object of a given size  $n$  from a combinatorial class (e.g. of trees, permutations, partitions, graphs, maps, . . .) uniformly at random. A prime motivation is to obtain simulations of large random objects and conjecture their asymptotic behavior, in particular for parameters that are difficult to analyze by generating function techniques (e.g. distance parameters). Providing a random sample obeying various sorts of syntactic and semantic constraints can also be useful to test correctness of software or of conjectures.

Random generation is closely related to counting (and to exhaustive generation): if the counting coefficients of a class have a simple closed-form formula (for instance Catalan numbers for binary trees) then combinatorial proofs of the formula can usually be turned into linear time random samplers. More generally, a class can be efficiently sampled if it can be set in bijection to another class that can be efficiently sampled; in the ALEA community this has proved particularly fruitful for the random generation of maps (embedded graphs) and of deterministic automata. This can also be combined with rejection techniques (e.g. if the correspondence is only injective), giving interesting complexity analysis problems [7]. In a broader context, symbolic combinatorics provides a general framework to formulate decompositions of combinatorial classes, which can be applied to exact and asymptotic enumeration on one hand, and to efficient random sampling on the other hand. Two random generation frameworks have been developed: the recursive method [32] that is based on the counting coefficients and recurrences, and Boltzmann samplers [26] that are based on generating functions. Boltzmann sampling achieves simple and faster (linear time) random samplers, at the cost of a slight loss of control on the output size. It has become a reference over the last 20 years and has been actively investigated by the ALEA community. Extensions [29, 12], improvements [23, 45] (e.g. to achieve linear time complexity for exact-size sampling, in particular for trees), and automatization [42, 41] (for the computation of the radius of convergence and evaluation of the generating functions) have been developed, as well as implementations in computer algebra systems such as Maple and SageMath.

Another well-known approach for random generation in a combinatorial class is via Markov chain Monte Carlo techniques, a wide-ranging method that can be applied to classes that are difficult to count (such as self-avoiding walks). Under easy-to-check conditions (ergodicity and symmetry), the distribution after  $t$  steps converges as  $t \rightarrow \infty$  to the uniform distribution on objects of size  $n$ . The mixing time is the number of steps required to be close to uniform, its dependency on  $n$  being often polynomially bounded (or even logarithmic, for instance in the famous card-shuffling dynamic). A so-called cutoff phenomenon is also frequently observed, namely a sharp transition (at the mixing time) between being far from uniform and being close to uniform. Showing the order of magnitude of the mixing time, and whether a cutoff occurs, can be very challenging. Recent progress has been made [44]

on exhibiting general properties of the Markov chain that determine whether cutoff occurs. Finally, coupling from the past, due to Propp and Wilson, is a generic technique to sample exactly from the limit distribution of an ergodic Markov chain satisfying some easy-to-check conditions. Its efficient implementation has been the subject of active research, in particular for structures encoded by a height function [25, 28].

#### 4 Understanding the behavior of large combinatorial structures

Gaining a better understanding of the statistical behavior of large combinatorial structures has long been an important goal of the ALEA community. One of the motivations is to analyze data structures – and the algorithms associated with them – on realistic data models. Three decades ago, most probabilistic results concerning random combinatorial structures focused on parameters of these structures. For example, it was proved that the height  $H_n$  of a tree of size  $n$  is of order  $\sqrt{n}$ , with a standard deviation of the same order. The most precise results concerned convergence in distribution: for instance, the law of  $H_n/\sqrt{n}$  converges as  $n \rightarrow +\infty$  ([30]).

A new point of view emerged in the 1990s, following the work of Aldous [4] and Pitman [40] on scaling limits of random trees: the entire tree, suitably rescaled (distances in the tree are divided by  $\sqrt{n}$ ), converges in distribution. The limit is a continuous random tree, which can already be perceived in the simulation, see Figure 1. These convergence results are global theorems, which imply – as by-products – an infinite number of more classical parameter convergence results. Moreover, the limit is the same for many different tree models, a phenomenon referred to as *universality*.

These results and techniques spread throughout the ALEA community in the early 21st century. Indeed, obtaining such results requires a very detailed understanding of the combinatorial structures under study, as well as expertise in probabilistic methods for proving convergence of complex structures.

The ALEA community now contains dozens of contributions in this specialty, and includes a significant proportion of the world’s leading experts in the field.

Among the most notable achievements are the scaling limit theorems – and, in particular, the identification of the Brownian map as the limit – for random maps such as uniform quadrangulations with  $n$  faces, together with numerous variants motivated by problems in theoretical physics ([21, 36]). Many other studies focus on graph limits (for example, the shape of giant components in random graphs [1]), random trees, families of paths, tableaux, and permutations (including the definition of permutons as limits of pattern-avoiding permutation tableaux, the analogue of graphons in graph theory [9]).

#### 5 Discrete differential equations

In enumerative combinatorics, we seek to determine the number  $a(n)$  of objects of size  $n$  in a combinatorial class (trees, permutations, paths, etc.). The most common approach consists of finding an unambiguous decomposition of these objects into smaller objects, which translates into a recurrence relation on the numbers  $a(n)$ . Often, we find a decomposition, but to deduce a recurrence, we are forced to consider, in addition to the size of the objects, an extra parameter (or even several ones). This parameter is sometimes called “catalytic”: without it, there is no recurrence. We must therefore work with an array of numbers,  $a(n; k)$ , rather than with a sequence  $a(n)$ .

In the most favorable cases, the recurrence will be expressed as a functional equation

on the associated generating function,  $A(t; x) = \sum_{n,k} a(n; k)t^n x^k$ . We would then like to answer, from such an equation, some natural questions about the power series  $A(t; x)$ , or about its specialization  $A(t; 1) = \sum_n a(n)t^n$ : is this generating function a rational function? Does it satisfy a polynomial equation, or at least a differential equation?

A particularly interesting class of functional equations, common in combinatorics, is that of *discrete differential equations*, that is, those using the operator  $\Delta$  defined by

$$\Delta A(t; x) := \frac{A(t; x) - A(t; 0)}{x}.$$

The first equations of this type appeared as early as the 1960s, for counting maps; then in the 1970s, for counting paths. Methods for solving them were then outlined, before it was established in algebra that their solutions were always *algebraic* (i.e., solutions of a polynomial equation). Members of the ALEA group subsequently contributed significantly to a better understanding of this result, to making it effective through adapted algorithms, and to exploiting it in extreme situations where the equations are particularly complex.

