AN INVITATION TO ANALYTIC COMBINATORICS IN SEVERAL VARIABLES – LECTURE 2

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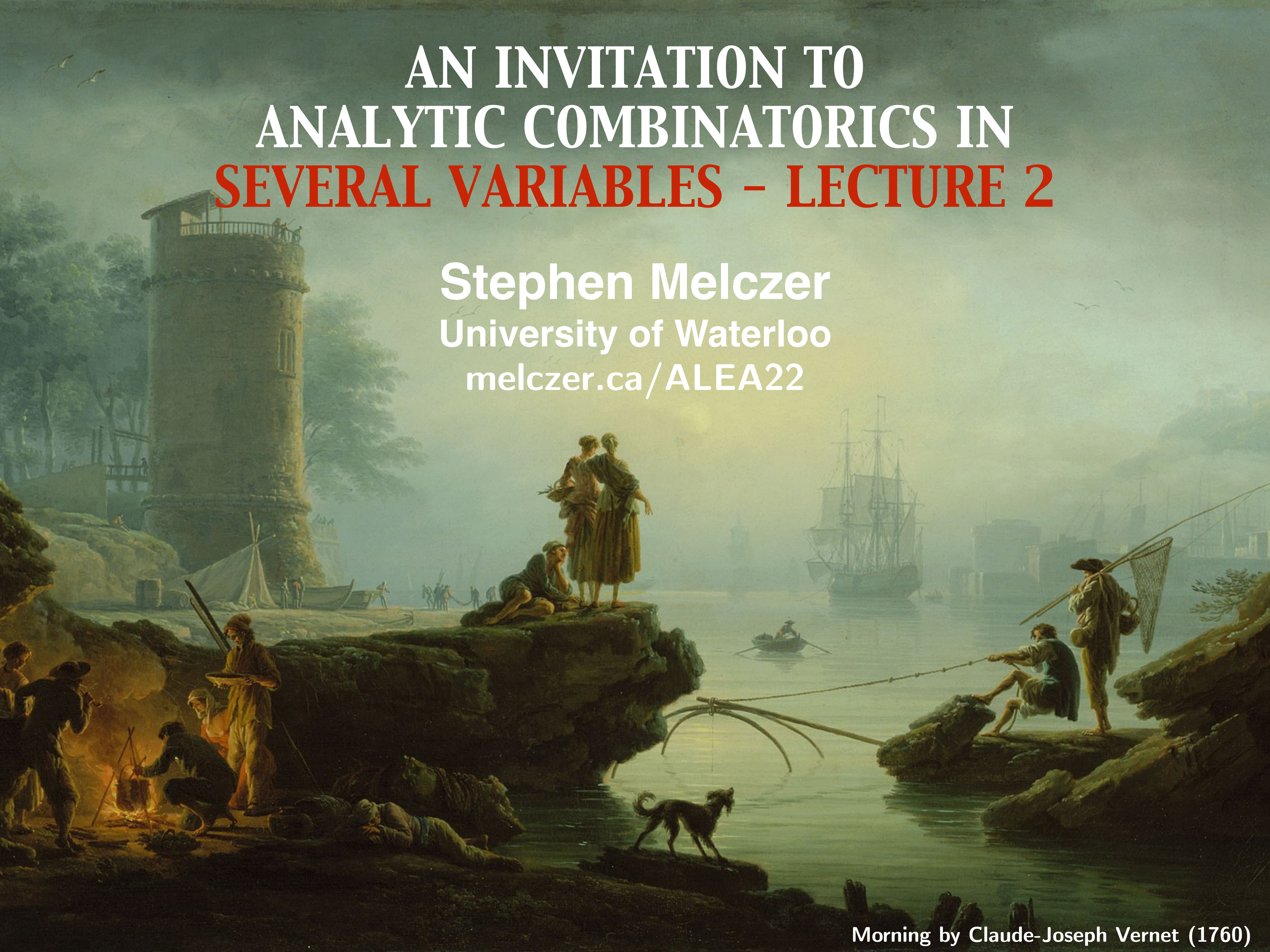

Morning by Claude-Joseph Vernet (1760)

Analytic Combinatorics in Several Variables

We study the **r**-diagonals of

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})} = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$$

Assume $H(\mathbf{z}) = H_{z_1}(\mathbf{z}) = \cdots = H_{z_d}(\mathbf{z}) = 0$ has no solution

Then the **singular variety** $\mathcal{V} = \{\mathbf{z} \in \mathbb{C}^d : H(\mathbf{z}) = 0\}$ is **smooth**

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Then the **singular variety** $\mathcal{V} = \{\mathbf{z} \in \mathbb{C}^d : H(\mathbf{z}) = 0\}$ is **smooth**

Minimal Points: Coordinate-wise smallest singularities

Critical Points: Solve $H = 0, \qquad r_j z_1 H_{z_1} = r_1 z_j H_{z_j} \quad (2 \leq j \leq d)$

Main Theorem of Smooth ACSV

Suppose that

$$H = 0, \qquad r_j z_1 H_{z_1} = r_1 z_j H_{z_j} \quad (2 \leq j \leq d)$$

admits a finite number of solutions. If

- there is exactly one minimal solution, $\mathbf{w} \in \mathbb{C}_*^d$
- $H_{z_d}(\mathbf{w})$ and $\det \mathcal{M}$ are non-zero,

then

$$[\mathbf{z}^{n\mathbf{r}}]\frac{G(\mathbf{z})}{H(\mathbf{z})} = \mathbf{w}^{-n\mathbf{r}}\ (nr_d)^{(1-d)/2}\ (2\pi)^{(1-d)/2} \det(\mathcal{M})^{-1/2} \left(\frac{-G(\mathbf{w})}{w_d\, H_{z_d}(\mathbf{w})} + O\left(\frac{1}{n}\right)\right)$$

If there are a finite number of **critical points** with the same coordinate-wise modulus as $\mathbf{w}$, all satisfying these conditions, then we can add their asymptotic contributions.

Topic 3
Higher-Order Terms

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Higher-Order Terms

Under the assumptions of the Main Theorem of Smooth ACSV, for any $M \in \mathbb{N}$ there is an **expansion**

$$[\mathbf{z}^{n\mathbf{r}}]F(\mathbf{z}) = \mathbf{w}^{-n\mathbf{r}}(2\pi n r_d)^{(1-d)/2} \det(\mathcal{M})^{-1/2} \left(\sum_{j=0}^{M} C_j (r_d n)^{-j} + O\left(n^{-M-1}\right) \right)$$

with each C_j **explicitly computable** from the derivatives of G and H up to order $2(j+2)$ evaluated at $\mathbf{w}$

Higher-Order Terms

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$$C_j = (-1)^j \sum_{0 \le \ell \le 2j} \left. \frac{\mathcal{E}^{\ell+j}\left(P(\theta)\,\psi(\theta)^{\ell}\right)}{2^{\ell+j}\ell!(\ell+j)!} \right|_{\theta=\mathbf{0}} \quad \text{for} \quad \mathcal{E} = -\sum_{1 \le i,j \le k} \left(\mathcal{M}^{-1}\right)_{ij} \partial_i \partial_j$$

and

$$P(\theta) = \frac{-G\left(\widehat{\mathbf{w}}e^{i\theta}, g\left(\widehat{\mathbf{w}}e^{i\theta}\right)\right)}{g\left(\widehat{\mathbf{w}}e^{i\theta}\right) H_{z_d}\left(\widehat{\mathbf{w}}e^{i\theta}, g\left(\widehat{\mathbf{w}}e^{i\theta}\right)\right)}$$

$$\psi(\theta) = \log\left(\frac{g\left(\widehat{\mathbf{w}}e^{i\theta}\right)}{g(\widehat{\mathbf{w}})}\right) + i(\hat{\mathbf{r}} \cdot \theta)/r_d - (1/2)\theta \cdot \mathcal{M} \cdot \theta^T$$

```
def smoothContrib(G,H,r,vars,CP,M,g):
    # Preliminary definitions
    dd = len(vars)
    field = SR
    tvars = list(var('t%d'%i) for i in range(dd-1))
    dvars = list(var('dt%d'%i) for i in range(dd-1))

    # Define differential Weyl algebra and set variable names
    W = DifferentialWeylAlgebra(PolynomialRing(field,tvars))
    WR = W.base_ring()
    T = PolynomialRing(field,tvars).gens()
    D = list(W.differentials())

    # Compute Hessian matrix and differential operator Epsilon
    HES = getHes(H,r,vars,CP)
    HESinv = HES.inverse()
    v = matrix(W,[D[k] for k in range(dd-1)])
    Epsilon = -(v * HESinv.change_ring(W) * v.transpose())[0,0]

    # Define quantities for calculating asymptotics
    tsubs = [v == v.subs(CP)*exp(I*t) for [v,t] in zip(vars,tvars)]
    tsubs += [vars[-1]==g.subs(tsubs)]
    P = (-G/g/diff(H,vars[-1])).subs(tsubs)
    psi = log(g.subs(tsubs)/g.subs(CP)) + I * add([r[k]*tvars[k] for k in range(dd-1)])/r[-1]
    v = matrix(SR,[tvars[k] for k in range(dd-1)])
    psiTilde = psi - (v * HES * v.transpose())[0,0]/2

    # Recursive function to convert symbolic expression to polynomial in t variables
    def to_poly(p,k):
        if k == 0:
            return add([a*T[k]^int(b) for [a,b] in p.coefficients(tvars[k])])
        return add([to_poly(a,k-1)*T[k]^int(b) for [a,b] in p.coefficients(tvars[k])])

    # Compute Taylor expansions to sufficient orders
    N = 2*M
    PsiSeries = to_poly(taylor(psiTilde,*((v,0) for v in tvars), N),dd-2)
    PSeries = to_poly(taylor(P,*((v,0) for v in tvars), N),dd-2)

    # Precompute products used for asymptotics
    EE = [Epsilon^k for k in range(3*M-2)]
    PP = [PSeries] + [0 for k in range(2*M-2)]
    for k in range(1,2*M-1):
        PP[k] = PP[k-1]*PsiSeries

    # Function to compute constants appearing in asymptotic expansion
    def Clj(l,j):
        return (-1)^j*SR(eval_op(EE[l+j],PP[l]))/(2^(l+j)*factorial(l)*factorial(l+j))

    # Compute different parts of asymptotic expansion
    var('n')
    ex = (prod([1/v^k for (v,k) in zip(vars,r)]).subs(CP).canonicalize_radical())^n
    pw = (r[-1]*n)^((1-dd)/2)
    se = sqrt((2*pi)^(1-dd)/HES.det()) * add([add([Clj(l,j) for l in range(2*j+1)])/(r[-1]*n)^j for j in range(M)])

    return ex, pw, se.canonicalize_radical()
```

Sage code available on melczer.ca/textbook/

Note: Requires g explicitly

Vanishing Terms

If $G(\mathbf{w}) \neq 0$ then higher order terms give more accuracy

$$[x^n y^n]\frac{1}{1-x-y} = 4^n \left(\frac{1}{\sqrt{\pi} n^{1/2}} - \frac{1}{8\sqrt{\pi} n^{3/2}} + \frac{1}{128\sqrt{\pi} n^{5/2}} + O\left(n^{-7/2}\right) \right)$$

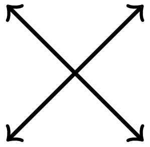
If $G(\mathbf{w}) = 0$ then higher order terms may give dominant asymptotics

$$[x^n y^n]\frac{x-2y^2}{1-x-y} = 4^n \left(\frac{1}{4\sqrt{\pi} n^{3/2}} + \frac{3}{32\sqrt{\pi} n^{5/2}} + O\left(n^{-7/2}\right) \right)$$

In the (rare) **worse case**, all terms may be zero!

$$[x^n y^n]\frac{x-y}{1-x-y} = O\left(\frac{4^n}{n^M}\right) \quad \text{for all } M > 0$$

Lattice Path Enumeration

The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\} =$  is

$$\left[(xyt)^n\right]\frac{(1+x)(1+y)}{1-txyS(x,y)} \sim \frac{2}{\pi}\cdot\frac{4^n}{n}$$

where $S(x,y) = xy + \frac{x}{y} + \frac{y}{x} + \frac{1}{xy}$.

