Chemins dans le quart de plan II

Kilian Raschel

Universität Bielefeld

Journées ALÉA 2011
1 Introduction and main results
   - Introduction
   - Results

2 Proofs
   - Explicit expression of the counting generating functions
     - Reduction to boundary value problems
     - Conformal gluing and uniformization
   - Nature of the counting generating functions

3 Conclusion
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3 Conclusion
Counting the numbers of walks confined to the quarter plane

Let $S$ be a step set

and let $q(i, j; n)$ be the number of paths:

- with increments in $S$;
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Let

$Q(x, y; t) = \sum_{i,j,n \geq 0} q(i,j;n)x^i y^j t^n$. 

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Counting the numbers of walks confined to the quarter plane

Let \( \mathcal{S} \) be a step set

and let \( q(i, j; n) \) be the number of paths:
- with increments in \( \mathcal{S} \);
- confined to the quarter plane;
- having length \( n \), starting from \((0, 0)\) and ending at \((i, j)\).

Let

\[
Q(x, y; t) = \sum_{i, j, n \geq 0} q(i, j; n)x^i y^j t^n.
\]

\( Q(x, y; t) \): explicit expression;
Counting the numbers of walks confined to the quarter plane

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Let

$$Q(x, y; t) = \sum_{i,j,n \geq 0} q(i, j; n)x^i y^j t^n.$$  

- $Q(x, y; t)$: explicit expression;
- $Q(x, y; t)$: dependence on $S$, e.g., its nature (rational, algebraic, (non-)holonomic).
Class of the walks with small steps

\[ S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}. \]

There are \(2^8\) such problems.
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Some of these \(2^8\) models are:

trivial;
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- simple;
Class of the walks with small steps

$S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$. There are $2^8$ such problems.

Some of these $2^8$ models are:

- trivial;
- simple;
- intrinsic to a half plane;
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Some of these \(2^8\) models are:

- trivial;
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- intrinsic to a half plane;
- symmetrical.
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There are \(2^8\) such problems.

Some of these \(2^8\) models are:

- trivial;
- simple;
- intrinsic to a half plane;
- symmetrical.

Finally, it remains 79 problems! [BMM]
The functional equation

The kernel:

\[ K(x, y; t) = \sum_{(k, \ell) \in S} x^k y^\ell - \frac{1}{t} \]  

The functional equation for \( Q(x, y; t) \):

\[ K(x, y; t)Q(x, y; t) = K(x, 0; t)Q(x, 0; t) + K(0, y; t)Q(0, y; t) - K(0, 0; t)Q(0, 0; t) - xy. \]
The functional equation

The kernel:

\[ K(x, y; t) = xyt \left[ \sum_{(k, \ell) \in S} x^k y^\ell - 1/t \right] . \]

The functional equation for \( Q(x, y; t) \):

\[ K(x, y; t)Q(x, y; t) = \]
\[ K(x, 0; t)Q(x, 0; t) + K(0, y; t)Q(0, y; t) - K(0, 0; t)Q(0, 0; t) - xy. \]
The group of the walk

\[
\sum_{(k, \ell) \in S} x^k y^\ell = \begin{cases} 
B_{-1}(y)x^{-1} + B_0(y) + B_{+1}(y)x^{+1} \\
A_{-1}(x)y^{-1} + A_0(x) + A_{+1}(x)y^{+1}
\end{cases}
\]
The group of the walk

\[ \sum_{(k, \ell) \in S} x^k y^\ell = \begin{cases} 
B_{-1}(y)x^{-1} + B_0(y) + B_{+1}(y)x^{+1} \\
A_{-1}(x)y^{-1} + A_0(x) + A_{+1}(x)y^{+1}
\end{cases} \]

is left unchanged by

\[ \psi(x, y) = \left( x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \Phi(x, y) = \left( \frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right) \]
The group of the walk

\[
\sum_{(k,\ell) \in S} x^k y^\ell = \begin{cases} 
B_{-1}(y)x^{-1} + B_0(y) + B_{+1}(y)x^{+1} \\
A_{-1}(x)y^{-1} + A_0(x) + A_{+1}(x)y^{+1}
\end{cases}
\]
is left unchanged by \(\psi(x, y) = (x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y})\), \(\phi(x, y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right)\)
and thus by any element of the group \(\langle \psi, \phi \rangle\).
### Examples