Beyond this, a broader class of equations, involving discrete derivatives in several variables, has been the subject of extensive work in our working group. As soon as derivatives with respect to two variables are involved, systematic algebraicity is lost — even if some cases do remain algebraic. Significant efforts to classify solutions have focused on a particular family of equations, considered typical, which involve two catalytic variables and only first-order derivatives. These equations arise from counting paths, with prescribed steps, confined to a quarter of a plane [16]. An astonishing variety of tools has been used to study them: probability theory, basic algebra on formal power series, computer algebra, complex analysis, and difference Galois theory [43, 24, 15]. It is striking how many people with diverse backgrounds this question now brings together. And the story is far from over: only the simplest range of equations is currently well understood. It should be noted that the tools developed in this study have already been adapted to similar equations that were not part of the initial set [8].

## 6 Interactions with physics

The interactions between the ALEA community and physics have grown significantly over the last 20 years. Among these fruitful interactions are combinatorial maps. Initially studied by Tutte in the 1960s in connection with the 4-color theorem, the study of maps resumed entirely independently in physics in the 1980s, within the context of two-dimensional quantum gravity. Indeed, as collections of discrete surfaces, maps provide a simple yet rich model of random geometries.

The advent of the Cori-Vauquelin-Schaeffer (CVS) bijection, between planar maps and decorated trees, triggered an unprecedented collaboration, which took place within the ALEA group, between combinatorists, physicists and probabilists. On the one hand, the CVS bijection allows access to information inaccessible through physicists' methods [18]; on the other hand, some of their methods, such as integrable hierarchies, are gradually making their way into combinatorics [19]. These interactions have taken the form of new meetings called the “Maps Days” (Journées Cartes) since 2012. They have given rise to an ever-expanding body of new results on map enumeration [17, 13], new bijections [10, 3, 20, 11], and the discovery of scaling limits [34].

These interactions continue today, for example, through the study of maps decorated with statistical physics models (i.e., matter coupled with 2-dimensional quantum gravity!) [2], and the study of higher-dimensional analogues of maps [14, 35].

Other aspects at the intersection of physics, probability, and combinatorics have also entered the GT, for example, probabilistic cellular automata, which are cellular automata whose states evolve according to rules determined by probabilities [27]. We can also mention particle models like the asymmetric simple exclusion process [22], also at the intersection with algebraic combinatorics and the GT CombAlg.

We conclude this text with a few more words about the connection with the GT CombAlg. Some objects commonly studied by the ALEA community also possess an algebraic facet: this is the case, for example, with maps through permutation factorizations and with Young tableaux via representations of classical groups. However, this facet is not central to ALEA's activities, but rather to those of CombAlg. Indeed, ALEA focuses on the statistical properties of combinatorial objects, while CombAlg uses these objects to represent algebraic structures and operations.

### Contributors.

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# Twenty Years of GdR IFM, seen from GT Algebraic Combinatorics

## 1 Introduction

Combinatorics is the study of discrete structures. These structures are ubiquitous in mathematics, computer science and their fields of application. They are used to model how data are structured within computer memory, to analyse algorithms, and also to understand larger-scale mechanisms, such as the organisation of telecommunications networks, fuel in a nuclear power plant, and varietal mixtures within cultivated fields.

Algebraic combinatorics, which is the GT CombAlg's area of expertise, has two facets: the study of discrete structures using algebraic tools and the study of algebraic structures on discrete objects. To illustrate the first perspective, consider a nuclear power plant. The uranium in this plant lasts for three cycles. In each cycle, the fuel must be present in specific locations. The type of questions studied by the members of the GT is therefore as follows: How can the fuel be repositioned in an optimal way while minimising the distance travelled by each rod?<sup>1</sup> This study involves a detailed examination of the symmetric group and its representations (encoded by symmetric functions). To illustrate the second point of view, let us consider a task scheduling problem, such as the production of different car models on an assembly line. In this type of problem, the order of precedence between tasks is fundamental (it makes no sense to start by painting the car body), and this problem is intrinsically linked to the partial order between tasks. Another example of a topic addressed by the GT is the study of (co-)products or products on combinatorial objects. The existence of these structures allows their study to be reduced to that of elementary building blocks. For example, words (or tensor algebra on letters) are obtained by concatenating letters (the elementary building blocks of words); if a property is stable by concatenation and true for letters, it will be true for words. Furthermore, each word can be written uniquely as a concatenation of its letters: this provides a way to generate all words once and only once. This problem is simple in terms of words, but leads to challenging questions when transposed to certain tree structures, or words that satisfy specific conditions, such as parking functions.

Over the past 20 years, there has been a remarkable expansion from symmetric functions, which were the focus of activity at the beginning of the millennium, towards more geometric (matroids, polytopes), algebraic (cluster algebras, Hopf algebras, categories) and bijective aspects. At the heart of these new research topics lies the study of partial orders, or posets, and their links with algebraic structures, such as algebras and operads and their geometric realisations (polytopes). We detail some of these aspects in the following sections, without aiming to be exhaustive: symmetric functions (Section 2), algebras and bialgebras (Section 3), posets (Section 4), operads and species (Section 5), geometric combinatorics (Section 6), bijective methods (Section 7), additive combinatorics and combinatorial number theory (Section 8) and finally, last but not least, formal proof and experimental algebraic combinatorics (Section 9).

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<sup>1</sup> see Martin Desombree's PhD thesis [35] for instance.

## 2 Symmetric functions and generalisations

### Symmetric functions and representations of the symmetric group

Symmetric functions are obtained as series of symmetric polynomials. They have been studied extensively for their connection to representations of the symmetric group. There are several bases for symmetric functions: elementary symmetric polynomials, power-sum polynomials, and Schur polynomials. Schur polynomials, indexed by partitions of integers, encode the irreducible representations of the symmetric group, while power-sum polynomials encode their characters. The change from one basis to another is a very difficult question that motivated the introduction of Macdonald polynomials, polynomials in several variables that instantiate into power sums and Schur polynomials in particular.