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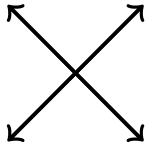
The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$ =  is

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The critical points are

$$\left(1, 1, \tfrac{1}{4}\right) \quad \left(-1, 1, \tfrac{1}{4}\right) \quad \left(1, -1, \tfrac{1}{4}\right) \quad \left(-1, -1, \tfrac{1}{4}\right)$$

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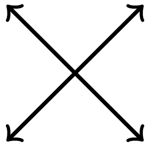
The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\} =$  is

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Numerator Vanishes

Lattice Path Enumeration

The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$ and **ending on the *x*-axis** is

$$\left[(xyt)^n\right]\frac{(1-x^2)(1+y)}{1-txyS(x,y)} \;\sim\; \frac{2\,(1+(-1)^n)}{\pi}\cdot\frac{4^n}{n^2}$$

The critical points are still

$$\left(1,1,\tfrac{1}{4}\right) \quad \left(-1,1,\tfrac{1}{4}\right) \quad \left(1,-1,\tfrac{1}{4}\right) \quad \left(-1,-1,\tfrac{1}{4}\right)$$

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$$\left(1,1,\tfrac{1}{4}\right) \quad \left(-1,1,\tfrac{1}{4}\right)$$

Numerator Vanishes to First Order

$$\left(1,-1,\tfrac{1}{4}\right) \quad \left(-1,-1,\tfrac{1}{4}\right)$$

Numerator Vanishes to Second Order

Lattice Path Enumeration

The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$ and **ending at the origin** is

$$\left[(xyt)^n\right] \frac{(1-x^2)(1-y^2)}{1-txyS(x,y)} \sim \frac{4\left(1+(-1)^n\right)}{\pi} \cdot \frac{4^n}{n^3}$$

The critical points are still

$$\left(1,1,\tfrac{1}{4}\right) \quad \left(-1,1,\tfrac{1}{4}\right) \quad \left(1,-1,\tfrac{1}{4}\right) \quad \left(-1,-1,\tfrac{1}{4}\right)$$

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Numerator Vanishes to Second Order

Topic 4
Perturbing Direction and Central Limit Theorems

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Main Theorem of Smooth ACSV

Suppose that

$$H = 0, \qquad r_j z_1 H_{z_1} = r_1 z_j H_{z_j} \quad (2 \leq j \leq d)$$

admits a finite number of solutions. If

- there is exactly one minimal solution, $\mathbf{w} \in \mathbb{C}_*^d$
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then

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then

$$[\mathbf{z}^{n\mathbf{r}}] \frac{G(\mathbf{z})}{H(\mathbf{z})} = \mathbf{w}^{-n\mathbf{r}} \, (n r_d)^{(1-d)/2} \, (2\pi)^{(1-d)/2} \det(\mathcal{M})^{-1/2} \left(\frac{-G(\mathbf{w})}{w_d \, H_{z_d}(\mathbf{w})} + O\left(\frac{1}{n}\right) \right)$$

If $\mathbf{r}$ varies in some compact set where the above conditions are still satisfied then the **error term** in this asymptotic estimate can be **uniformly bounded**.

Irrational Directions

Recall from Lecture 1 that

$$[x^{rn}y^{sn}]\frac{1}{1-x-y} \sim \left(\frac{r+s}{r}\right)^{rn}\left(\frac{r+s}{s}\right)^{sn}\frac{\sqrt{r+s}}{\sqrt{2rs\pi\, n}}$$

Irrational Directions

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Only makes sense if $rn, sn \in \mathbb{N}$

Valid for any $r, s > 0$

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What if we put $(r,s)=(\pi,1)$ into the approximation?
What does this correspond to?

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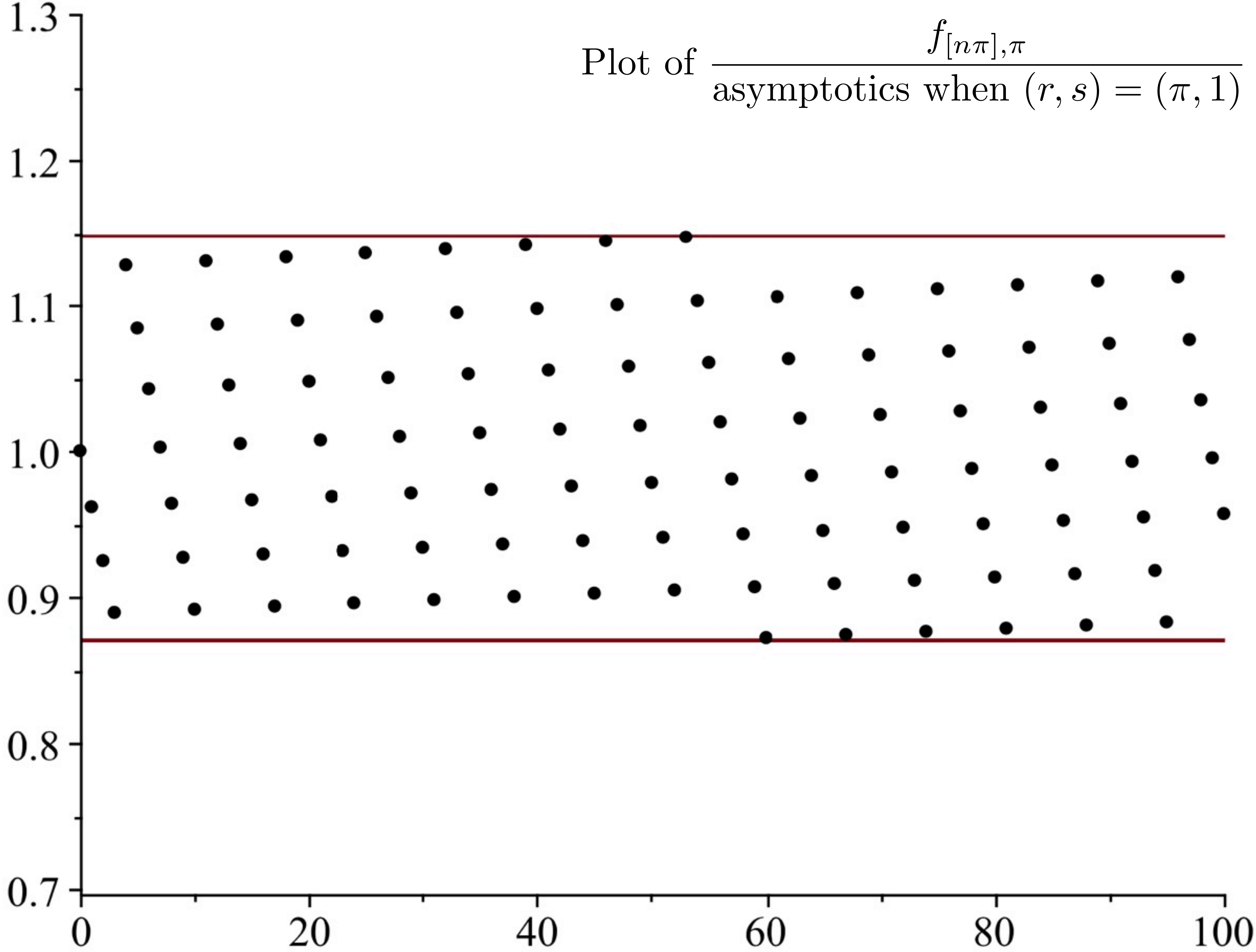
What if we put $(r,s)=(\pi,1)$ into the approximation?
What does this correspond to?

Compare to $f_{[n\pi],n}$ where $[z]=$ closest integer to z



Plot of $\frac{f_{[n\pi],\pi}}{\text{asymptotics when } (r,s)=(\pi,1)}$
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0
20
40
60
80
100

Irrational Directions

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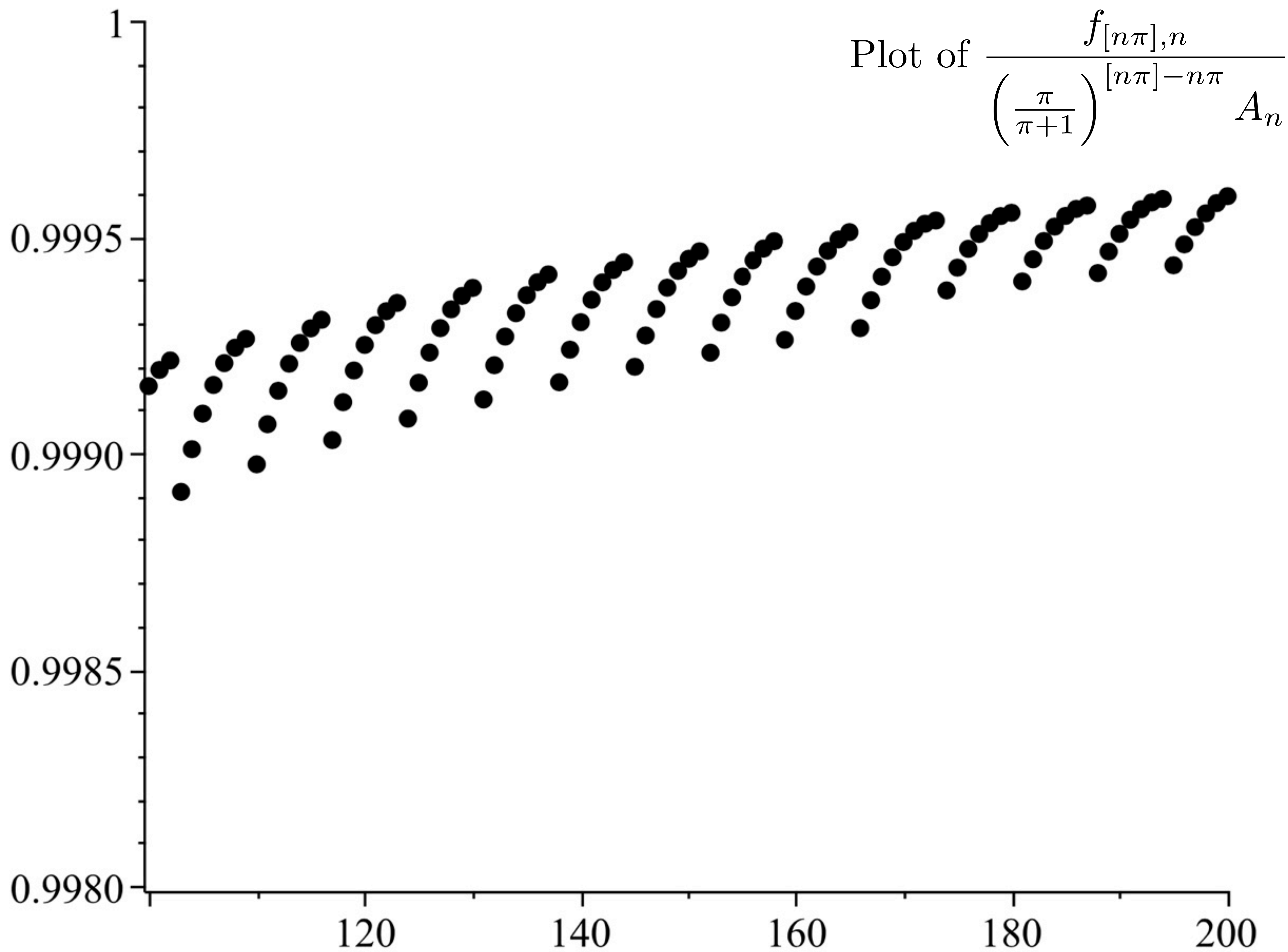
If A_n is this approximation with $(r,s)=(\pi,1)$ then

$$f_{[n\pi],n} \sim \underbrace{\left(\frac{\pi}{\pi+1}\right)^{[n\pi]-n\pi}}_{\text{bounded factor}} A_n$$


Plot of $\frac{f_{[n\pi],n}}{\left(\frac{\pi}{\pi+1}\right)^{[n\pi]-n\pi} A_n}$
1
0.9995
0.9990
0.9985
0.9980
120
140
160
180
200