<table>
<thead>
<tr>
<th>Order 4;</th>
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```latex
\text{Order 4;}
```

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```latex
\text{Order 4;}
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Examples

Order 4; order 6;
Examples

- Order 4;
- order 6;
- order 8;
Examples

Order 4;  order 6;  order 8;  order $\infty$. 

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Examples

Order 4;
order 6;
order 8;
order $\infty$.

Classification of the 79 models [BMM]

- For 23 walks, $\langle \Psi, \Phi \rangle$ is finite;
- For 56 walks, $\langle \Psi, \Phi \rangle$ is infinite.
### Existing results for the 23 finite group cases

<table>
<thead>
<tr>
<th>Group</th>
<th>Covariance</th>
<th>Walks</th>
<th>$Q(x, y; t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$= 0$</td>
<td>and 14 others</td>
<td>holonomic [BMM]</td>
</tr>
<tr>
<td>6</td>
<td>$&lt; 0$</td>
<td></td>
<td>holonomic [BMM]</td>
</tr>
<tr>
<td>8</td>
<td>$&lt; 0$</td>
<td></td>
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<tr>
<td>6</td>
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<td></td>
<td>algebraic [BK]</td>
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</table>
Existing results for the 56 infinite group cases

- $5 = 2 + 3$ singular walks:
Existing results for the 56 infinite group cases

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Expression & nature (non-holonomic) of $Q(x, y; t)$ in [MR].
Existing results for the 56 infinite group cases

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In progress.
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Expression & nature (non-holonomic) of $Q(x, y; t)$ in [MR].

In progress.

- 51 non-singular walks?
Existing results for the 56 infinite group cases

- $5 = 2 + 3$ singular walks:

  Expression & nature (non-holonomic) of $Q(x, y; t)$ in [MR].

  In progress.

- 51 non-singular walks [KR].
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For all walks, 

\[ K(x,0;t)Q(x,0;t) - K(0,0;t)Q(0,0;t) = \]
For all walks,

\[ K(x,0; t) Q(x,0; t) - K(0,0; t) Q(0,0; t) = x Y_0(x; t) + \]
\[ \frac{1}{2\pi i} \int_{x_1(t)}^{x_2(t)} u [Y_0(u; t) - Y_1(u; t)] \left[ \frac{\partial_u w(u; t)}{w(u; t) - w(x; t)} - \frac{\partial_u w(u; t)}{w(u; t) - w(0; t)} \right] du, \]

where:

* \( Y_0(x; t) \) and \( Y_1(x; t) \) are the two roots of the kernel \( y \mapsto K(x,y; t) \).

* \( x_1(t) \) and \( x_2(t) \) are branch points of the algebraic function \( Y(x; t) \).

* \( w \) will be defined soon.
For all walks,

$$K(x, 0; t) Q(x, 0; t) - K(0, 0; t) Q(0, 0; t) = x Y_0(x; t) +$$

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- $w$ will be defined soon.
For all walks,

\[ K(x,0; t)Q(x,0; t) - K(0,0; t)Q(0,0; t) = x Y_0(x; t) + \]

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- \( w \) will be defined soon.
For all walks,

\[ K(x,0; t)Q(x,0; t) - K(0,0; t)Q(0,0; t) = xY_0(x; t) + \]
\[ \frac{1}{2\pi i} \int_{x_1(t)}^{x_2(t)} u [Y_0(u; t) - Y_1(u; t)] \left[ \frac{\partial_u w(u; t)}{w(u; t) - w(x; t)} - \frac{\partial_u w(u; t)}{w(u; t) - w(0; t)} \right] du, \]

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A similar identity holds for

\[ K(0, y; t)Q(0, y; t) - K(0, 0; t)Q(0, 0; t); \]
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\( Q(0, 0; t) \) is deduced by evaluation;
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\[ K(0, y; t) Q(0, y; t) - K(0,0; t) Q(0,0; t); \]

- \( Q(0,0; t) \) is deduced by evaluation;

- \( Q(x, y; t) \) is obtained thanks to the functional equation:

\[
K(x, y; t) Q(x, y; t) = K(x, 0; t) Q(x, 0; t) + K(0, y; t) Q(0, y; t) - K(0,0; t) Q(0,0; t) - xy.
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It remains to find \( w \)!
Results (3/4)

\[ \langle \Psi, \Phi \rangle \text{ finite } \]
\[ \sum_{(k, \ell) \in S} k\ell \leq 0 \] \[ \Rightarrow \] \[ w \text{ rational} \]
\[ \langle \Psi, \Phi \rangle \text{ finite } \left\{ \sum_{(k,\ell) \in S} k\ell \leq 0 \right\} \Rightarrow w \text{ rational} \]
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\[ \langle \Psi, \Phi \rangle \text{ finite } \left\{ \sum_{(k, \ell) \in S} k \ell > 0 \right\} \Rightarrow w \begin{cases} \text{algebraic} \\ \text{non-rational} \end{cases} \]
Results (3/4)

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Results (3/4)

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\[ \langle \Psi, \Phi \rangle \text{ infinite } \Rightarrow \ w \text{ non-holonomic} \]
Results (3/4)

- \( \left\langle \Psi, \Phi \right\rangle \) finite \( \sum_{(k, \ell) \in S} k\ell \leq 0 \) \( \Rightarrow \) \( w \) rational