### MacDonald polynomials, coinvariant spaces and Schubert polynomials

Research on Macdonald polynomials led to the introduction of covariant spaces whose characters are given by these polynomials. The first example of a covariant space is the covariant algebra  $R_1(n)$ , which is the algebra of polynomials in  $n$  variables quotiented by the elementary symmetric polynomials  $e_1, \dots, e_n$ . It is a central element of algebraic combinatorics through its links with the theory of representations of the symmetric group, algebraic geometry and combinatorics. This space has been generalised with  $k$  sets of variables (spaces  $R_k(n)$ ), and more recently by adding  $l$  sets of antisymmetric variables, giving multigraded spaces  $R_{k,l}(n)$ . The case  $R_{2,0}(n)$  of diagonal coinvariants (see [11, 60, 59, 64, 61, 7, 62]) has been a driving force with Haiman's calculation of the Frobenius characteristic and its combinatorial interpretation (the shuffle conjecture proved by Carlsson and Mellit [21]). For the rest,  $R_{1,1}(n)$  is now partially understood [78] and some results are known (see [32, 31, 30, 29, 67]), but many conjectures remain unproven. This has led to the development of  $q, t$ -combinatorics [58, 63].

Schubert polynomials indexed by permutations of  $S_n$  form a basis for the space  $R_1(n)$ , which extends to a basis for polynomials in an infinite number of variables. The combinatorics of these polynomials has been the subject of constant activity [Knutson-Miller, Lascoux, Weigandt, ...] with extensions motivated by geometry. Algebraically, recent work [73] allows us to reformulate their definition and extend it. In another direction, quasi-symmetric polynomials were introduced by Stanley and then Gessel more than forty years ago in the context of  $P$ -partition theory. They form an algebra that is the terminal object in the theory of combinatorial Hopf algebras [3]. They form a natural and useful intermediate space between polynomials and symmetric polynomials, and have been studied intensively since their introduction [73].

## 3 Combinatorial algebras and bialgebras

### Connes-Kreimer Hopf algebra, renormalisation and applications to physics

The theory of Hopf algebras and their applications, particularly in combinatorics, experienced a significant revival in the late 1990s, thanks to the work of Connes, Kreimer and their collaborators on renormalisation. This procedure, used in quantum field theory and described recursively in the 1960s, allows the extraction of poles from iterative integrals defined on certain graphs, known as Feynman diagrams. The ultimate goal is to predict the value of certain physical quantities such as the mass of an electron very precisely. In addition to providing theoretical clarifications, this work has renewed the approach, particularly in

computer science, to renormalisation, as well as to Dyson-Schwinger equations [Kreimer, Yeats]. Furthermore, the ideas behind this discovery have been successfully applied in other fields. Combined with Terry Lyons' rough path theory (a theory widely interpreted in a Hopf-algebraic framework, see also [Zambotti]), an approach using tree-based Hopf algebras led Martin Hairer to construct a solution to the KPZ stochastic differential equation and then to develop regularity structures [15], work for which he was awarded the **Fields Medal in 2014**. Regularity structures are based on a type of Hopf algebras known as 'double' Hopf algebras, i.e. equipped with a second coproduct with good compatibilities: the first example of such an object [20] is based on rooted trees. The double bialgebras introduced have enabled the study of Butcher series in numerical analysis (solving PDEs). Other objects have been equipped with a double structure: graphs, posets, matrices, hypergraphs, words, etc. [Foissy, Manchon, . . .], leading to an intrinsic and purely algebraic definition of certain invariants, such as chromatic or Ehrhart polynomials.

### Combinatorial Hopf algebras

In parallel with this work, many other combinatorial Hopf algebras have been discovered, based on different types of graphs, words, matrices, etc., originating from different areas of mathematics: theoretical computer science [66, 55, 36, 65], control theory (Hopf algebras associated with Fliess groups [Grey]), probability theory (free probability theory, which can be largely formulated in terms of Hopf algebras of noncrossing partitions [Arizmendi, Nica, Speicher]), etc.

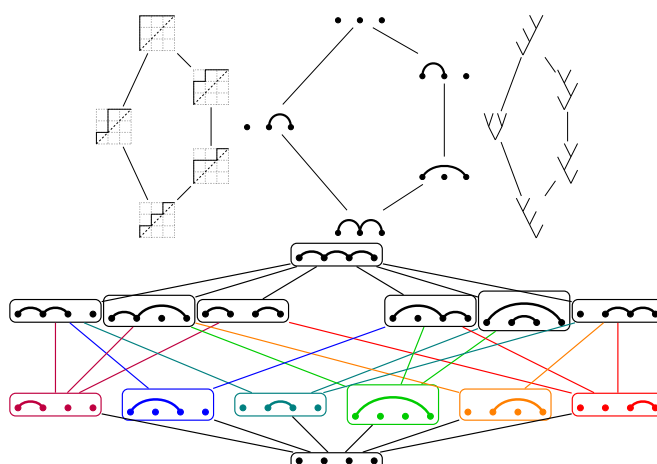
### Cluster algebras

The theory of cluster algebras, introduced around 2000 by Fomin and Zelevinsky [51, 49, 50, 10, 52], has since undergone rapid development, including in algebraic combinatorics. Among the notable results is the spectacular proof of their conjectures [56]. It involves regular graphs whose vertices are the clusters at the heart of this theory. In a number of important cases, these are finite graphs, which are identified with the graphs formed by the edges and vertices of certain polytopes. Finally, cluster theory allows us to equip these finite graphs with natural orientations, which define partial orders. This provides a very rich context in which polyhedral geometry, order theory and cluster mutation interact in subtle ways. This context has served as a model and inspiration for many developments. One example is a translation of all this cluster combinatorics in terms of finite Coxeter groups, initiated by Reading, under the name of Cambrian lattices. This leads in particular to a major extension to the case of finite non-crystallographic Coxeter groups, but also to new perspectives on Coxeter groups. The field has been given the name of Coxeter-Catalan combinatorics. A remarkable point is the new interpretation given to very classical combinatorial objects. Thus, the lattice of non-crossing partitions of Kreweras and the lattice of binary trees of Tamari are now understood to be associated, by a general construction, with Coxeter groups of type A, which are the symmetric groups.

## 4 Posets

### Tamari lattices, weak order and generalisations

Around 2005, a particular interest in the algebra of planar binary trees (PBT) [71, 65] and its connection to other combinatorial Hopf algebras gave rise to several articles highlighting its central role in numerous applications in combinatorics and mathematics. In particular, the



■ **Figure 1** From left to right: Tamari lattice on Dyck paths, noncrossing partitions and planar binary trees, lattice of noncrossing partitions on four elements.

study of different bases of PBT revealed a very natural and profound partial order structure (poset) on binary trees: the Tamari order. This order coincides with the order introduced by Dov Tamari in 1962 on the correct parenthesisations of an expression. Loday and Ronco notably showed the link between Tamari's order and the Hopf algebra of tree shuffles. Since then, a great deal of research has been devoted to a better understanding of this order and its many generalisations (other quotients of the weak order, generalisation to other Coxeter types, generalisation of Dyck paths of multiple types, etc.) (see, for example [24, 26, 5, 23, 34, 74].)