Smooth Variation of Coefficients

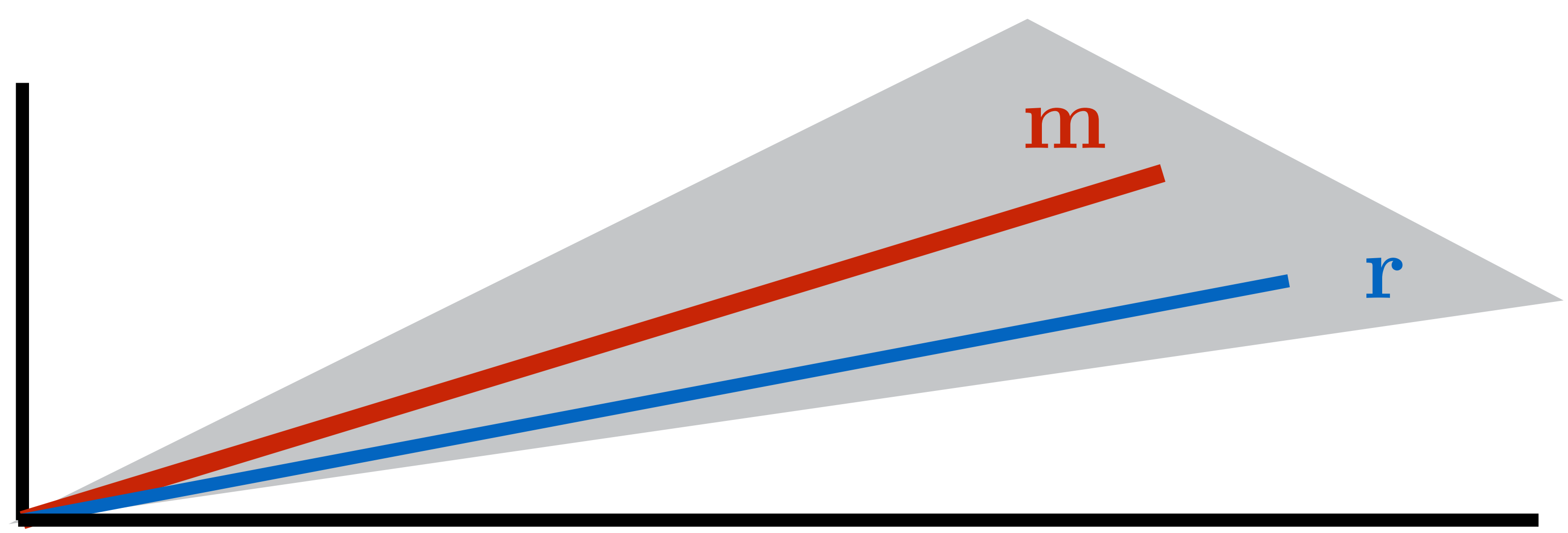
Fix direction $\mathbf{m} = (\widehat{\mathbf{m}}, 1)$ and suppose that for all $\mathbf{r} = (\hat{\mathbf{r}}, 1)$ in a neighbourhood of $\mathbf{m}$ there is a smoothly varying minimal critical point $\mathbf{w}(\mathbf{r})$ such that

1. no other singularity has the same coordinate-wise modulus as $\mathbf{w}(\mathbf{r})$
2. $H_{z_d}(\mathbf{w}(\mathbf{r}))$ and $G(\mathbf{w}(\mathbf{r}))$ are non-zero
3. the matrix $\mathcal{M}_{\mathbf{w}(\mathbf{r})}$ is non-singular

If $\hat{\mathbf{s}} = \hat{\mathbf{s}}(n)$ is a sequence in $\mathbb{N}^{d-1}$ with each coordinate of $|\hat{\mathbf{s}} - n\widehat{\mathbf{m}}|$ in $o(n^{2/3})$ then

$$f_{\hat{\mathbf{s}},n} \sim \mathbf{w}^{-n\mathbf{m}} n^{(1-d)/2} \left(\frac{-G(\mathbf{w})(2\pi)^{(1-d)/2}}{w_d H_{z_d}(\mathbf{w})\sqrt{\det \mathcal{M}}} \right) \widehat{\mathbf{w}}^{-(\hat{\mathbf{s}}-n\widehat{\mathbf{m}})} \exp\left[-\frac{(\hat{\mathbf{s}} - n\widehat{\mathbf{m}})^T \mathcal{M}^{-1} (\hat{\mathbf{s}} - n\widehat{\mathbf{m}})}{2n} \right]$$

where $\mathbf{w} = \mathbf{w}(\mathbf{m})$ and $\mathcal{M} = \mathcal{M}_{\mathbf{w}}$.



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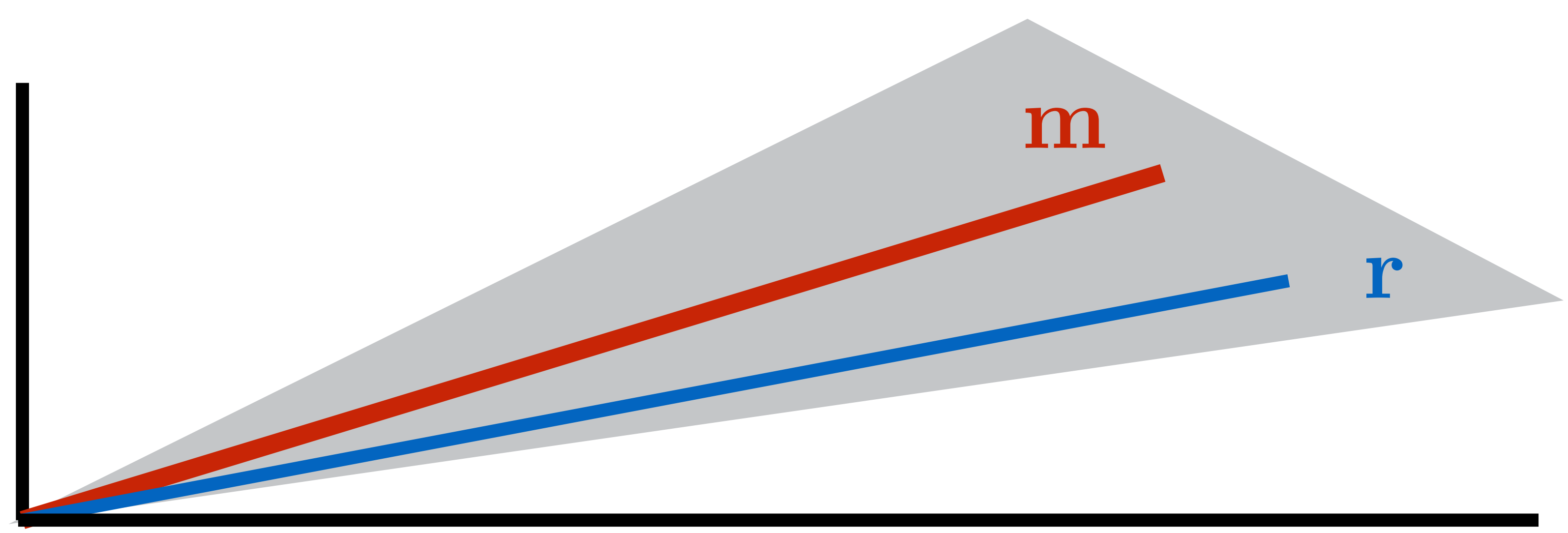
where $\mathbf{w} = \mathbf{w}(\mathbf{m})$ and $\mathcal{M} = \mathcal{M}_{\mathbf{w}}$.



Local Central Limit Theorem

Suppose that in some direction $(\mathbf{m}, 1)$ there is a minimal critical point $\mathbf{w} = (\mathbf{1}, t)$ with $t > 0$ such that

1. no other singularity has the same coordinate-wise modulus as $\mathbf{w}$
2. $H_{z_d}(\mathbf{w})$ and $G(\mathbf{w})$ are non-zero
3. the explicit matrix $\mathcal{M}$ is non-singular

Then as $n \to \infty$ the coefficients of $[z_d^n]F(\mathbf{z})$ approach a multivariate normal distribution with density

$$\nu_n(\mathbf{s}) = -\frac{G(\mathbf{w})}{w_d H_{z_d}(\mathbf{w})} \frac{(2\pi n)^{(1-d)/2}}{\sqrt{\det \mathcal{M}}} \exp\left[-\frac{(\mathbf{s} - n\mathbf{m})^T \mathcal{M}^{-1} (\mathbf{s} - n\mathbf{m})}{2n}\right].$$

$$F(x, y, z) = \square + \cdots + \left(\square + \square x + \square y + \square xy + \cdots\cdots\right) z^n + \cdots$$

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Then

$$\sup_{\mathbf{s}\in\mathbb{Z}^{d-1}} n^{(d-1)/2} \left| t^n f_{\mathbf{s},n} - \nu_n(\mathbf{s}) \right| \to 0$$

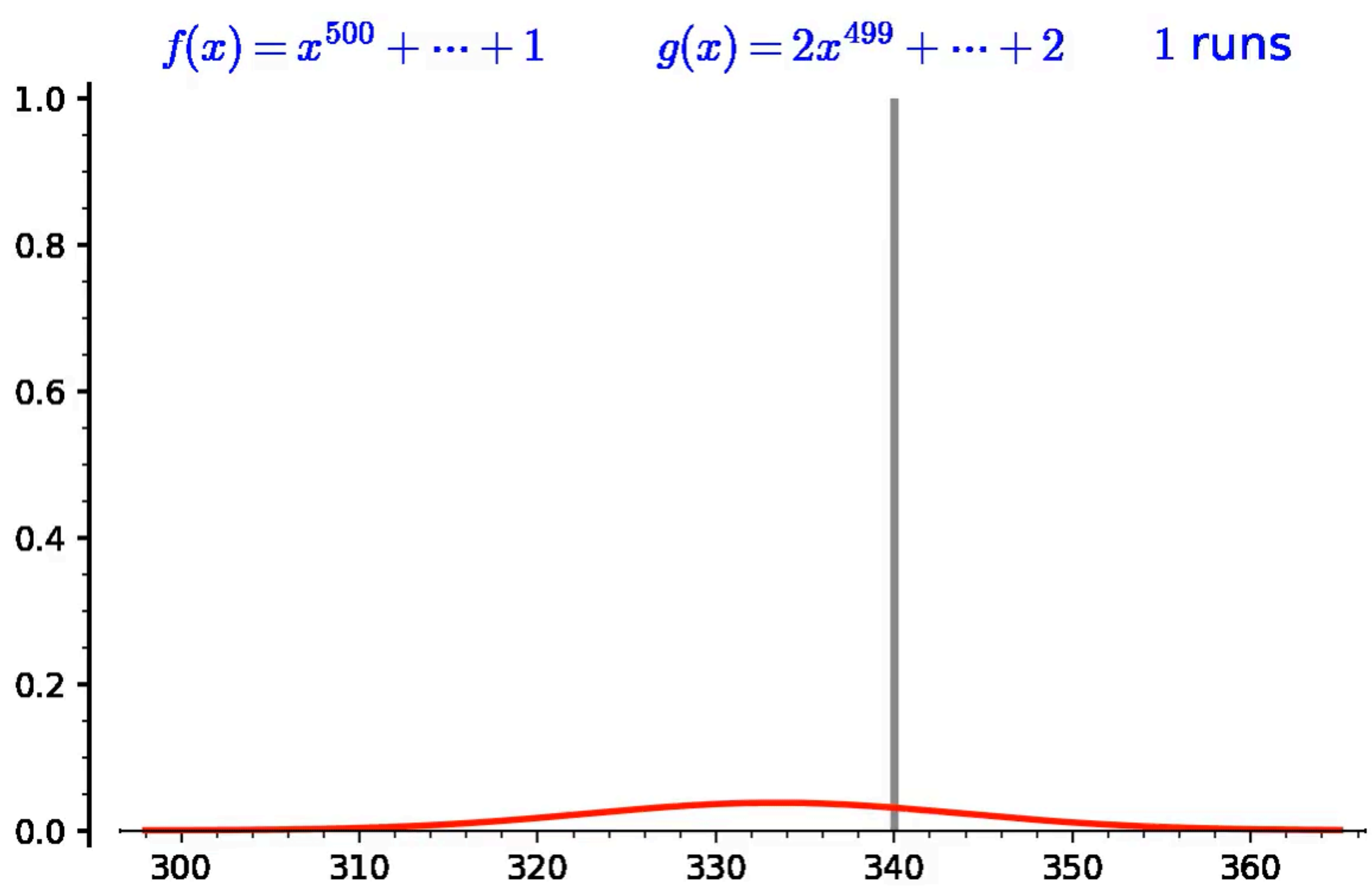
as $n \to \infty$, where

$$\nu_n(\mathbf{s}) = -\frac{G(\mathbf{w})}{w_d H_{z_d}(\mathbf{w})} \frac{(2\pi n)^{(1-d)/2}}{\sqrt{\det \mathcal{M}}} \exp\left[-\frac{(\mathbf{s} - n\mathbf{m})^T \mathcal{M}^{-1} (\mathbf{s} - n\mathbf{m})}{2n} \right].$$

CLT for the Euclidean Algorithm

If $e_{k,n}$ denotes the number of pairs of polynomials $f_0, f_1 \in \mathbb{F}_p[x]$ such that $\deg(f_1) < \deg(f_0) = n$ and the Euclidean algorithm performs k divisions then

$$F(u,z) = \sum_{n,k \geq 0} e_{k,n} u^k z^n = \frac{1}{1 - pz - p(p-1)uz}$$



A running count of the number of divisions performed when running the Euclidean algorithm on pairs of polynomials in $\mathbb{Z}_3[x]$ with the higher degree polynomial monic of degree 500, compared to the expected distribution from the limit curve.