- \( \left\langle \Psi, \Phi \right\rangle \) finite \( \sum_{(k, \ell) \in S} k\ell > 0 \) \( \Rightarrow \) \( w \) \( \{ \) algebraic non-rational \( \} \)

- \( \left\langle \Psi, \Phi \right\rangle \) infinite \( \Rightarrow \) \( w \) non-holonomic
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\[ \langle \Psi, \Phi \rangle \text{ infinite } \Rightarrow w \text{ non-holonomic} \]

\[ w \text{ explicit } [\wp\text{-Weierstrass functions}] \]
Comparison between the nature of $Q$ and that of $w$ & $\tilde{w}$:

<table>
<thead>
<tr>
<th>Group</th>
<th>Covariance</th>
<th>$Q(x, y; t)$</th>
<th>$w(x; t) &amp; \tilde{w}(y; t)$</th>
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<tbody>
<tr>
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<tr>
<td>$\infty$</td>
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<td>?</td>
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Conjecture of Bousquet-Mélou and Mishna: “? = non-holonomic”
Comparison between the nature of $Q$ and that of $w$ & $\tilde{w}$:

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Conjecture of Bousquet-Mélou and Mishna: "? = non-holonomic"

Proof of the conjecture: [KR]
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3. Conclusion
A 3-steps method

- Continuation of the generating functions $Q(x, 0; t)$ and $Q(0, y; t)$

- Boundary value problems

\[ KQ(x, 0; t) - KQ(x, 0; t) = [...] \]
\[ KQ(x, 0; t) = [...] \]

- Analysis of $w(x; t)$ [uniformization]
A 3-steps method

- Continuation of the generating functions $Q(x, 0; t)$ and $Q(0, y; t)$

- Boundary value problems [Unit circle: topic of an exercise]

\[ KQ(x, 0; t) - \\ KQ(\overline{x}, 0; t) = [\ldots] \]

- Analysis of $w(x, t)$ [uniformization]
A 3-steps method

- Continuation of the generating functions \( Q(x,0; t) \) and \( Q(0,y; t) \)
- Boundary value problems

\[
KQ(x,0; t) - KQ(x,0; t) = [...]
\]

\[
KQ(x,0; t) - KQ(x,0; t) = [...]
\]

- Analysis of \( w(x; t) \) [uniformization]
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- Continuation of the generating functions \( Q(x, 0; t) \) and \( Q(0, y; t) \)
- Boundary value problems

\[
KQ(x, 0; t) - KQ(\overline{x}, 0; t) = [\ldots]
\]

\[
KQ([w^{-1}]^+(u), 0; t) - KQ([w^{-1}]^-(u), 0; t) = [\ldots]
\]
A 3-steps method

- Continuation of the generating functions $Q(x, 0; t)$ and $Q(0, y; t)$
- Boundary value problems
  
- Analysis of $w(x; t)$ [uniformization]
Introduction and main results

- Introduction
- Results

Proofs

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Conclusion
Boundary value problem of Riemann-Carleman type

There exists a curve $\mathcal{M}_t$, symmetrical w.r.t. the horizontal axis,

such that: $\forall u \in \mathcal{M}_t$,

$$K(u, 0; t) Q(u, 0; t) - K(\overline{u}, 0; t) Q(\overline{u}, 0; t) = uX_0^{-1}(u; t) - \overline{u}X_0^{-1}(\overline{u}; t),$$

$X_0$ being a root of the kernel $x \mapsto K(x, y; t) = xyt \left[ \sum_{(k, \ell) \in S} x^k y^\ell - 1/t \right]$. 

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How to obtain this Riemann-Carleman problem? (1/2)

The functional equation:

\[ K(x, y; t)Q(x, y; t) = K(x, 0; t)Q(x, 0; t) \]
\[ + K(0, y; t)Q(0, y; t) - K(0, 0; t)Q(0, 0; t) - xy. \]
How to obtain this Riemann-Carleman problem? (1/2)

The functional equation:

\[ K(x, y; t)Q(x, y; t) = K(x, 0; t)Q(x, 0; t) + K(0, y; t)Q(0, y; t) - K(0, 0; t)Q(0, 0; t) - xy. \]

Roots of the kernel:

\[ K(x, y; t) = xyt \left( \sum_{(k, \ell) \in S} x^k y^\ell - 1/t \right) = 0 \iff x = X_0(y; t) \text{ or } X_1(y; t). \]
How to obtain this Riemann-Carleman problem? (1/2)

The functional equation:

\[ K(x, y; t)Q(x, y; t) = K(x, 0; t)Q(x, 0; t) \]

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A new functional equation:

\[ 0 = K(X_\ell(y; t), 0; t)Q(X_\ell(y; t), 0; t) \]

\[ + K(0, y; t)Q(0, y; t) - K(0, 0; t)Q(0, 0; t) - X_\ell(y; t)y. \]
How to obtain this Riemann-Carleman problem? (1/2)

The functional equation:

\[ K(x, y; t)Q(x, y; t) = K(x, 0; t)Q(x, 0; t) \]
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Roots of the kernel:

\[ K(x, y; t) = xyt \left[ \sum_{(k,\ell) \in S} x^k y^\ell - 1/t \right] = 0 \iff x = X_0(y; t) \text{ or } X_1(y; t). \]

A new functional equation:

\[ 0 = K(X_\ell(y; t), 0; t)Q(X_\ell(y; t), 0; t) \]
\[ + K(0, y; t)Q(0, y; t) - K(0, 0; t)Q(0, 0; t) - X_\ell(y; t)y. \]

We get:

\[ K(X_0(y; t), 0; t)Q(X_0(y; t), 0; t) - K(X_1(y; t), 0; t)Q(X_1(y; t), 0; t) \]
\[ = X_0(y; t)y - X_1(y; t)y. \]
How to obtain this Riemann-Carleman problem? (2/2)

\[ K(X_0(y; t), 0; t)Q(X_0(y; t), 0; t) - K(X_1(y; t), 0; t)Q(X_1(y; t), 0; t) \]

\[ = X_0(y; t)y - X_1(y; t)y. \]
How to obtain this Riemann-Carleman problem? (2/2)

\[ K(X_0(y; t), 0; t)Q(X_0(y; t), 0; t) - K(X_1(y; t), 0; t)Q(X_1(y; t), 0; t) \]