### Lattice properties

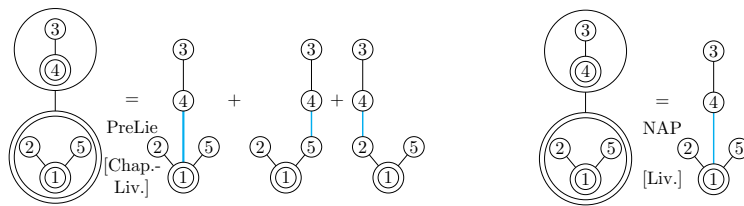
The research conducted also covers classical concepts in the field of partial orders and lattices, such as semi-distributivity and congruence-uniformity, but has also led to new concepts and constructions, including H. Thomas's definition of thin lattices and the Reading-Mühle fragment order, which generalises lattices of non-crossing partitions, or 'middle' orders, which are partial orders that are both finer than the weak (Bruhat) order and coarser than the strong Bruhat order, and which are distributive lattices [14].

### Incidence algebra of a poset

A relatively new direction in representation theory focusses on incidence algebras of finite partial orders. These algebras have the advantage of having a subset of representations of a highly combinatorial nature, which offers useful leverage. One of the questions that arises is the study of partial orders whose derived category has a certain periodicity of the Coxeter functor. This property, which defines fractional Calabi-Yau categories, is expected for all Cambrian lattices and has only been proven for Tamari lattices. A surprising relationship has recently been obtained [72] between this periodicity and combinatorial dynamics.

## 5 Operads and species

Species of structure are a categorical construction introduced in the 1980s by Joyal. They are analogous to Flajolet's combinatorial classes and have recently made a comeback in the



■ **Figure 2** Two ways of sending a tree of trees to a linear combination of trees, corresponding to two operads on the species of trees: the PreLie operad [25] and the NAP operad [69]

combinatorial arsenal. They are moreover linked with linear logic (GT LHC) [47] and operad theory.

### Operads

Operads have emerged as a fundamental tool in algebraic topology, and their development in this field continues. They have recently begun to spread more widely into different areas of mathematics. The theory of operads could be described as the algebra of trees, by analogy with the algebra of words. The interest of operads in algebraic combinatorics is thus similar to that long recognised in simpler and more classical algebraic structures, such as associative algebras or Hopf algebras. Defining an operad on combinatorial objects allows or requires these objects to be organised and better understood. Conversely, the study of a given operad can lead to the discovery of new combinatorial objects. The theory of operads and its many variants (cyclic operads, modular operads, properads, reconnectads, etc.) also has important connexions with Kontsevich’s graph complexes, which are very important and still mysterious structures.

### Generalised bialgebras

Numerous links between Hopf algebras, bialgebras and operads have also been highlighted. One of them is the notion of triple of operads, introduced by Loday [70], based on the fact that free algebras can be endowed with a bialgebraic structure in many cases. The primitive elements then inherit an additional structure, based on an operad that can be difficult to grasp [16, 18, 17, 19, 48, 69], but knowledge of which provides very rich information about the bialgebras thus obtained.

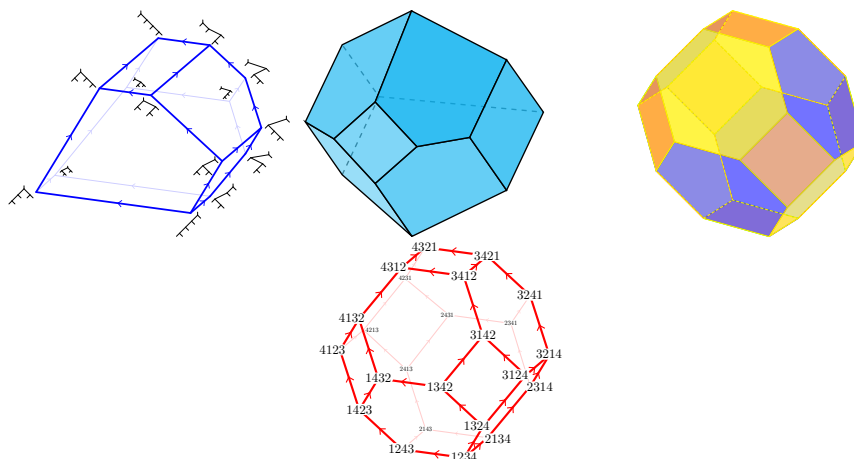
### Topology of posets and operads

One final aspect worth mentioning is the link between the topology of posets and operads. This link, established in the case of partition posets by Fresse [53] and extended to decorated partition posets by Vallette [80], was initially used to demonstrate algebraic properties on operads (Koszulity). Today, it is one of the tools used to prove topological properties of posets (Cohen-Macaulay character, shellability).

## 6 Geometric combinatorics

### Polytope combinatorics

Polytope combinatorics has been very fruitful in recent years. Among the results obtained, we can mention Santos’ counterexample to Hirsch’s conjecture in connection with the complexity



■ **Figure 3** Some polytopes :associahedron (on the left) and permutohedron (on the right)

of the simplex algorithm [79], **awarded the Fulkerson Prize**, the study of Postnikov's generalised permutohedra [77] and all their combinatorial and algebraic ramifications, and the discovery of the amplituhedron [6] in relation to particle physics, and the multiple works on positive geometries.

### Link with posets

The field of research linking posets and polytopes has also become very active in recent years. Indeed, the weak order and Tamari lattice mentioned in the previous section form the 1-skeleton of polytopes (the permutohedron and the associahedron, respectively). Among the directions explored is the construction of new examples by a simultaneous quotient procedure of the lattice and the polytope, such as quotientopes [76] or permutrees [75], which encompass and generalise the Cambrian lattices associated with symmetric groups.

### Matroids

Matroid theory was introduced in 1935, as an abstraction of the combinatorial properties of linear dependence in vector spaces. The finer structure of oriented matroids, taking into account the signs in these dependencies, has been developed since the 1970s. These structures satisfy various axioms and can be seen as combinatorial abstractions of graph theory, linear algebra, and convex geometry. Over the past thirty years, applications have been developed in a wide variety of fields.