Cycles Lengths in Permutations w/ Restricted Positions

P. Diaconis: "My latest paper has an explicit multivariate rational generating function. I'm pretty sure a CLT holds..."

Permanental generating functions and sequential importance sampling

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62D99

A B S T R A C T

We introduce techniques for deriving closed form generating functions for enumerating permutations with restricted positions keeping track of various statistics. The method involves evaluating permanents with variables as entries. These are applied to determine the sample size required for a novel sequential importance sampling algorithm for generating random perfect matchings in classes of bipartite graphs.

Cycles Lengths in Permutations w/ Restricted Positions

Fix $t \in \mathbb{N}$.
Let $\mathcal{F}_t(n)$ be the set of permutations $\sigma \in S_n$ with $i - t \leq \sigma(i) \leq i + 1$

Theorem 1. If $a_i(\sigma)$ denotes the number of i cycles in σ,

$$F(\mathbf{x}, z) = \sum_{\substack{n \geq 0 \\ \sigma \in \mathcal{F}_t(n)}} \mathbf{x}^{\mathbf{a}(\sigma)} z^n = \frac{1}{1 - x_1 z - x_2 z^2 - \cdots - x_{t+1} z^{t+1}}$$

Cycles Lengths in Permutations w/ Restricted Positions

Conditions for a CLT to hold

1. no other singularity has the same coordinate-wise modulus as $\mathbf{w}$
2. $H_{z_d}(\mathbf{w})$ and $G(\mathbf{w})$ are non-zero
3. the explicit matrix $\mathcal{M}$ is non-singular

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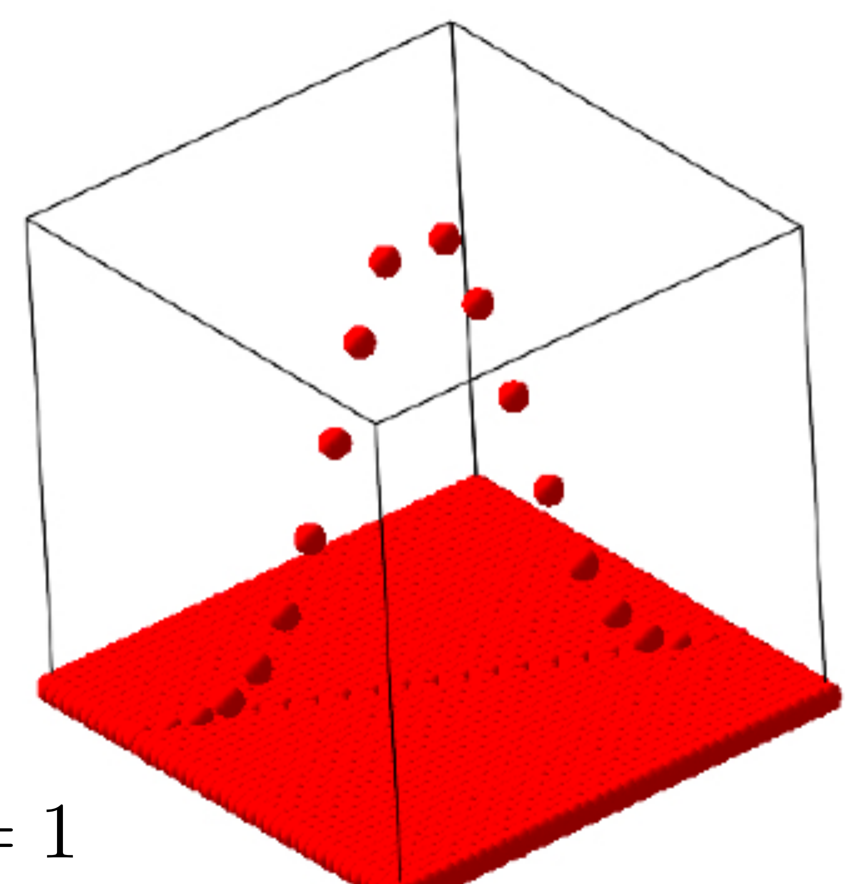

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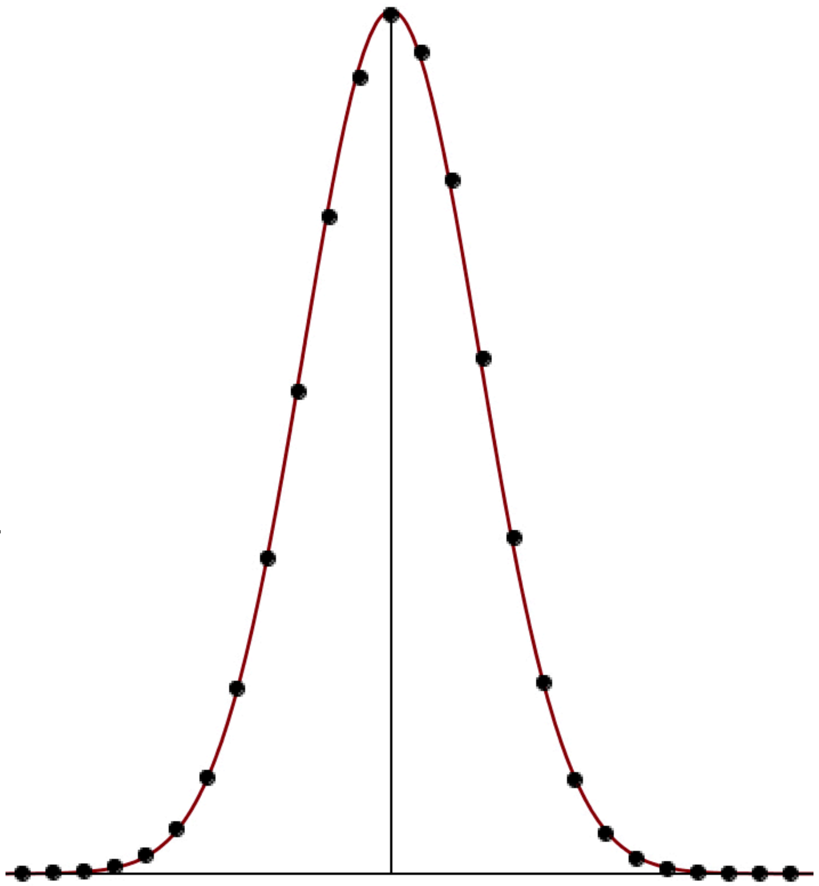
$$F(1, x_2, \ldots, x_{t+1}, z) = \frac{1}{1 - z - x_2 z^2 - \cdots - x_{t+1} z^{t+1}}$$



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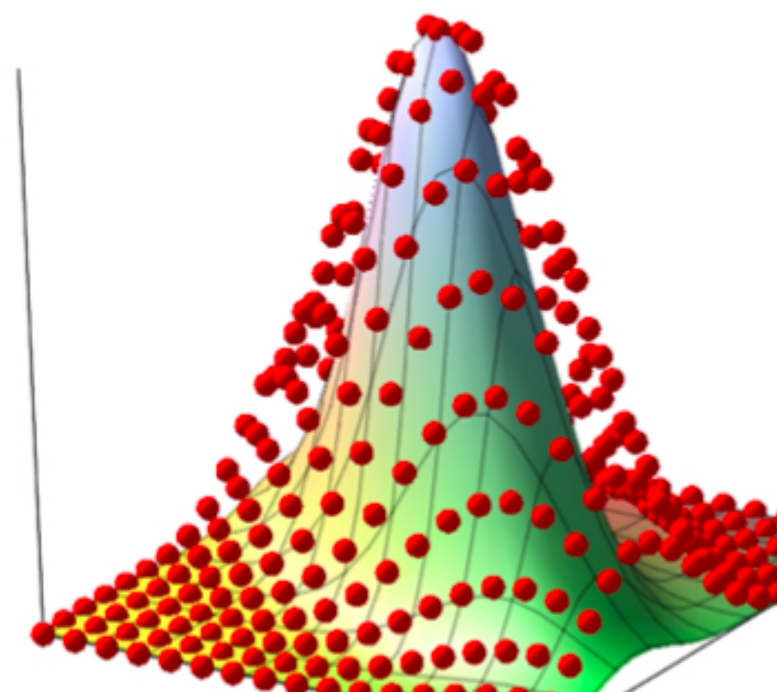
$t = 2$



Bivariate Multinomial ML-Degree

The **maximum likelihood degree (ML-degree)** is a measure of the complexity of the statistical *maximum likelihood method* for estimating parameters in a multivariate probability model with missing data.

Theorem (Hosten, Sullivant 2010)
If ML(n,k) denotes the ML-degree for multinomial random variables $X_1 \in \{1, \ldots, n\}$ and $X_2 \in \{1, \ldots, k\}$ then

$$\sum_{n,k \geq 0} \mathrm{ML}(n,k) \frac{x^n y^k}{n!k!} = \frac{e^{-x-y}}{e^{-x} + e^{-y} - 1}$$

Bivariate Multinomial ML-Degree

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Theorem (Khera, Lundberg, M. 2021)
For all fixed $K > 0$,

$$\sup_{|k-n/2|\leq K\sqrt{n}} \left| \frac{(2\log 2)^n}{n!}\mathrm{ML}(n-k,k) - \frac{2^{-\frac{2(k-n/2)^2}{n(1-\log 2)}}}{(4\log 2)\sqrt{1-\log 2}} \right| \to 0$$

Topic 5
Beyond Smoothness

melczer.ca/ALEA22

Higher Order Poles

What if there are solutions to

$$H(\mathbf{z}) = H_{z_1}(\mathbf{z}) = \cdots = H_{z_d}(\mathbf{z}) = 0$$

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Easiest Case: $H(\mathbf{z}) = P(\mathbf{z})^k$ for some $k > 1$ where

$$P(\mathbf{z}) = P_{z_1}(\mathbf{z}) = \cdots = P_{z_d}(\mathbf{z}) = 0$$

has no solutions.

Then $\mathcal{V} = \{\mathbf{z} : H(\mathbf{z}) = 0\} = \{\mathbf{z} : P(\mathbf{z}) = 0\}$ is still a manifold
Minimal points unchanged, and critical points defined by P

The residue computation in the Main Theorem of Smooth ACSV has a minor modification to account for the **higher order pole**

Higher Order Poles

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has no solutions.

If the usual assumptions hold with the smooth critical point equations for P then

$$f_{n\mathbf{r}} = \mathbf{w}^{-n\mathbf{r}}\, n^{k-1+(1-d)/2}\, (2\pi r_d)^{(1-d)/2}\, \det(\mathcal{M})^{-1/2} \cdot \frac{(-1)^k G(\mathbf{w})}{(k-1)!\,(w_d P_{z_d}(\mathbf{w}))^k} \left(1 + O\left(n^{-1}\right)\right)$$

Non-Smooth Points

More pathologically, $H(\mathbf{z}) = H_{z_1}(\mathbf{z}) = \cdots = H_{z_d}(\mathbf{z}) = 0$ if $\mathcal{V}$ self-intersects at **non-smooth points**.

This does not happen generically, but **does come up** in combinatorial examples.

I'd *estimate* 75% of naturally occurring combinatorial examples have $\mathcal{V}$ smooth.

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Simplest Non-Smooth Case:

$$F(x,y) = \frac{G(x,y)}{\ell_1(x,y)\ell_2(x,y)}$$

where ℓ_1 and ℓ_2 are linear.