\[ = X_0(y; t)y - X_1(y; t)y. \]

\( X_0(y; t) \) and \( X_1(y; t) \) are complex conjugate

\( Y_1(t) \) \( Y_2(t) \) \( Y_3(t) \) \( Y_4(t) \)

\( X_0(y; t) \) and \( X_1(y; t) \) are real
How to obtain this Riemann-Carleman problem? (2/2)

\[ K(X_0(y; t), 0; t)Q(X_0(y; t), 0; t) - K(X_1(y; t), 0; t)Q(X_1(y; t), 0; t) = X_0(y; t)y - X_1(y; t)y. \]

\( X_0(y; t) \) and \( X_1(y; t) \) are complex conjugate

\( X_0(y; t) \) and \( X_1(y; t) \) are real

\( X_0([y_1(t), y_2(t)]; t) \)

\( X_1([y_1(t), y_2(t)]; t) \)
Boundary value problem of Riemann-Carleman type

There exists a curve $\mathcal{M}_t$, symmetrical w.r.t. the horizontal axis, such that: $\forall u \in \mathcal{M}_t$,

$$K(u, 0; t)Q(u, 0; t) - K(\bar{u}, 0; t)Q(\bar{u}, 0; t) = uX_0^{-1}(u; t) - \bar{u}X_0^{-1}(\bar{u}; t),$$

$X_0$ being a root of the kernel $x \mapsto K(x, y; t) = xyt \left[ \sum_{(k,\ell) \in S} x^k y^\ell - 1/t \right]$. 

$\mathcal{M}_t$
Conformal gluing function

\[ w(x; t) \]
Conformal gluing function

Resolution of this boundary value problem of Riemann-Carleman type

\[ K(x, 0; t)Q(x, 0; t) - K(0, 0; t)Q(0, 0; t) = \]
\[ \frac{1}{2\pi i} \int_{\mathcal{M}_t} uX_0^{-1}(u; t) \left[ \frac{\partial_u w(u; t)}{w(u; t) - w(x; t)} - \frac{\partial_u w(u; t)}{w(u; t) - w(0; t)} \right] du. \]
1. Introduction and main results
   - Introduction
   - Results

2. Proofs
   - Explicit expression of the counting generating functions
     - Reduction to boundary value problems
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   - Nature of the counting generating functions

3. Conclusion
Example: the unit circle
Example: the unit circle

\[ w(x) = \frac{1}{2} \left(x + \frac{1}{x}\right) \text{ is a good CGF} \]
Example: the unit circle

\[ w(x) = \frac{1}{2} \left(x + \frac{1}{x}\right) \] is a good CGF:

\[ w(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2} = w(e^{-i\theta}). \]
Example: the unit circle

\[ w(x) = \frac{1}{2} \left( x + \frac{1}{x} \right) \] is a good CGF: \[ w(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2} = w(e^{-i\theta}). \]

Main idea: transforming the curve \( M_t \) into a simple curve

In our case:

\[ \begin{array}{c}
\omega_1^t \\
\omega_1^t / 2 \\
0 \\
\omega_2^t / 2 + \omega_3^t / 2 \\
\omega_2^t \\
\end{array} \]
Different formulations for $\mathcal{K}_t$

$$\mathcal{K}_t = \{ (x, y) \in \mathbb{C}^2 : \sum_{(k, \ell) \in S} x^k y^\ell - 1/t = 0 \}$$

$$= \{ (u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - gt^2 - 2u - 2 \}.$$
Different formulations for $\mathcal{K}_t$

$$\mathcal{K}_t = \{(x, y) \in \mathbb{C}^2 : \sum_{(k, \ell) \in S} x^k y^\ell - 1/t = 0\}$$

$$= \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2^t u - g_3^t\}.$$
Different formulations for $\mathcal{K}_t$

\[\mathcal{K}_t = \{(x, y) \in \mathbb{C}^2 : \sum_{(k, \ell) \in S} x^k y^\ell - 1/t = 0\}\]

\[= \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2^t u - g_3^t\}.\]

Riemann surface of the square root of a third degree polynomial

Let $g_2^3 - 27g_3^2 \neq 0$ and $\mathcal{L} = \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2 u - g_3\}.$
Different formulations for $\mathcal{K}_t$

$$\mathcal{K}_t = \{(x, y) \in \mathbb{C}^2 : \sum_{(k, \ell) \in S} x^k y^\ell - 1/t = 0\}$$

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Riemann surface of the square root of a third degree polynomial

Let $g_2^3 - 27g_3^2 \neq 0$ and $\mathcal{L} = \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2 u - g_3\}$.

- $\mathcal{L} \cong \mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})$.
- $\mathcal{L} \cong \{(1/n, \text{ph}(1/n)) : n \in \mathbb{Z}/(2 \mathbb{Z} + 3 \mathbb{Z})\}$.
- If $e_1 < e_2 < e_3$ are the roots of $4u^3 - g_2 u - g_3$, then

$$\mathcal{L} \text{ is stable by } (u, v) \mapsto (u, -v) \iff \mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}) \text{ is stable by } \phi(u) = -u \iff [u + \omega_1] = 0.$$
Different formulations for $\mathcal{K}_t$

\[
\mathcal{K}_t = \{(x, y) \in \mathbb{C}^2 : \sum_{(k, \ell) \in S} x^k y^\ell - 1/t = 0\}
= \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g^t_2 u - g^t_3\}.
\]

Riemann surface of the square root of a third degree polynomial

Let $g^3_2 - 27g^2_3 \neq 0$ and $\mathcal{L} = \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g^t_2 u - g^t_3\}$.

- $\mathcal{L} \cong \mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})$.
- $\mathcal{L} = \{(\varphi(\omega; \omega_1, \omega_2), \varphi'(\omega; \omega_1, \omega_2)) : \omega \in \mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})\}$.
  - If $\omega_1 < \omega < \omega_2$ are the roots of $4u^3 - g^t_2 u - g^t_3$, then
    \[
    \omega_1 \quad \omega_2
    \]
    \[
    0