Internationally, matroids have recently been in the spotlight thanks to the work of June Huh, **recipient of the Fields Medal in 2022**. In particular, Adiprasito, Huh and Katz proved the famous unimodality conjecture ([1]), according to which the sequence of coefficients of the characteristic polynomial of a matroid is log-concave, establishing a link with Hodge theory in algebraic geometry. Some of the recent research in the field has followed this lead by combining these two approaches. From a combinatorial point of view, this research falls within the traditional field of matroid polynomials, which has been studied continuously since the 1950s (notably the Tutte polynomial, which generalises the above-mentioned polynomials, see [46]), but also beyond. On aspects related to Tutte's polynomial, in an enumerative and bijective approach, a theory of 'active bijection' describing the properties of (oriented) matroids on an ordered set has been developed in a series of articles since 2002 by Gioan

and Las Vergnas. Other specific results on these aspects include, for example: a proof of the Merino-Welsh conjecture in a special case [68]; a generalisation in Hopf algebras [39]; a theory of ‘matroid set functions’ by Lass in 1997 and 2024.

On geometric aspects, in hyperplane arrangement theory, oriented matroids are classic underlying structures, both as rich and practical combinatorial tools and as potential topological generalisations with, for example, various works on polytopes and geometric representations of combinatorial structures by Padrol or Pilaud, and double pseudolines by Pocchiola. Furthermore, it is worth mentioning that matroids are also related in algebraic geometry to the stratification of Grassmannians and thus appeared in the work of Laurent Lafforgue (which earned him the **Fields Medal in 2002**), leading to the still open question of when a matroid polytope can be divided into matroid polytopes, for which results in specific cases have been obtained [27, 28]. Finally, we should mention a nascent theory developed since 2015 by Chepoi, Knauer, Chalopin, Philibert et al. of ‘Oriented Matroid Complexes’ which generalises both oriented matroids and lopsided sets, fundamental objects in learning theory.

## 7 Bijjective methods

### Planar maps

Planar map enumeration has its roots in the seminal work of W. Tutte in the 1960s. It is at the heart of modern developments in probability theory concerning random maps (explored by the GT Alea) and is also linked to questions in mathematical physics. An unexpected connection has emerged and is still developing between the enumeration of different types of maps and the enumeration of intervals in posets, for most variants of Tamari lattices, in particular greedy Tamari orders [34]. This has led in particular to new elegant bijections and a better understanding of the important classical bijections that connect blossoming trees and maps.

### Polyominoes

A polyomino is a finite union of square unit cells in the plane, whose interior is connected. The problem of enumerating polyominoes according to the number of cells is a famous and difficult problem. This is why different subclasses of polyominoes have been introduced in order to push the limits of enumeration techniques. Convexity, i.e. the fact that the intersection of the polyomino with any column or row is connected, is one of the best-known restrictions that have been studied in the literature. Many different methods have been successfully applied to enumerate convex polyominoes and some subfamilies: classic approaches include linear constructions involving generating trees, grammar-type decompositions, and bijective constructions.

In recent years, in the context of discrete tomography, a new classification of convex polyominoes has been proposed: a polyomino is said to be  $k$ -convex if each pair of cells can be connected by a monotonic path with at most  $k$  changes of direction. The problem of enumerating these  $k$ -convex polyominoes according to perimeter raised in [22] has led, for example, for  $k = 2$ , to the development of a new method that consists of generating these polyominoes by ‘inflating’ smaller polyominoes [38], or, in the directed case, to the introduction of an original bijective approach [13]. Guttman and Massazza recently obtained a set of recurrences to calculate the number of  $k$ -convex polyominoes in polynomial time and in  $O(n^4)$  space [57]. Another notable result is that of Bacher [8], who introduces new

bijjective methods for explicitly calculating the average value of three parameters of directed polyominoes with given area and perimeter. His constructions are based on a Viennot bijection between directed animals and coin stacks and also apply to polyominoes on the triangular lattice. Finally, the class of fighting fish was introduced [37], which can be seen as surfaces that generalise directed polyominoes without holes by allowing branches: fighting fish are counted as non-separable planar maps and thus provide a link to the very active field of maps. Most recently, new subclasses of vertically convex polyominoes have been studied in relation to Motzkin or Fibonacci words [9].

### Permutations

Over the last 20 years, the study of excluded patterns has seen a sharp acceleration, in particular with the exact enumeration of numerous classes of permutations that avoid one or more patterns, and with the emergence of new systematic approaches, such as Vatter's enumeration schemes [81] or the method of combinatorial exploration [4]. Vatter's enumeration schemes are an algorithmic method based on the natural recursive structure of permutation classes, which allows a recurrence equation to be constructed automatically from local construction rules. Combinatorial exploration consists of systematically exploring how pattern restrictions interact with the usual decomposition operations of the objects studied. This method has also been applied to certain classes of polyominoes, words, and matrices. Although the famous open problem of enumerating permutations excluding the pattern 1324 has not been solved, significant progress has been made thanks to the exact enumeration of well-chosen subclasses [12]. Finally, even if these are asymptotic results rather than exact enumerations, it is worth mentioning the question of constructing natural limit objects for large random objects: indeed, this line of research has motivated, beyond questions of excluded patterns, numerous enumeration works for the statistics of pattern occurrence numbers in permutation classes.

## 8 Additive combinatorics and combinatorial number theory

### Combinatorial identities and $q$ -series

In recent years, combinatorial identities and  $q$ -series have come under the spotlight thanks to new connections. In particular, it is possible to prove quadratic duality relations between solutions of  $q$ -hypergeometric equations without using  $q$ -difference Galois theory, in a manner completely analogous to the classical case of hypergeometric differential equations [Beukers, Jouhet, Roques]. Furthermore, new considerations of graded quotient rings related to arc spaces in algebraic geometry have led to the discovery of Andrews-Gordon-type partition identities generalising those of Rogers-Ramanujan: this discovery [2] came as a surprise given the abundance of literature in the field. Finally, it was discovered that the modular structure of the generating series of partition cores that appear naturally in representation theory made it possible to enumerate them, thus opening up new fields of exploration in both algebra and combinatorics (see, for example, the work of Alpoge, Brunat, Chapelier-Laget, Gerber, Granville, Jacon, Jouhet, Lecouvey, Ono, and Wahiche).