Hyperplane Example

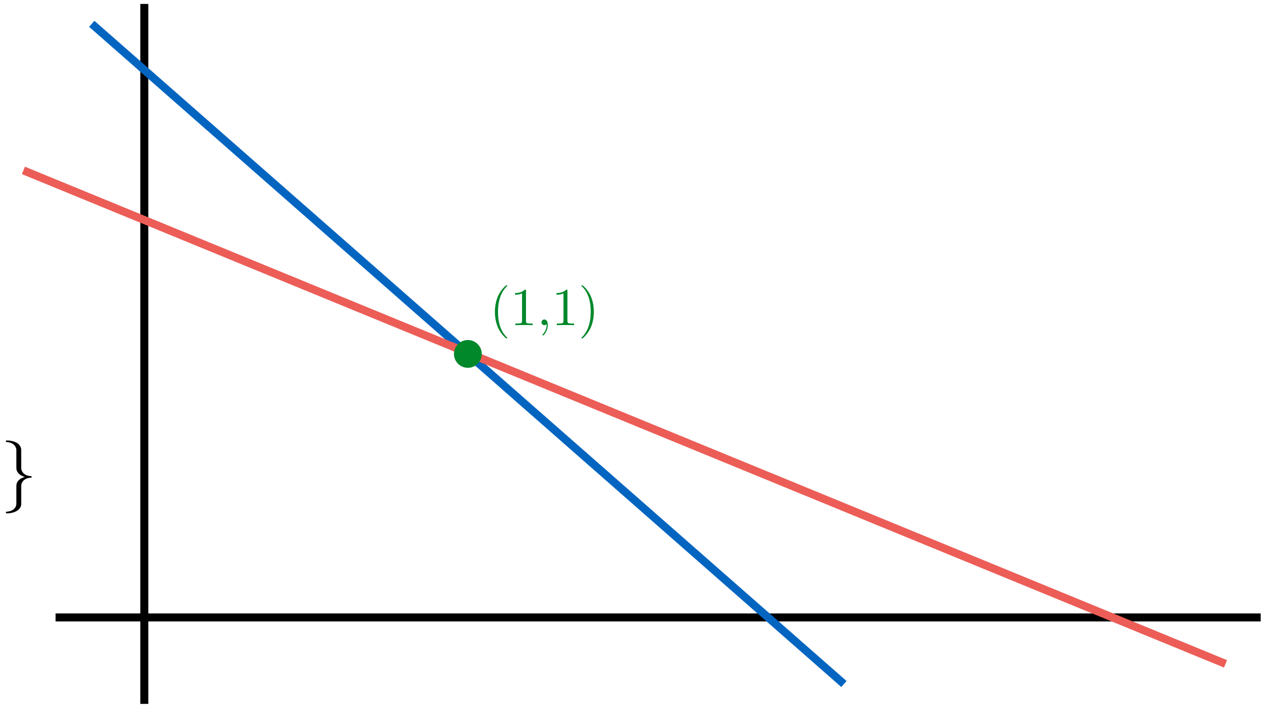
Let

$$F(x,y) = \frac{G(x,y)}{\ell_1(x,y)\ell_2(x,y)}$$

for $\quad \ell_1(x,y) = 1 - \dfrac{2x+y}{3} \qquad$ and $\qquad \ell_2(x,y) = 1 - \dfrac{3x+y}{4}$

Then $\mathcal{V} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_{1,2}$ for smooth sets

$$\begin{aligned} \mathcal{S}_1 &= \{\ell_1(x,y) = 0\} \setminus \mathcal{S}_{1,2} \\ \mathcal{S}_2 &= \{\ell_2(x,y) = 0\} \setminus \mathcal{S}_{1,2} \\ \mathcal{S}_{1,2} &= \{\ell_1(x,y) = \ell_2(x,y) = 0\} = \{(1,1)\} \end{aligned}$$



Hyperplane Example

We compute **critical points** on each *stratum*

On $\mathcal{S}_1$ $\quad \ell_1(x,y) = sx(\ell_1)_x(x,y) - ry(\ell_1)_y(x,y) = 0$

has the unique solution $\boldsymbol{\sigma}_1 = \dfrac{1}{r+s}\left(\dfrac{3r}{2}, 3s\right)$

$r \neq 2s$

On $\mathcal{S}_2$ $\quad \ell_2(x,y) = sx(\ell_2)_x(x,y) - ry(\ell_2)_y(x,y) = 0$

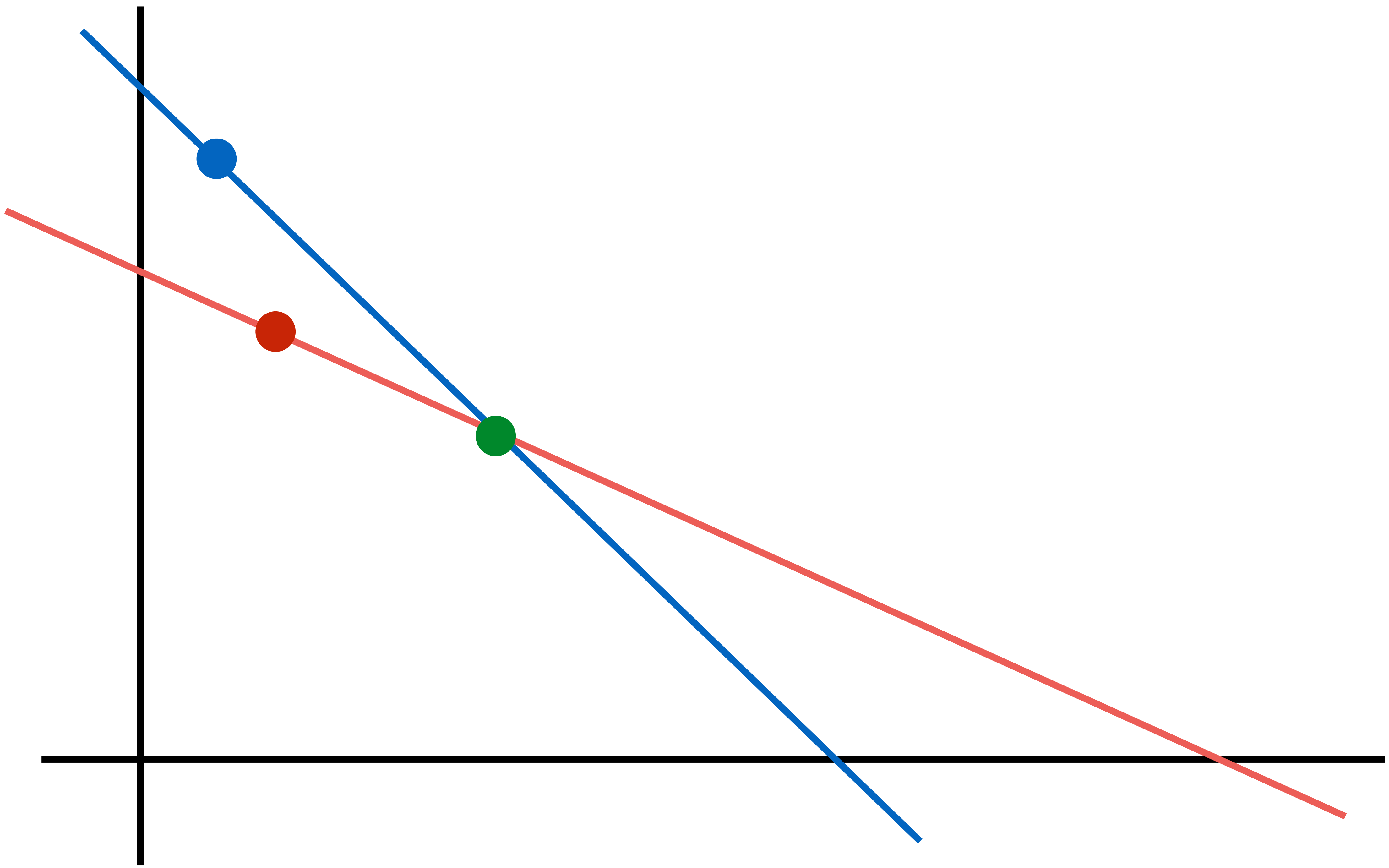
has the unique solution $\boldsymbol{\sigma}_2 = \dfrac{1}{r+s}\left(\dfrac{4r}{3}, 4s\right)$

$r \neq 3s$

On $\mathcal{S}_{1,2}$ the only point $\boldsymbol{\sigma}_{1,2} = (1,1)$ is trivially critical

Case 1: $0 < \frac{r}{r+s} < 2/3$

The point σ_1 is a smooth minimal critical point
We can compute asymptotics as before



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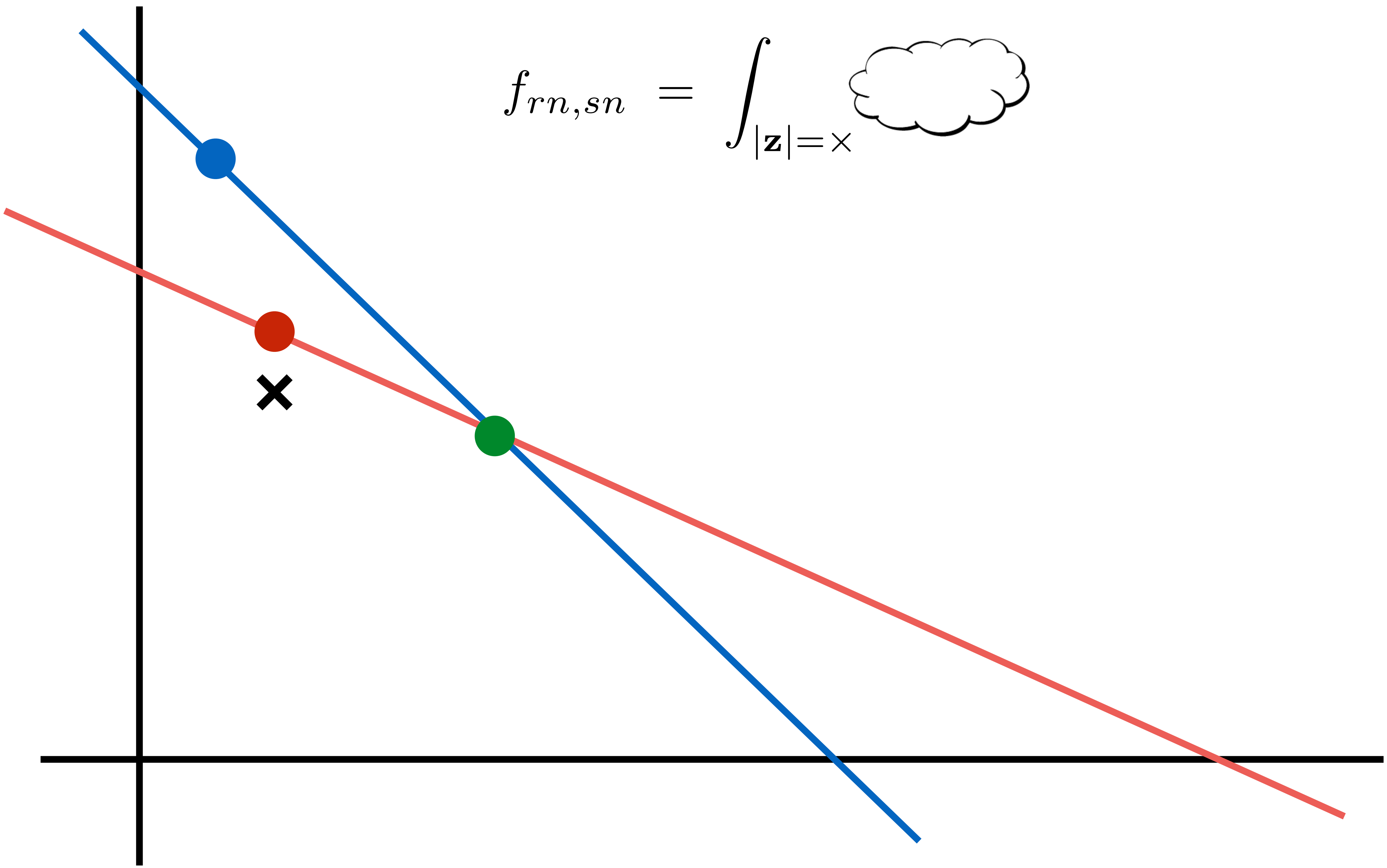