    \]

- $\mathcal{L}$ is stable by $(u, v) \mapsto (u, -v)$, $\mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})$ is stable by \[
\psi(\omega) = -\omega + [\omega_1 + \omega_2].
\]
Different formulations for $\mathcal{K}_t$

$$\mathcal{K}_t = \{(x, y) \in \mathbb{C}^2 : \sum_{(k, \ell) \in S} x^k y^\ell - 1/t = 0\}$$

$$= \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2^t u - g_3^t\}.$$ 

Riemann surface of the square root of a third degree polynomial

Let $g_2^3 - 27g_3^2 \neq 0$ and $\mathcal{L} = \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2 u - g_3\}$.

- $\mathcal{L} \cong \mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})$.
- $\mathcal{L} = \{(\wp(\omega; \omega_1, \omega_2), \wp'(\omega; \omega_1, \omega_2)) : \omega \in \mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})\}$.
- If $e_3 < e_2 < e_1$ are the roots of $4u^3 - g_2 u - g_3$, then

$$\begin{array}{|c|c|}
\hline
\omega_1/2 & \omega_1/2 \\
\hline
\hline
0 & \omega_2/2 \\
\hline
\hline
\omega_2 & \omega_2 \\
\hline
\hline
0 & \omega_2/2 \\
\hline
\end{array}$$

- $\mathcal{L}$ is stable by $(u, v) \mapsto (u, -v)$ and $\mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})$ is stable by $\wp(\omega) = -\omega + \omega_1 + \omega_2$. 

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Different formulations for $\mathcal{K}_t$

\[
\mathcal{K}_t = \left\{ (x, y) \in \mathbb{C}^2 : \sum_{(k, \ell) \in S} x^k y^\ell - 1/t = 0 \right\} = \left\{ (u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2^t u - g_3^t \right\}.
\]

Riemann surface of the square root of a third degree polynomial

Let $g_2^3 - 27g_3^2 \neq 0$ and $\mathcal{L} = \left\{ (u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2 u - g_3 \right\}$.

- $\mathcal{L} \cong \mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})$.
- $\mathcal{L} = \left\{ (\wp(\omega; \omega_1, \omega_2), \wp'(\omega; \omega_1, \omega_2)) : \omega \in \mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}) \right\}$.
- If $e_3 < e_2 < e_1$ are the roots of $4u^3 - g_2 u - g_3$, then

\[
\begin{array}{c|c|c}
\omega_1 & \psi(\omega) \\
\hline
\omega_1/2 & e_3 & e_2 \\
\hline
\omega_2/2 & \omega & e_1 \\
\hline
0 & \infty & \omega_2
\end{array}
\]

- $\mathcal{L}$ is stable by $(u, v) \mapsto (u, -v) \iff \mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})$ is stable by $\psi(\omega) = -\omega + [\omega_1 + \omega_2]$. 
A symmetric view point

\[ \psi(\omega) \]

\[ x_3(t) \]

\[ x_4(t) \]

\[ x_2(t) \]

\[ x_1(t) \]

\[ 0 \quad \omega_t^2 / 2 \quad \omega_t^2 \]

\[ \omega_1^t \quad \omega_1^t / 2 \]
A symmetric view point

\begin{align*}
\omega_1^t & \quad \psi(\omega) \\
\omega_1^t/2 & \quad \omega \\
\omega_2^t/2 & \quad x_2(t)
\end{align*}

\begin{align*}
\omega_1^t & \quad \phi(\omega) \\
\omega_1^t/2 & \quad \omega \\
\omega_2^t/2 & \quad y_2(t)
\end{align*}

\begin{align*}
\omega_1^t & \quad x_3(t) \\
\omega_1^t/2 & \quad x_4(t)
\end{align*}

\begin{align*}
\omega_1^t & \quad y_3(t) \\
\omega_1^t/2 & \quad y_4(t)
\end{align*}
Explicit expression of the counting generating functions

Nature of the counting generating functions

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Explicit expression of the counting generating functions

Nature of the counting generating functions

A symmetric view point

\[ \psi(\omega) = -\omega + [\omega_1^t + \omega_2^t] \]
A symmetric view point

\[\psi(\omega) = -\omega + [\omega_1^t + \omega_2^t]\]

\[\phi(\omega) = -\omega + [\omega_1^t + \omega_2^t]\]
Explicit expression of the counting generating functions

\[ \psi(\omega) = -\omega + [\omega_1^t + \omega_2^t] \]

\[ \phi(\omega) = -\omega + [\omega_1^t + \omega_2^t] \]

\[ \psi(\omega) = -\omega + [\omega_1^t + \omega_2^t + \omega_3^t] \]

\[ \phi \circ \psi(\omega) = \omega + \omega_3^t \]
Conformal gluing function

\[ X_0([y_1(t), y_2(t)]; t) \]

\[ X_1([y_1(t), y_2(t)]; t) \]

\[ w(u; t) = w(u; t) \]

\[ w(u; t) = w(u; t) \]

\[ w(u; t) = w(u; t) \]
Conformal gluing function

\[ X_0([y_1(t), y_2(t)]; t) \]

\[ X_1([y_1(t), y_2(t)]; t) \]

\[ w(u; t) = w(\bar{u}; t) \]

\[ w(x(\omega); t) \parallel w(x(-\omega + [\omega_1^t + \omega_2^t + \omega_3^t]); t) \]
Expression of the CGFs $w$ & $\tilde{w}$

We have

$$w(x(\omega); t) = \varphi(\omega - \frac{\omega_1^t + \omega_2^t}{2}; \omega_1^t, \omega_3^t).$$
Expression of the CGFs $w$ & $\tilde{w}$

We have

$$w(x(\omega); t) = \varphi(\omega - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t).$$

By projection:

$$w(u; t) = \varphi(x^{-1}(u) - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t).$$
Expression of the CGFs $w$ & $\tilde{w}$

We have

$$w(x(\omega); t) = \phi(\omega - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t).$$