### Combinatorial number theory

Work in combinatorial number theory has mainly focused on *numerical semigroups*, i.e. submonoids of natural numbers with finite complement. These semigroups can be described as the results of the coin problem: what sums can be formed with 2 and 5 euro coins?

Introduced in the 19th century by Sylvester, these remain the subject of many open problems, including Wilf's conjecture<sup>2</sup> (1987) and that of Bras-Amoros (2008) [43, 40, 54, 41, 42, 33]. Another aspect concerns the growth of cardinals of iterated sum sets  $A_n = \{\sum_{i=1}^n x_i, x_i \in A\}$  for  $A$  a subset of an abelian group or semigroup. The first general bounds date back to 1970, with Plünnecke's inequalities. These classical inequalities have been improved recently [44, 45].

## 9 Formal proof and experimental algebraic combinatorics

### Combinatorics and formal proof

In recent years, the algebraic combinatorics community has grown closer to the mathematical formalisation community in several ways: first, algebraic combinatorics benefits from its unique position, studying both advanced algebraic structures (Hopf algebras, Operads, etc.) and non-trivial algorithmic objects. This intertwining of mathematics and algorithmics poses formalisation challenges that push proof systems and their libraries to their limits. Secondly, the question of how much trust can be placed in computing systems is becoming increasingly pressing as their use in proofs increases. It is becoming important to explore how to perform combinatorics more reliably. As published proofs are also becoming increasingly complex, we must also look at how they can be verified. One example is the proof of Littlewood-Richardson's rule, where many false proofs have been accepted, some for more than two decades.

### Software environment

From the perspective of algebraic combinatorics, the last 20 years have been characterised by significant developments in the software environment and an increasing use of machine experimentation for the study and understanding of discrete combinatorial objects. Experimental algebraic combinatorics offers an innovative approach to exploring discrete structures through an ingenious combination of theoretical results, algorithms and experiments. For example, it allows us to construct mathematical objects with certain desired properties or to conjecture certain phenomena. To pool the software development efforts required, the community has gradually structured itself to share libraries (SF, guess, ACE, muEC for instance). However, the underlying general-purpose calculation systems remained closed, imposing ethical, scientific, technical and practical limitations. A major breakthrough was the launch in 2005, spearheaded by William Stein, of the open-source general-purpose computing system SageMath, based on Python and a large number of specialised libraries. The contribution to our community is considerable, both in practical terms (calculation, experimentation, prototyping) and in structural terms (creation of an open, sustainable, reproducible ecosystem). At the heart of these advances is the Sage-Combinat community (initially MuPAD-Combinat), which since 2000 has brought together combinatorial researchers from around the world who wish to turn to open development. Among all the significant contributions, let us cite Nicolas Thiéry's, who led the design and development of the category infrastructure that allows the management of large hierarchies of algebraic structures (groups, rings, fields, Hopf algebras, ...).

<sup>2</sup> See <https://paysages.math.cnrs.fr/Semigroupes-numeriques-et-conjecture-de-Wilf-3865.html> and <https://images.math.cnrs.fr/les-extraordinaires-predictions-du-reverend-walker/> for instance.

### High-performance computing

Typical search spaces in algebraic combinatorics grow exponentially with the size of the input, which is known as the combinatorial explosion. In this regard, algorithmic efficiency is of the utmost importance. Recent technological developments in processors (multi-core, vector units) and computing environments (distributed computing, use of GPUs) offer several ways to improve machine experimentation. To take full advantage of these technologies, it is necessary to design new algorithms for algebraic combinatorics that are adapted to the various technical constraints. This algorithmic research will require a complete rethinking of existing algorithms and the introduction of new ways of representing the underlying mathematical objects.

### Contributors.

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# Twenty Years of GdR IFM, seen from GT Dynamical Systems, Automata, and Algorithms

## 1 Introduction

The main research topics studied by the SDA2 workgroup are at the frontier between theoretical computer science, combinatorics, number theory, discrete geometry, ergodic theory and topological dynamics. Some of them are common with others groups of the GDR, mainly GT Alea and GT Calculabilités. In addition to their inherent appeal, scientific motivations also arise from physical questions — such as those related to quasicrystals — and biological ones, like protein folding or DNA/RNA assembly.

We highlight here some major results and research directions of the last 20 years.

The notion of *symbolic system* is at the core of the studies of SDA2. Symbolic systems are discrete time dynamical systems, whose elements are colorings of  $\mathbb{Z}^d$  by a finite alphabet which respect some local constraints (such a set of colorings is called a *subshift*). The dynamics may be the *shift* action or more generally a *cellular automaton* (also called sliding block-code): an update rule which is the same for each cell and only depends on a finite neighborhood. These systems are also computational models.

## 2 Cellular Automata

Problems related to cellular automata are at the heart of the research activities of the members of the SDA2 group. Recent advances have changed the way these objects are viewed, in particular their connection to the structure of the subshift on which they act.

A fruitful approach to studying cellular automata (CA) is to interpret them within the framework of dynamical systems, i.e., as continuous shift-commuting functions, thereby establishing a canonical link with symbolic dynamics. This perspective, initiated by Hedlund, makes it possible to use tools from topology and ergodic theory and reveals that cellular automata exhibit a wide diversity of dynamical behaviors, ranging from the simplest to the most complex.

A classical problem consists in studying all reversible cellular automata (called automorphisms) that preserve a given subshift. Recently, a series of papers by S. Petite, F. Durand, A. Maass, and S. Donoso, V. Cyr, B. Kra [17, 16] have led to a good understanding of these automorphisms. In particular, they show that the growth rate of the complexity of the subshift restricts the algebraic properties of its automorphisms, such as their growth rate and the presence of distortion elements. It follows that the automorphism group of subshifts with linear complexity is genuinely constrained, and that neither the Baumslag–Solitar group  $BS(1, n)$ ,  $n \geq 2$ , nor  $SL(3, \mathbb{Z})$  can occur as subgroups of the automorphism group of a zero-entropy subshift. This constitutes the first known algebraic restriction for a zero-entropy system.

By contrast, A. Callard and V. Salo [11] have shown that the automorphism group of the full shift contains a distorted automorphism. For multidimensional subshifts of finite type, the situation is rather mysterious: P. Guillon, E. Jeandel, J. Kari, and P. Vanier [29] present an example of such a system whose automorphism group has an undecidable word problem.