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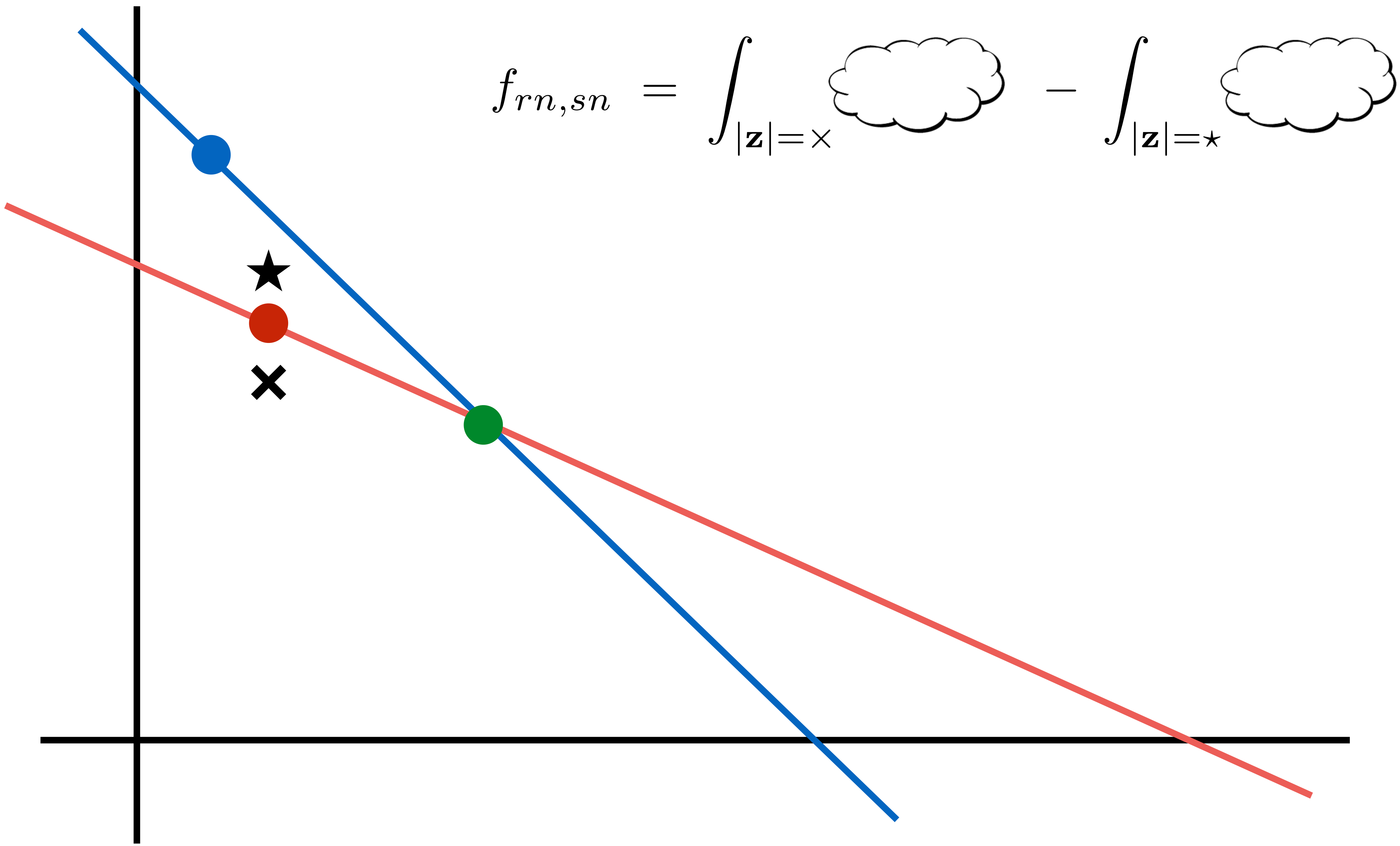

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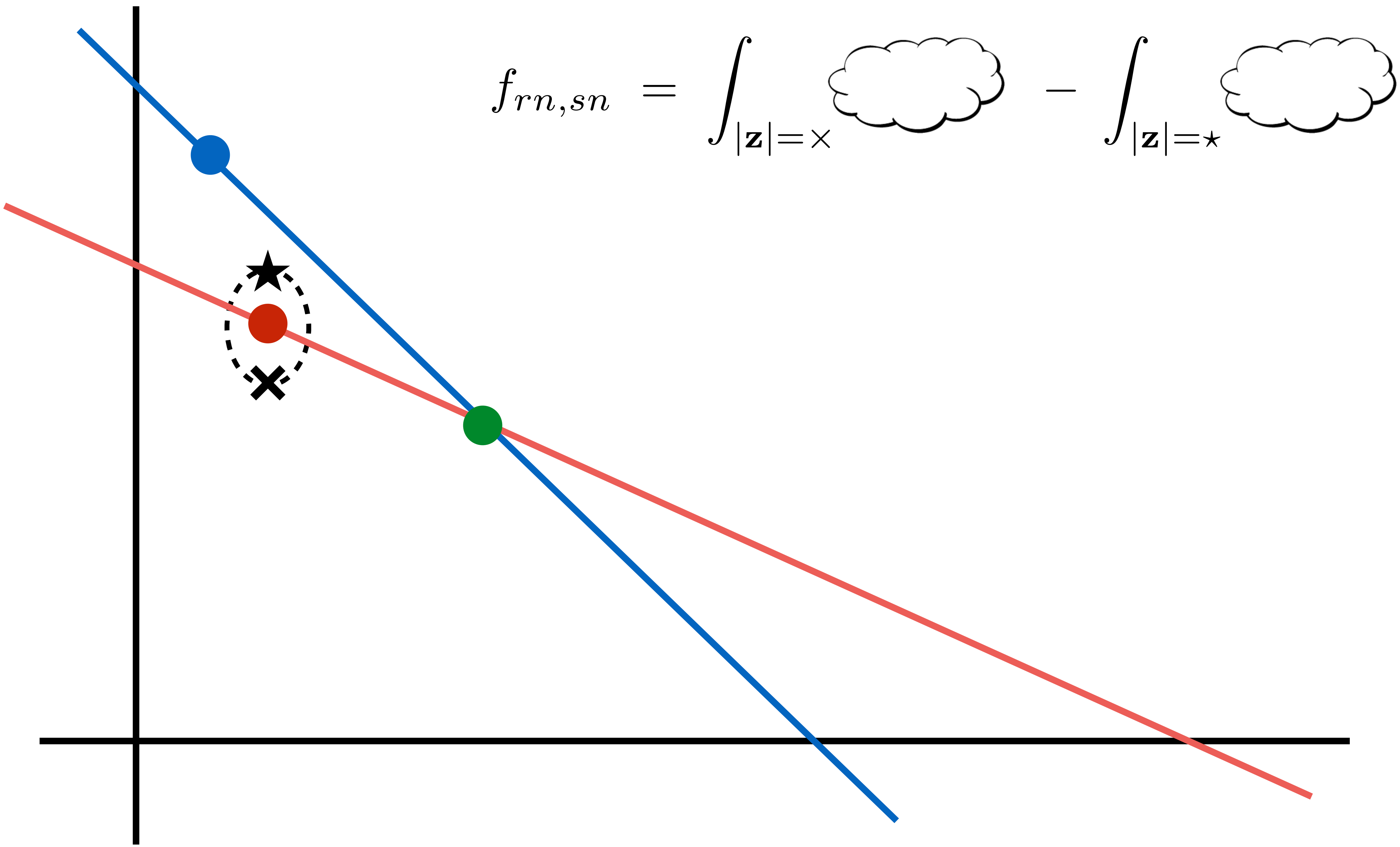

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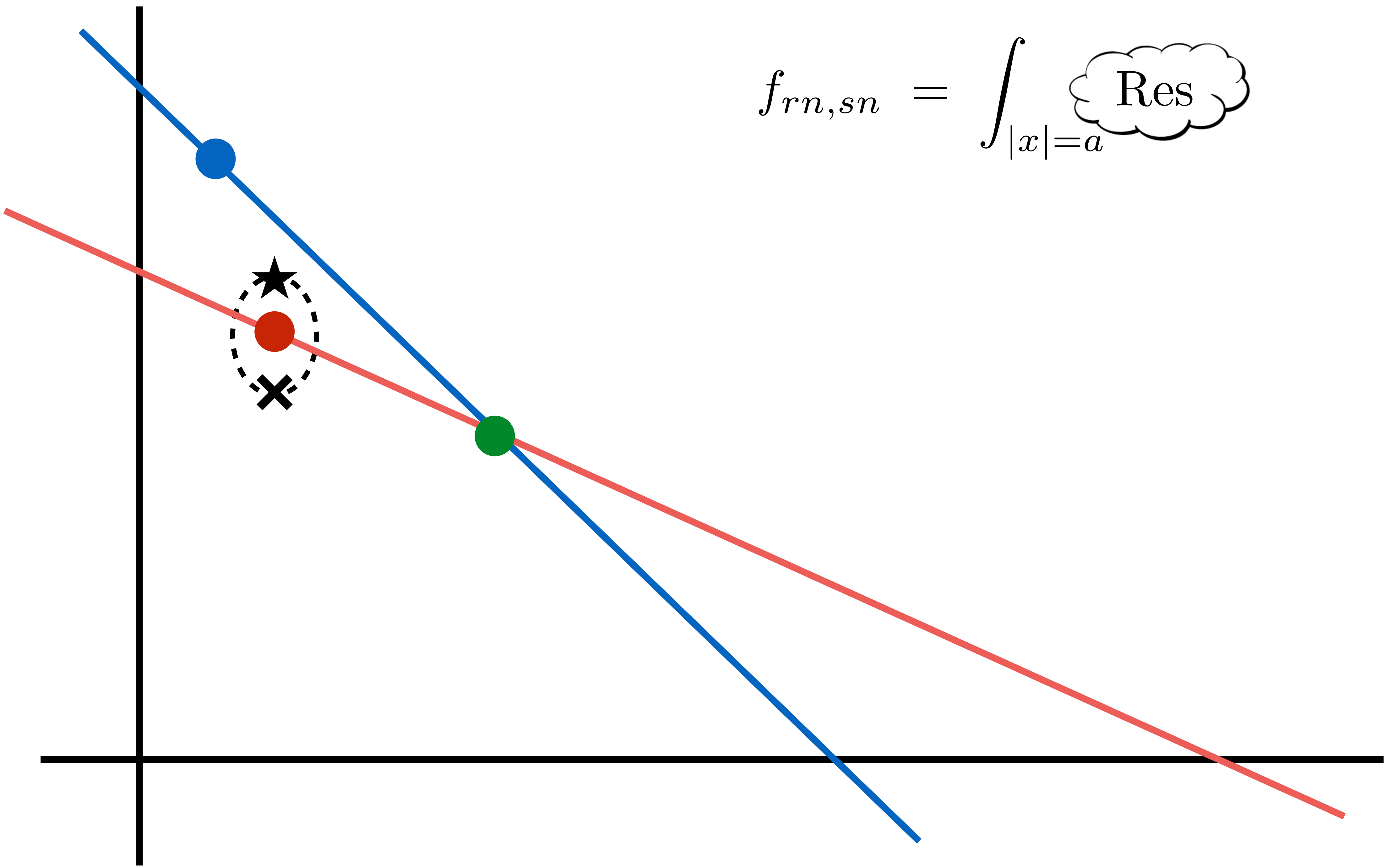
$$f_{rn,sn} = \int_{|x|=a} \text{Res}$$



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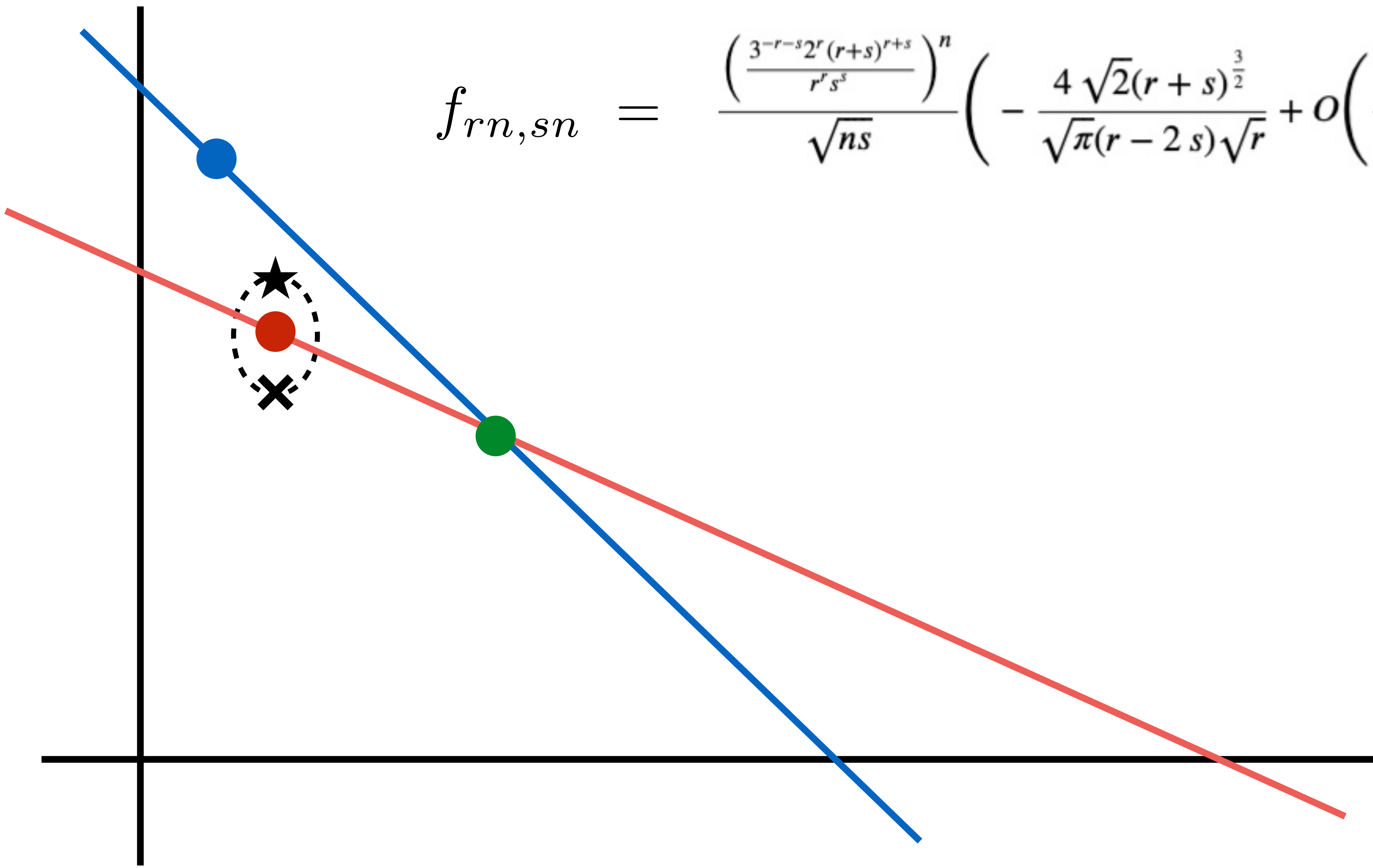

$$f_{rn,sn} = \frac{\left(\frac{3^{-r-s}2^r(r+s)^{r+s}}{r^r s^s}\right)^n}{\sqrt{ns}}\left(-\frac{4\sqrt{2}(r+s)^{\frac{3}{2}}}{\sqrt{\pi}(r-2\,s)\sqrt{r}} + O\left(\frac{1}{n}\right)\right)$$