By projection:

$$w(u; t) = \phi(x^{-1}(u) - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t).$$

Nature of the CGFs $w$ & $\tilde{w}$

- $x^{-1}(u) = \phi^{-1}(f(u); \omega_1^t, \omega_2^t)$, $f$ affine function;
Expression of the CGFs $w$ & $\tilde{w}$

We have

$$w(x(\omega); t) = \varphi(\omega - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t).$$

By projection:

$$w(u; t) = \varphi(x^{-1}(u) - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t).$$

Nature of the CGFs $w$ & $\tilde{w}$

- $x^{-1}(u) = \varphi^{-1}(f(u); \omega_1^t, \omega_2^t)$, $f$ affine function;

- Nature of $\varphi(\varphi^{-1}(u; \omega_1^t, \omega_2^t) - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t)$;
Expression of the CGFs \( w \) & \( \tilde{w} \)

We have

\[
    w(x(\omega); t) = \wp(\omega - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t).
\]

By projection:

\[
    w(u; t) = \wp(x^{-1}(u) - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t).
\]

Nature of the CGFs \( w \) & \( \tilde{w} \)

- \( x^{-1}(u) = \wp^{-1}(f(u); \omega_1^t, \omega_2^t), \) \( f \) affine function;
- Nature of \( \wp(\wp^{-1}(u; \omega_1^t, \omega_2^t) - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t); \)
- Nature of \( \sin^2 \left( \frac{\omega_2^t}{\omega_3^t} \arcsin(u) \right) \)
Expression of the CGFs $w$ & $\tilde{w}$

We have

$$w(x(\omega); t) = \varphi(\omega - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t).$$

By projection:

$$w(u; t) = \varphi(x^{-1}(u) - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t).$$

Nature of the CGFs $w$ & $\tilde{w}$

- $x^{-1}(u) = \varphi^{-1}(f(u); \omega_1^t, \omega_2^t)$, $f$ affine function;

- Nature of $\varphi(\varphi^{-1}(u; \omega_1^t, \omega_2^t) - [\omega_1^t + \omega_2^t]/2; \omega_1^t, \omega_3^t)$;

- Nature of $\sin^2 \left( \frac{\omega_2^t}{\omega_3^t} \arcsin(u) \right)$ [Topic of an exercise].
1. Introduction and main results
   - Introduction
   - Results

2. Proofs
   - Explicit expression of the counting generating functions
     - Reduction to boundary value problems
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   - Nature of the counting generating functions

3. Conclusion
An enlightening example: the logarithm function

- The function
  \[ \log_0(x) = \log(|x|) + \imath \arg(x) \]
  
is holomorphic for \( x \in \mathbb{C} \setminus \mathbb{R}_- \);
An enlightening example: the logarithm function

- The function
  \[ \log_0(x) = \log(|x|) + \Im \arg(x) \]
  is holomorphic for \( x \in \mathbb{C} \setminus \mathbb{R}_- \);
- \( \log_0 \) doesn't admit a direct meromorphic continuation through \( \mathbb{R}_- \);
An enlightening example: the logarithm function

The function

$$
\log_0(x) = \log(|x|) + \mathbf{i} \arg(x)
$$

is holomorphic for $x \in \mathbb{C} \setminus \mathbb{R}_-$;

$\log_0$ doesn’t admit a direct meromorphic continuation through $\mathbb{R}_-$;

$\log_0$ has a meromorphic continuation along a path going through $\mathbb{R}_-$, say $\log_1$.
An enlightening example: the logarithm function

The function

\[ \log_0(x) = \log(|x|) + \imath \arg(x) \]

is holomorphic for \( x \in \mathbb{C} \setminus \mathbb{R}_- \);

\( \log_0 \) doesn't admit a \textbf{direct} meromorphic continuation through \( \mathbb{R}_- \);

\( \log_0 \) has a meromorphic continuation \textbf{along a path} going through \( \mathbb{R}_- \), say \( \log_1 \);

\( \log_1 \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R}_- \) and

\[ \log_1 = \log_0 + 2\pi \imath; \]
An enlightening example: the logarithm function

- The function
  \[ \log_0(x) = \log(|x|) + \imath \arg(x) \]
  is holomorphic for \( x \in \mathbb{C} \setminus \mathbb{R}_- \);
- \( \log_0 \) doesn’t admit a **direct** meromorphic continuation through \( \mathbb{R}_- \);
- \( \log_0 \) has a meromorphic continuation **along a path** going through \( \mathbb{R}_- \), say \( \log_1 \);
- \( \log_1 \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R}_- \) and
  \[ \log_1 = \log_0 + 2\pi \imath; \]
- **Similarly**, we can define \( \log_\ell \) for any \( \ell \in \mathbb{Z} \), and
  \[ \log_\ell = \log_0 + 2\ell \pi \imath; \]
An enlightening example: the logarithm function

- The function
  \[ \log_0(x) = \log(|x|) + i \arg(x) \]
  is holomorphic for \( x \in \mathbb{C} \setminus \mathbb{R}_- \);
- \( \log_0 \) doesn’t admit a direct meromorphic continuation through \( \mathbb{R}_- \);
- \( \log_0 \) has a meromorphic continuation along a path going through \( \mathbb{R}_- \), say \( \log_1 \);
- \( \log_1 \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R}_- \) and
  \[ \log_1 = \log_0 + 2\pi i; \]
- Similarly we can define \( \log_\ell \) for any \( \ell \in \mathbb{Z} \), and