Classification, i.e. whether two systems are isomorphic, is a central problem in dynamical systems, in particular for families that arise in a wide range of contexts, such as substitution subshifts. Usually invariants are used for this purpose, but currently known invariants are not

sufficient. In 2022, F. Durand and J. Leroy [22] showed that it is possible to decide whether a given cellular automaton defines a map between two prescribed minimal substitution subshifts. Building on these results, they provide an algorithm which, given two minimal substitution subshifts, determines whether one is a factor of the other and, as a direct consequence whether they are isomorphic.

### 3 Links between arithmetic and coding

The coding in a (finite memory) computer of an (irrational) real number or the coordinates of a vector is a classical but fundamental problem in computer science. Proceeding by approximations even with a very high number of digits is not enough when we have to iterate computations using this real, e.g. when computing one single orbit of a translation on the  $d$ -dimensional torus.

Substitutions are combinatorial objects (one replaces a letter by a word) which produce sequences by iteration and generate symbolic dynamical systems. These systems, produced by this elementary algorithmic process, have a highly ordered self-similar structure. Substitutions occur in many mathematical fields (combinatorics on words, digital geometry, ergodic theory and spectral theory, Diophantine approximation and transcendence, as well as in theoretical computer science or physics). The connections with numeration systems are numerous and natural. For example, the analogy between substitutions and beta-numeration is underscored by the work of Rauzy and Thurston, which led to the *Pisot conjecture*. This conjecture posits that a substitution satisfying certain arithmetic conditions generates a symbolic system that is measure-theoretically equivalent (i.e. up to a statistically negligible set) to a rotation on a torus.

Inspired by Rauzy's approach and generalizing substitutions to  $S$ -adic systems (given by sequence of substitutions), V. Berthé, W. Steiner and J. Thuswaldner [7], as well as the N. Pytheas Fogg and C. Noûs group [27], give codings in terms of multidimensional continued fractions algorithms.

In particular, given a continued fraction algorithm with exponential convergence, these sequences lead to renormalization schemes that naturally produce symbolic codings of toral translations and bounded remainder sets at all scales. Moreover, these examples have the lowest possible factor complexity (more precisely, the number of factor of length  $n$  is  $2n + 1$ ). These examples include classical algorithms such as the Jacobi–Perron, Brun, Cassaigne–Selmer, and Arnoux–Rauzy algorithms.

### 4 Conjectures in symbolic dynamics

We present here several well-known long-standing conjectures that have been resolved over the past twenty years by members of SDA2.

#### 4.1 Cobham's Theorem

Cobham's theorem (1969) is a deep result in theoretical computer science, linking finite automata, formal languages, and number theory. It concerns automatic sets: subsets of integers recognizable by a finite automaton reading their digits in base  $p$ . Cobham's theorem states that the only sequences recognizable in two multiplicatively independent bases (e.g., 2 and 3) are ultimately periodic. This theorem reveals a profound connection between automata, numeration bases, and periodicity, highlighting the difficulty of transcribing an automatic sequence from one base to another.

S. Eilenberg criticized the original proof of Cobham’s theorem (1969) as “correct, long, and difficult” [23], prompting the search for a more accessible proof. Cobham later clarified that  $p$ -recognizable sets are the images, under letter-to-letter morphisms, of fixed points of constant-length substitutions [14]. The theorem was extended by Semenov to recognizable subsets of  $\mathbb{N}^d$ , giving rise to the Cobham–Semenov theorem, and was then generalized to non-standard numeration systems (such as the Fibonacci numeration system). More accessible proofs subsequently emerged, notably via logical approaches (Michaux, Villemaire [42]) and dynamical approaches (Durand [21]), linking automaticity, logic, and dynamical systems. Finally, more recent works (Fabre, Bruyère-Point, Bès [44]) generalized this framework to non-standard numeration systems associated with Pisot numbers, further extending the scope of the theorem.

The Cobham theorem has been extended far beyond its original setting, touching geometry, dynamics, algebra, and even analysis, with applications to tilings, formal power series, and real numbers. In particular, the works of Boigelot, Brusten, and Bruyère [8] generalized Cobham’s theorem to recognizable subsets of  $\mathbb{R}$  or  $\mathbb{R}^d$  in standard numeration systems, paving the way for a geometric and dynamical interpretation. From the viewpoint of algebraic geometry, the work of Christol (1979) characterized algebraic formal power series in  $\mathbb{F}_q((t))$  by  $p$ -automatic sequences [38]. Subsequently, Kedlaya generalized this result to Hahn series  $\mathbb{F}_q((t^{\mathbb{Q}}))$  via quasi-automatic functions. Adamczewski and Bell extended Cobham’s theorem (2008) to quasi-automatic functions [1], showing that a sequence of coefficients represents two algebraic power series in distinct characteristics if and only if these series are rational. Beyond finite fields, and more recently (2024), B. Adamczewski and C. Faverjon proved that an irrational real number can be automatic in two multiplicatively independent integer bases using a quasi-automatic function and Mahler’s method [2].

## 4.2 The $\mathcal{S}$ -adic Conjecture

Recent approaches show that zero-entropy systems have a very restrictive combinatorial structure. This viewpoint, initiated by Morse and Hedlund for Sturmian sequences, has been enriched by other examples (substitutive systems, Toeplitz systems, interval exchange transformations, dendric sequences, etc.). This intuition, developed between the 1970s and 1990s, led to the  $\mathcal{S}$ -adic conjecture for subshifts with the lowest possible complexity.

*There exists an  $\mathcal{S}$ -adic characterization (i.e., in terms of concatenations of monoid morphisms) of subshifts with linear complexity.*

Formulated by S. Ferenczi and attributed to B. Host and the Marseille school, this conjecture was resolved in 2023 thanks to the work of B. Espinoza (a PhD student of F. Durand), who solved it in the minimal case. His characterization is explicit. He also proposes a characterization of systems with non-superlinear complexity [26]. These structural theorems will allow for a better understanding and interpretation of these systems.

## 4.3 Nivat’s Conjecture

It has long been observed that low complexity systems exhibit a rigid structure. One of the earliest illustrations of this principle was highlighted by Morse and Hedlund in 1938, who showed that a bi-infinite sequence with few distinct words (more precisely, when for some length  $n$  the number of factors of length  $n$  is less than  $n$ ) must necessarily be periodic. Motivated by the discovery of quasicrystals, which display strong long-range order

while remaining aperiodic, the question arises as to the weakest condition that still implies aperiodicity.