Case 2: $1/3 < \dfrac{r}{r+s}$

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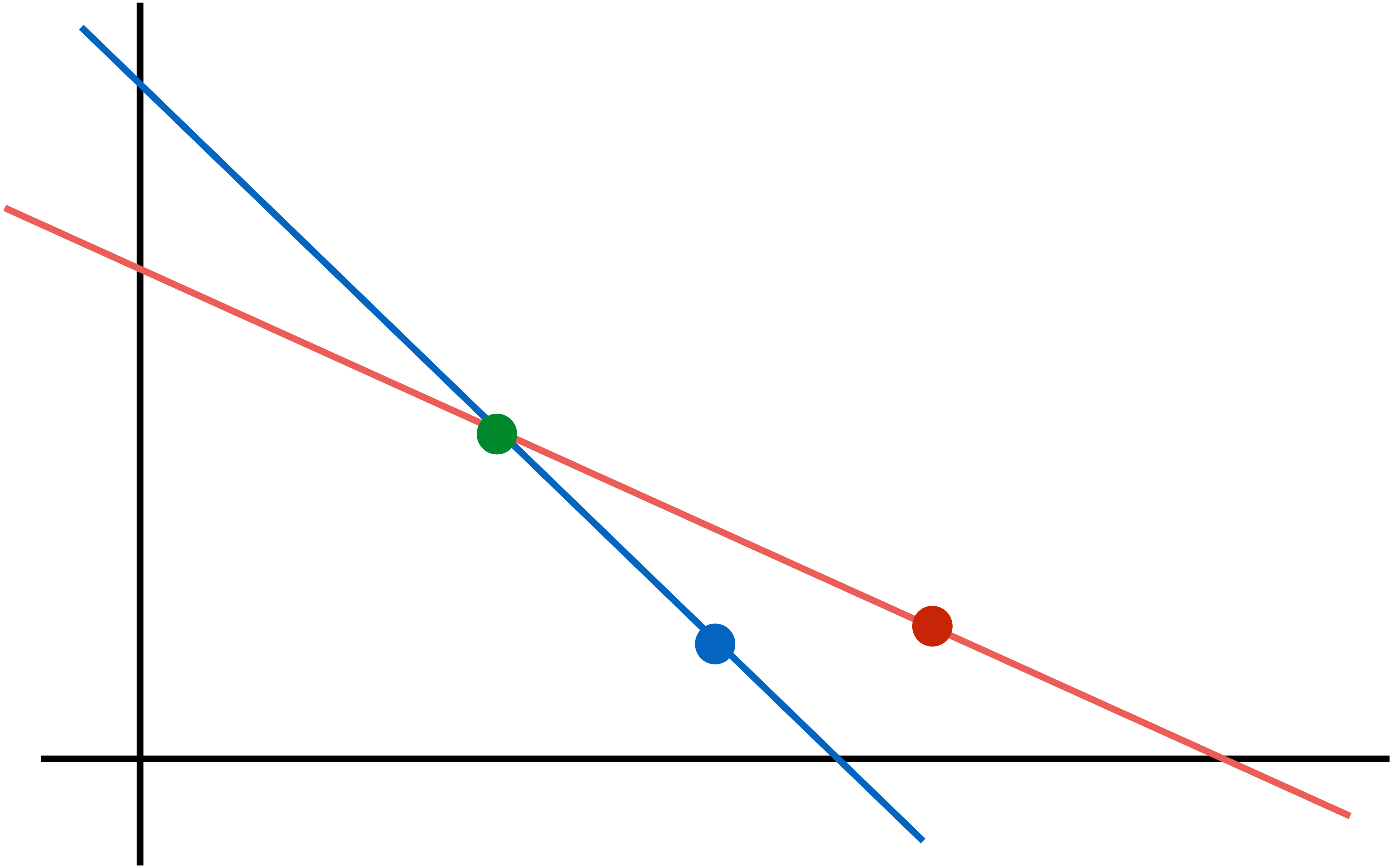

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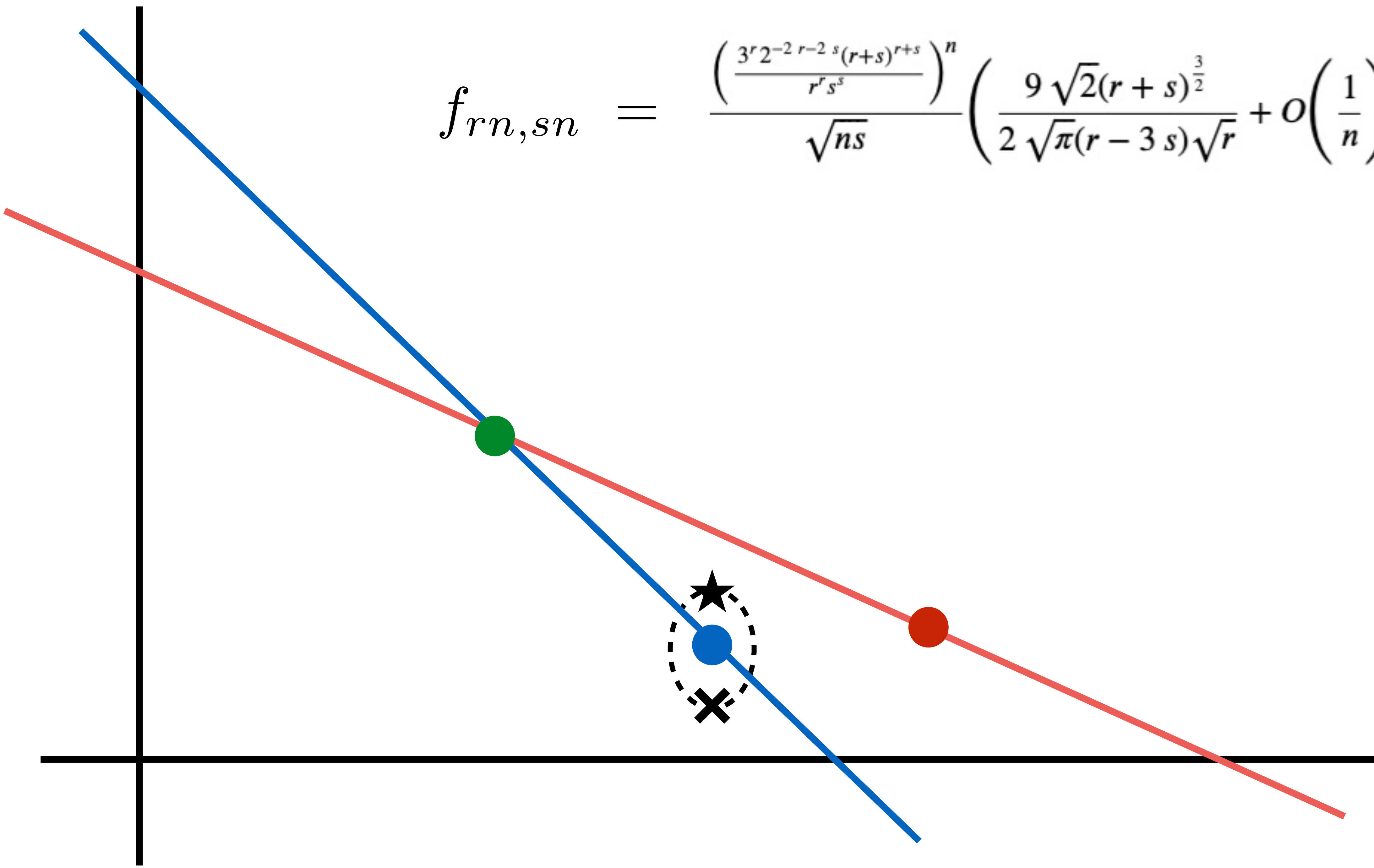
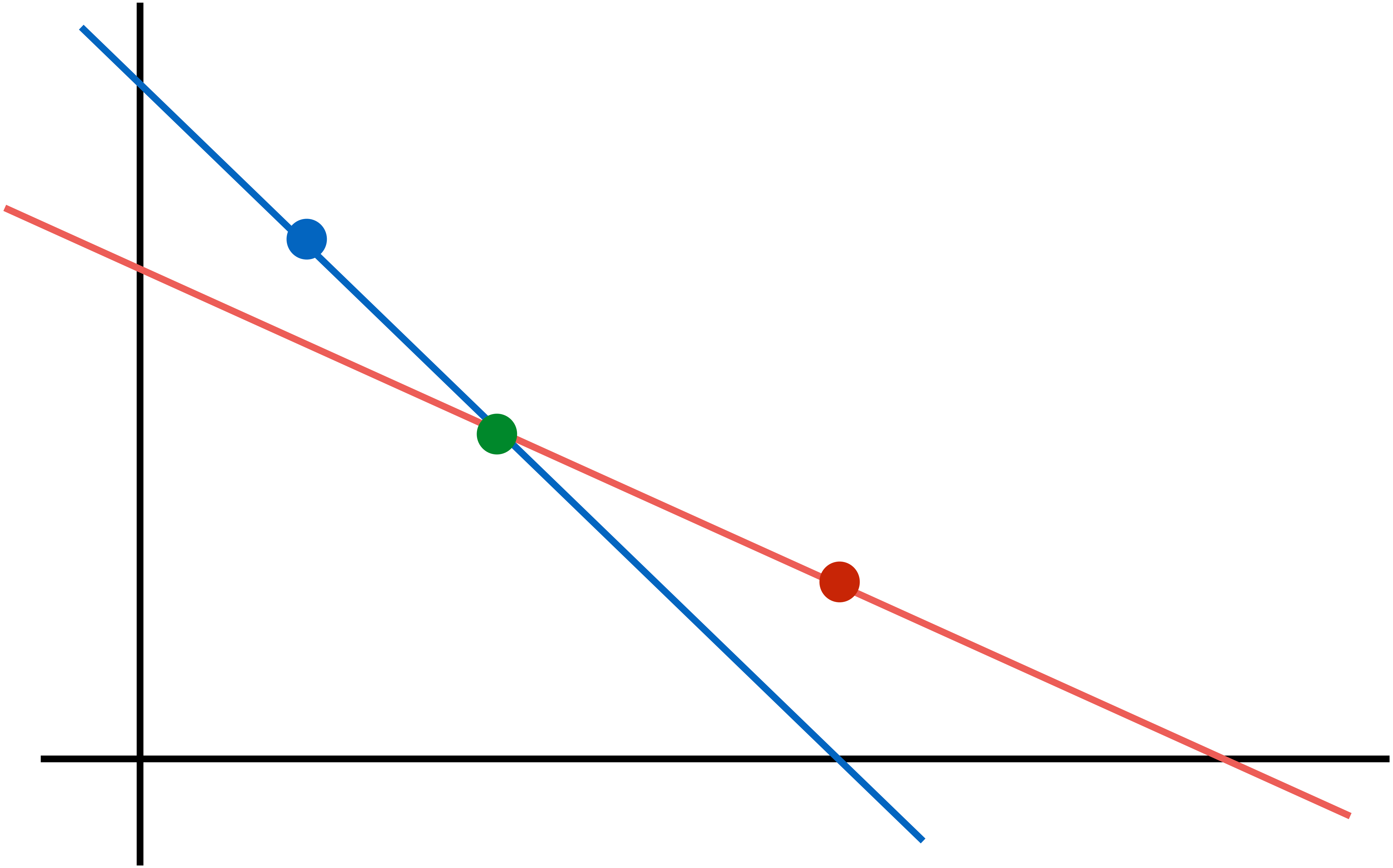
$$f_{rn,sn} = \frac{\left(\frac{3^r 2^{-2r-2s}(r+s)^{r+s}}{r^r s^s}\right)^n}{\sqrt{ns}} \left(\frac{9\sqrt{2}(r+s)^{\frac{3}{2}}}{2\sqrt{\pi}(r-3s)\sqrt{r}} + O\left(\frac{1}{n}\right)\right)$$