  \[ \log_\ell = \log_0 + 2\ell \pi i; \]
- \( \log_0 \) is holonomic, with vanishing differential equation \( xy' - 1 = 0 \);
An enlightening example: the logarithm function

- The function
  \[ \log_0(x) = \log(|x|) + \iota \arg(x) \]
  
is holomorphic for \( x \in \mathbb{C} \setminus \mathbb{R}_- \);
- \( \log_0 \) doesn’t admit a direct meromorphic continuation through \( \mathbb{R}_- \);
- \( \log_0 \) has a meromorphic continuation along a path going through \( \mathbb{R}_- \), say \( \log_1 \);
- \( \log_1 \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R}_- \) and
  \[ \log_1 = \log_0 + 2\pi \iota; \]
- Similarly we can define \( \log_\ell \) for any \( \ell \in \mathbb{Z} \), and
  \[ \log_\ell = \log_0 + 2\ell \pi \iota; \]
- \( \log_0 \) is holonomic, with vanishing differential equation \( xy' - 1 = 0 \);
- For all \( \ell \in \mathbb{Z} \), \( \log_\ell \) is also holonomic, with the same equation.
An enlightening example: the logarithm function

- The function
  \[ \log_0(x) = \log(|x|) + i \arg(x) \]
  is holomorphic for \( x \in \mathbb{C} \setminus \mathbb{R}_- \);
- \( \log_0 \) doesn’t admit a **direct** meromorphic continuation through \( \mathbb{R}_- \);
- \( \log_0 \) has a meromorphic continuation **along a path** going through \( \mathbb{R}_- \), say \( \log_1 \);
- \( \log_1 \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R}_- \) and
  \[ \log_1 = \log_0 + 2\pi i; \]
- Similarly we can define \( \log_\ell \) for any \( \ell \in \mathbb{Z} \), and
  \[ \log_\ell = \log_0 + 2\ell \pi i; \]
- \( \log_0 \) is holonomic, with vanishing differential equation \( xy' - 1 = 0 \);
- For all \( \ell \in \mathbb{Z} \), \( \log_\ell \) is also holonomic, with the same equation:
  - **Expression of \( \log_\ell \) in terms of \( \log_0 \);**
An enlightening example: the logarithm function

- The function
  \[ \log_0(x) = \log(|x|) + \imath \arg(x) \]

  is holomorphic for \( x \in \mathbb{C} \setminus \mathbb{R}_- \);

- \( \log_0 \) doesn’t admit a direct meromorphic continuation through \( \mathbb{R}_- \);

- \( \log_0 \) has a meromorphic continuation along a path going through \( \mathbb{R}_- \), say \( \log_1 \);

- \( \log_1 \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R}_- \) and

  \[ \log_1 = \log_0 + 2\pi \imath; \]

- Similarly we can define \( \log_\ell \) for any \( \ell \in \mathbb{Z} \), and

  \[ \log_\ell = \log_0 + 2\ell \pi \imath; \]

- \( \log_0 \) is holonomic, with vanishing differential equation \( xy' - 1 = 0 \);

- For all \( \ell \in \mathbb{Z} \), \( \log_\ell \) is also holonomic, with the same equation:
  - Expression of \( \log_\ell \) in terms of \( \log_0 \);
  - Reasoning via a meromorphic continuation along a path.
Our reasoning

The branches of $Q(x, 0; t)$:

$Q_0(x, 0; t)$

$Q_1(x, 0; t)$

$Q_2(x, 0; t)$

$Q_3(x, 0; t)$
Our reasoning

- The branches of $Q(x, 0; t)$:

$$Q_0(x, 0; t) \quad Q_1(x, 0; t) \quad Q_2(x, 0; t) \quad Q_3(x, 0; t)$$
Our reasoning

- The branches of $Q(x, 0; t)$:

  \[ Q_0(x, 0; t) \]

  \[ Q_1(x, 0; t) \]

  \[ Q_2(x, 0; t) \]

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Our reasoning

- The branches of $Q(x, 0; t)$:

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Kilian Raschel
Chemins dans le quart de plan
Our reasoning

The branches of \( Q(x, 0; t) \):

\[ Q_0(x, 0; t) \]

\[ Q_1(x, 0; t) \]

\[ Q_2(x, 0; t) \]

\[ Q_3(x, 0; t) \]
Our reasoning

- The branches of \( Q(x, 0; t) \):

  - \( Q_0(x, 0; t) \)
  - \( Q_1(x, 0; t) \)
  - \( Q_2(x, 0; t) \)
  - \( Q_3(x, 0; t) \)

- There are infinitely many poles.
Our reasoning

- The branches of $Q(x, 0; t)$:
  
  \[
  Q_0(x, 0; t) \quad Q_1(x, 0; t) \\
  Q_3(x, 0; t) \quad Q_2(x, 0; t)
  \]

- There are infinitely many poles.
- If $Q(x, 0; t)$ satisfies a differential equation, all its branches satisfy the same equation.
The universal covering

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</table>
The universal covering

The diagram illustrates the universal covering of a specific portion of the plane, with axes labeled as follows:

- The vertical axis is labeled from $0$ to $2\omega^t_1$, with intermediate labels $\omega^t_1/2$.
- The horizontal axis is labeled from $0$ to $2\omega^t_2$, with intermediate labels $\omega^t_3/2, \omega^t_2/2, \omega^t_2$.

Key points and labels include:
- $x_3(t)$, $y_3(t)$
- $x_2(t)$, $y_2(t)$
- $x_4(t)$, $y_4(t)$
- $x_1(t)$, $y_1(t)$

These labels indicate the coordinates or values associated with specific points on the plane.
A functional equation for $q_x(\omega) = Q(x(\omega), 0; t)$ on the universal covering

We have: $q_x(\omega + \omega_3^t) = q_x(\omega) + xy(\omega + \omega_3^t) - xy(-\omega)$. 