In 1997, Maurice Nivat formulated a conjecture concerning two-dimensional configurations (colorings of the integer lattice  $\mathbb{Z}^2$ ) and their local patterns. More precisely, if for some integers  $n, m \geq 1$  the number  $P_c(m, n)$  of distinct patterns (restrictions) of a configuration  $c$  on arbitrary  $n \times m$  rectangles is at most  $nm$ , then the conjecture asserts that  $c$  must be periodic in some direction.

Since then, Nivat's conjecture has been the subject of extensive research. It is, however, known to be false in dimension  $d \geq 3$  (Sander–Tijdeman) and for non-convex domains (Cassaigne [13]). In particular, Cassaigne exhibited periodic examples that do not satisfy the complexity assumption. Numerous partial results toward the conjecture have been established. For instance, there are proofs showing the existence of a period whenever  $P_c(n, m) \leq \alpha nm$  for various constants  $\alpha < 1$ : Epifanio–Koskas–Mignosi [24] obtained  $\alpha = 1/144$ , later improved to  $\alpha = 1/16$  by Quas–Zamboni [45]. At present, the best bound for  $\alpha$  was obtained by Cyr and Kra, with  $\alpha = 1/2$ . The optimization of this constant with respect to the size of the alphabet was achieved by Colle and Garibaldi. Sander and Tijdeman established periodicity when  $P_c(2, m) \leq 2m$  for some  $m \geq 1$ , and similarly Cyr and Kra obtained the same result when  $P_c(3, m) \leq 3m$ .

Jarkko Kari and Balázs Szabados achieved a major breakthrough by introducing an algebraic approach to Nivat's conjecture. Their method relies on tools from algebraic geometry, notably Hilbert's Nullstellensatz. In 2019, Kari and Étienne Moutot further developed this algebraic approach [37]. They proved that any two-dimensional configuration  $c$  of low complexity (with respect to a convex shape) contains, in its orbit closure (the smallest subshift containing  $c$ ), a periodic configuration. This result means that  $c$  contains arbitrarily large periodic regions, even if it is not globally periodic. To date, these works represent the strongest progress toward Nivat's conjecture.

All these results suggest that the appropriate framework for understanding low-complexity sequences is to interpret them in terms of first-order formulas in Presburger arithmetic.

#### 4.4 In search of the smallest set of Wang tiles

The domino problem, which asks whether a set of tiles can tile the plane  $\mathbb{Z}^2$ , was first formulated by Wang in the 1960s. A key ingredient in the proof of undecidability is the existence of an aperiodic set of tiles. While the initial proof paid little attention to the number of tiles (the first construction used more than 20,000 tiles), the search for the smallest tileset soon became a goal. The number of tiles was gradually reduced: first by Berger himself to 104 tiles, then by others over time (92 tiles [39], 35 tiles [46], 16 tiles [3], 14 tiles [36], 13 tiles [15]). Most of these constructions are based on self-similarity, but perhaps the most remarkable is that of [36], which introduces a completely new method for creating aperiodic tilesets based on representing real numbers along the rows of the tilings.

For a long time, the 13-tile bound could not be improved, but a definitive answer was given by Jeandel and Rao [34] recently: there exists a set of 11 tiles that tiles the plane aperiodically, and no such set exists with 10 tiles. This result was obtained both through an exhaustive computer search and by a manual proof on the candidate aperiodic tilesets identified. The resulting 11 tile tileset turned out to be self-similar, like most of the previously constructed aperiodic tilesets.

## 5 Computability and dynamical systems

A major shift in perspective occurred around 2010 with the use of computability theory to study discrete dynamical systems. Whereas previously the tool of choice had been language theory and computability was seen as preventing their study<sup>1</sup>, it is now considered one of the standard tools.

This shift in perspective was initiated by the article [32], which introduced tilings with computably enumerable constraints: effective subshifts. One of the major results of that paper, later refined in [20, 5], is that 1-dimensional effective subshifts are exactly the projections of rows of shifts of finite type, i.e., subshifts defined by a finite set of forbidden patterns. This result later became a tool to easily construct subshifts satisfying certain properties.

Shortly after [32], computability was used for the first time to characterize a dynamical aspect of subshifts: their entropy [33]. This line of research has been fruitful, and many results from the SDA2 community follow this approach and characterize isomorphism invariants of subshifts or cellular automata through computability: invariants based on pattern growth [12, 30], projective subdynamics [28], limit sets of cellular automata [10, 25, 9, 31], characterization of periods using complexity classes [35], and more.

## 6 Tilings and groups

The structures on which tilings are traditionally considered are the discrete plane  $\mathbb{Z}^2$  or the continuous plane  $\mathbb{R}^2$ . A major difference with one-dimensional tilings (tilings on  $\mathbb{Z}$ ) is the undecidability of the domino problem in the plane. A recent line of research, whose goal is in part to understand the boundary between decidability and undecidability of the domino problem, is to study it on groups.

The development of this topic followed the article [6], in which it is conjectured that the domino problem is decidable on a group if and only if the group is virtually free. An overview of these topics was written by members of the working group [4], and many articles in this direction have been produced by the SDA2 community. This line of research has also led to the introduction of new notions on groups originating from symbolic dynamics, such as the notion of a self-simulable group [47], that is, groups for which the classes of effective and sofic subshifts coincide.

## 7 Self-assembly and pumping

Another topic present in the working group concerns bio-inspired models of computation, and in particular self-assembly. This is a model close to Wang tiles, where colors are called “glues” and are assigned a certain strength. These tiles are placed one after another on the grid  $\mathbb{Z}^2$ , starting from a seed which is a pre-assembly of several tiles. The assembly takes place at a given temperature  $\tau$ , which determines the minimal glue strength required for attachment: each tile must attach to one or more existing tiles in such a way that the sum of the strengths exceeds the temperature. Several variants of these models exist (see the surveys [43, 18]).

The study of the computational power of this model has been a major question over the past two decades. This model is intrinsically universal starting at temperature 2 [19],

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<sup>1</sup> One of the authors of the reference book *An Introduction to Symbolic Dynamics and Coding* [40], Doug Lind, even wrote: “The swamp of undecidability: It’s a place you don’t want to go.”

meaning that there exists a tileset capable of simulating all other tilesets. One of the main results obtained by members of the SDA2 community is that, by contrast, at temperature 1 it is not universal, and there exists a pumping lemma showing that such assemblies admit a finite description [41].

## Contributors

Samuel Petite & Pascal Vanier.

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