Consequence of the functional equation for \( q_x(\omega) = Q(x(\omega), 0; t) \)

Remember: \( q_x(\omega + \omega_3 t) = q_x(\omega) + xy(\omega + \omega_3 t) - xy(-\omega) \).
Consequence of the functional equation for $q_x(\omega) = Q(x(\omega), 0; t)$

Remember: $q_x(\omega + \omega_3^t) = q_x(\omega) + xy(\omega + \omega_3^t) - xy(-\omega)$.

- Known results in the finite group case;
Consequence of the functional equation for $q_x(\omega) = Q(x(\omega), 0; t)$

Remember: $q_x(\omega + \omega_3^t) = q_x(\omega) + xy(\omega + \omega_3^t) - xy(-\omega)$.

- Known results in the finite group case;
- Poles and non-holonomy in the infinite group case.
Consequence of the functional equation for $q_x(\omega) = Q(x(\omega), 0; t)$

Remember: $q_x(\omega + \omega_3^t) = q_x(\omega) + xy(\omega + \omega_3^t) - xy(-\omega)$.

- Known results in the finite group case;
- Poles and non-holonomy in the infinite group case.

Proof of the functional equation on the universal covering

$$KQ(x, y; t) = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy.$$
Consequence of the functional equation for $q_x(\omega) = Q(x(\omega), 0; t)$

Remember: $q_x(\omega + \omega_3^t) = q_x(\omega) + xy(\omega + \omega_3^t) - xy(-\omega)$.

- Known results in the finite group case;
- Poles and non-holonomy in the infinite group case.

Proof of the functional equation on the universal covering

$$KQ(x, y; t) = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy.$$  

If $K(x, y; t) = 0$,  

$$0 = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy,$$
$$0 = KQ(\Phi(x, 0); t) + KQ(0, y; t) - KQ(0, 0; t) - \Phi(xy).$$
Consequence of the functional equation for \( q_x(\omega) = Q(x(\omega), 0; t) \)

Remember: \( q_x(\omega + \omega^t_3) = q_x(\omega) + xy(\omega + \omega^t_3) - xy(-\omega) \).

- Known results in the finite group case;
- Poles and non-holonomy in the infinite group case.

Proof of the functional equation on the universal covering

\[
KQ(x, y; t) = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy.
\]

If \( K(x, y; t) = 0 \),

\[
0 = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy,
0 = KQ(\Phi(x, 0); t) + KQ(0, y; t) - KQ(0, 0; t) - \Phi(xy).
\]

Making the difference gives

\[
KQ(\Phi(x, 0); t) - KQ(x, 0; t) = \Phi(xy) - xy.
\]
Consequence of the functional equation for $q_x(\omega) = Q(x(\omega), 0; t)$

Remember: $q_x(\omega + \omega t^3) = q_x(\omega) + xy(\omega + \omega t^3) - xy(-\omega)$.

- Known results in the finite group case;
- Poles and non-holonomy in the infinite group case.

Proof of the functional equation on the universal covering

\[ KQ(x, y; t) = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy. \]

If $K(x, y; t) = 0$,

\[ 0 = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy, \]
\[ 0 = KQ(\Phi(x, 0); t) + KQ(0, y; t) - KQ(0, 0; t) - \Phi(xy). \]

Making the difference gives

\[ KQ(\Psi \circ \Phi(x, 0); t) - KQ(x, 0; t) = \Phi(xy) - xy. \]
Consequence of the functional equation for \( q_x(\omega) = Q(x(\omega), 0; t) \)

Remember: \( q_x(\omega + \omega^t_3) = q_x(\omega) + xy(\omega + \omega^t_3) - xy(-\omega). \)

- Known results in the finite group case;
- Poles and non-holonomy in the infinite group case.

Proof of the functional equation on the universal covering

\[
KQ(x, y; t) = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy.
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If \( K(x, y; t) = 0 \),

\[
0 = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy,
\]

\[
0 = KQ(\Phi(x, 0); t) + KQ(0, y; t) - KQ(0, 0; t) - \Phi(xy).
\]

Making the difference gives

\[
KQ(\Psi \circ \Phi(x, 0); t) - KQ(x, 0; t) = \Phi(xy) - xy.
\]

Remember:

\[
\Psi \circ \Phi \longleftrightarrow \omega \mapsto \omega - \omega^t_3, \quad \Phi \longleftrightarrow \omega \mapsto -\omega + [\omega^t_2 + \omega^t_3].
\]
1 Introduction and main results
   • Introduction
   • Results

2 Proofs
   • Explicit expression of the counting generating functions
     • Reduction to boundary value problems
     • Conformal gluing and uniformization
   • Nature of the counting generating functions

3 Conclusion
Perspectives

- Non-holonomy of the counting generating functions in the variable $z$;
Perspectives

- Non-holonomy of the counting generating functions in the variable $z$;
- Slight extensions of the model (like weighted paths or more general behavior on the boundary);
Perspectives

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- More general jumps;
Perspectives

- Non-holonomy of the counting generating functions in the variable $z$;
- Slight extensions of the model (like weighted paths or more general behavior on the boundary);
- More general jumps;
- Higher dimension.
Thanks for your attention!