Case 3: $2/3 < \frac{r}{r+s} < 3/4$

The **non-smooth** point $\sigma_{1,2}$ is the only minimal critical point

Only now can we introduce **three** new integrals with **small error**



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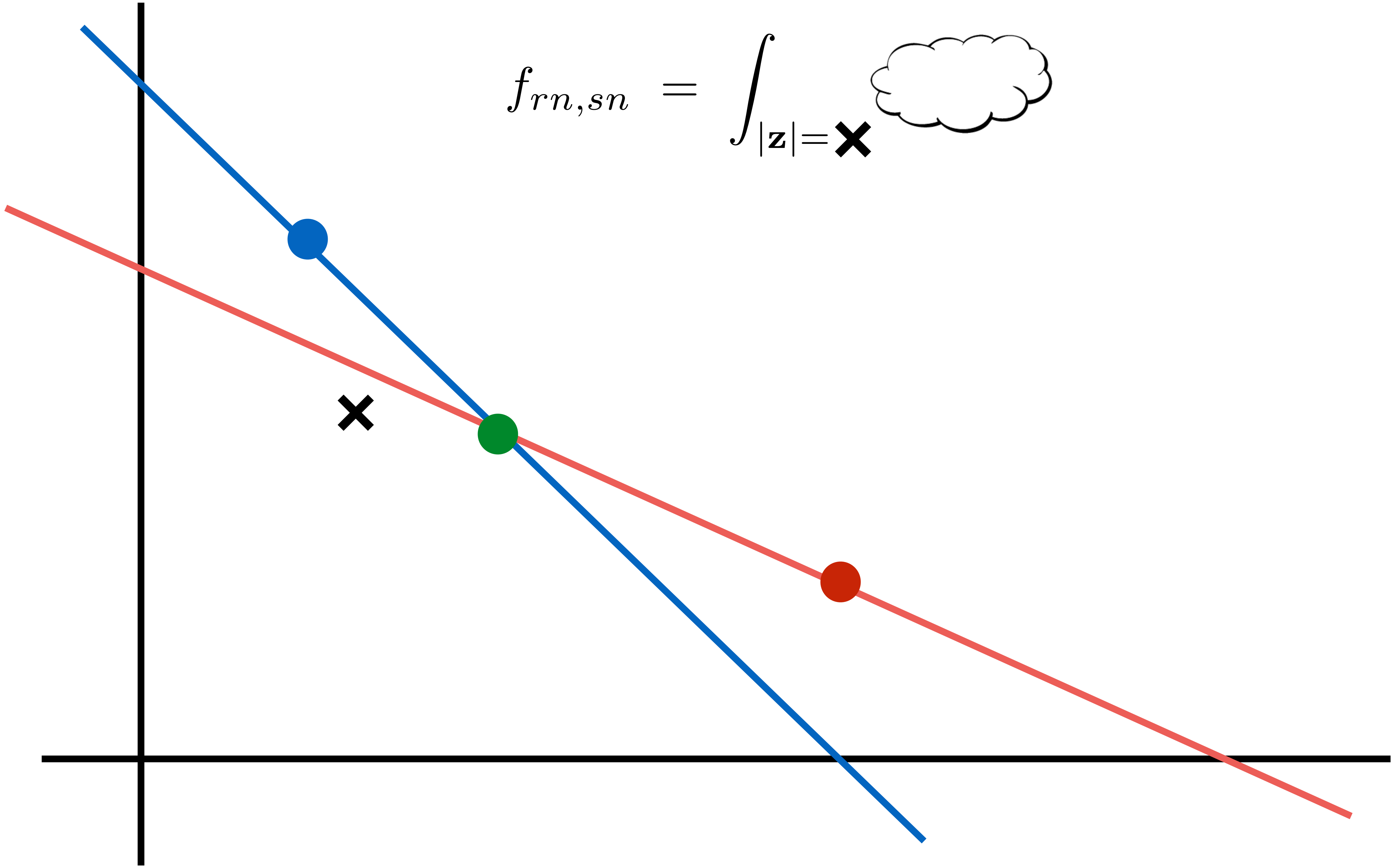
$$f_{rn,sn} = \int_{|\mathbf{z}|=\times}$$



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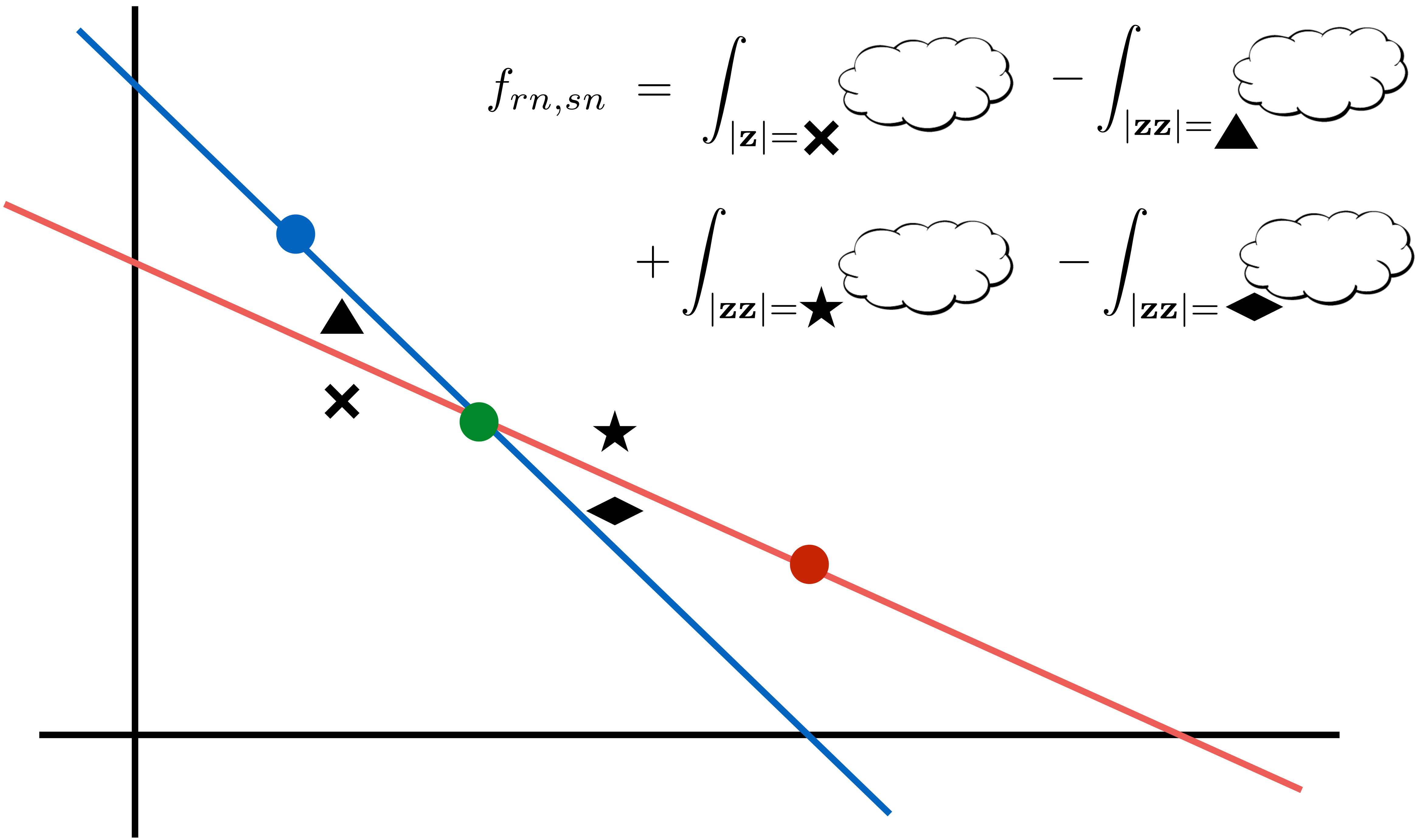

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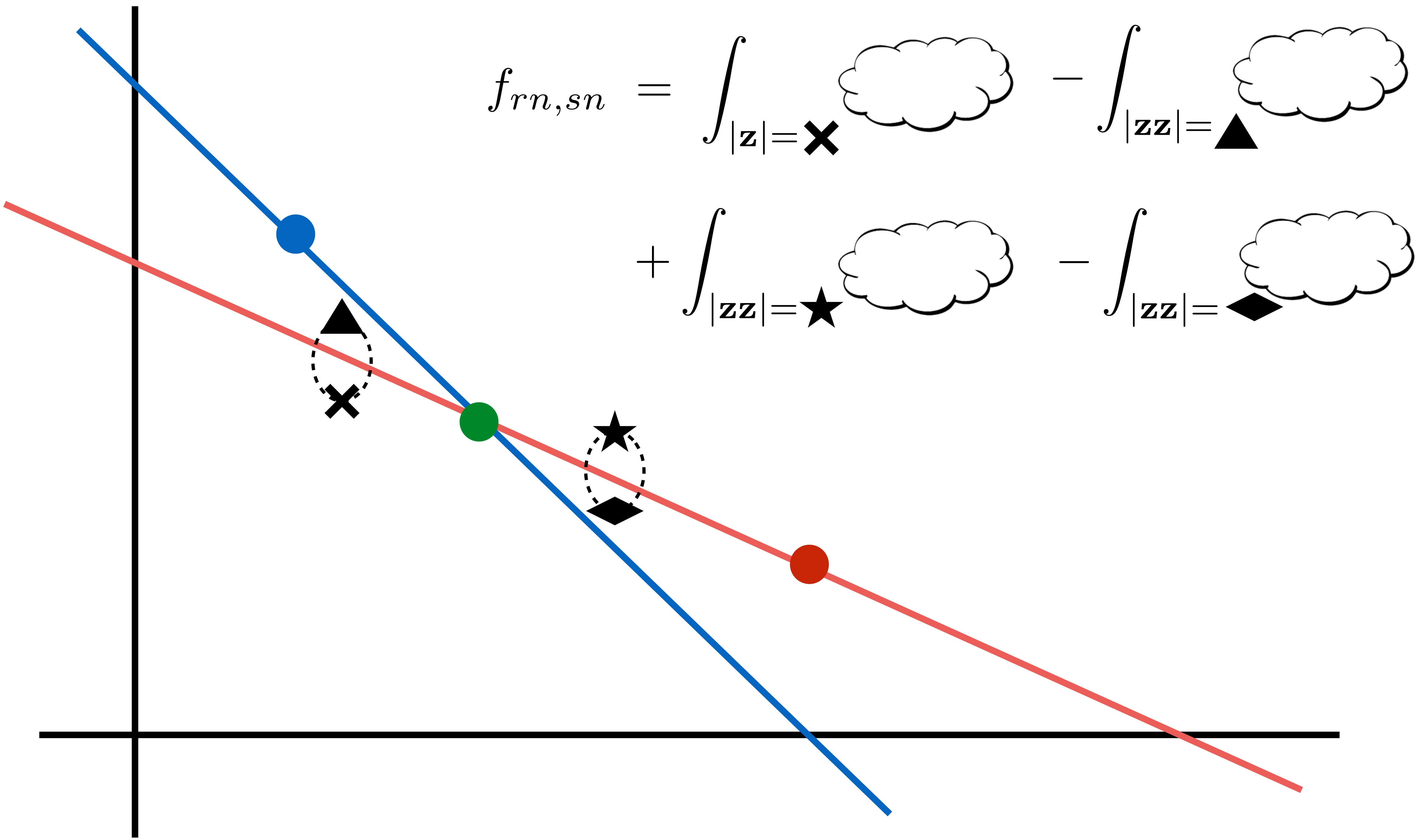

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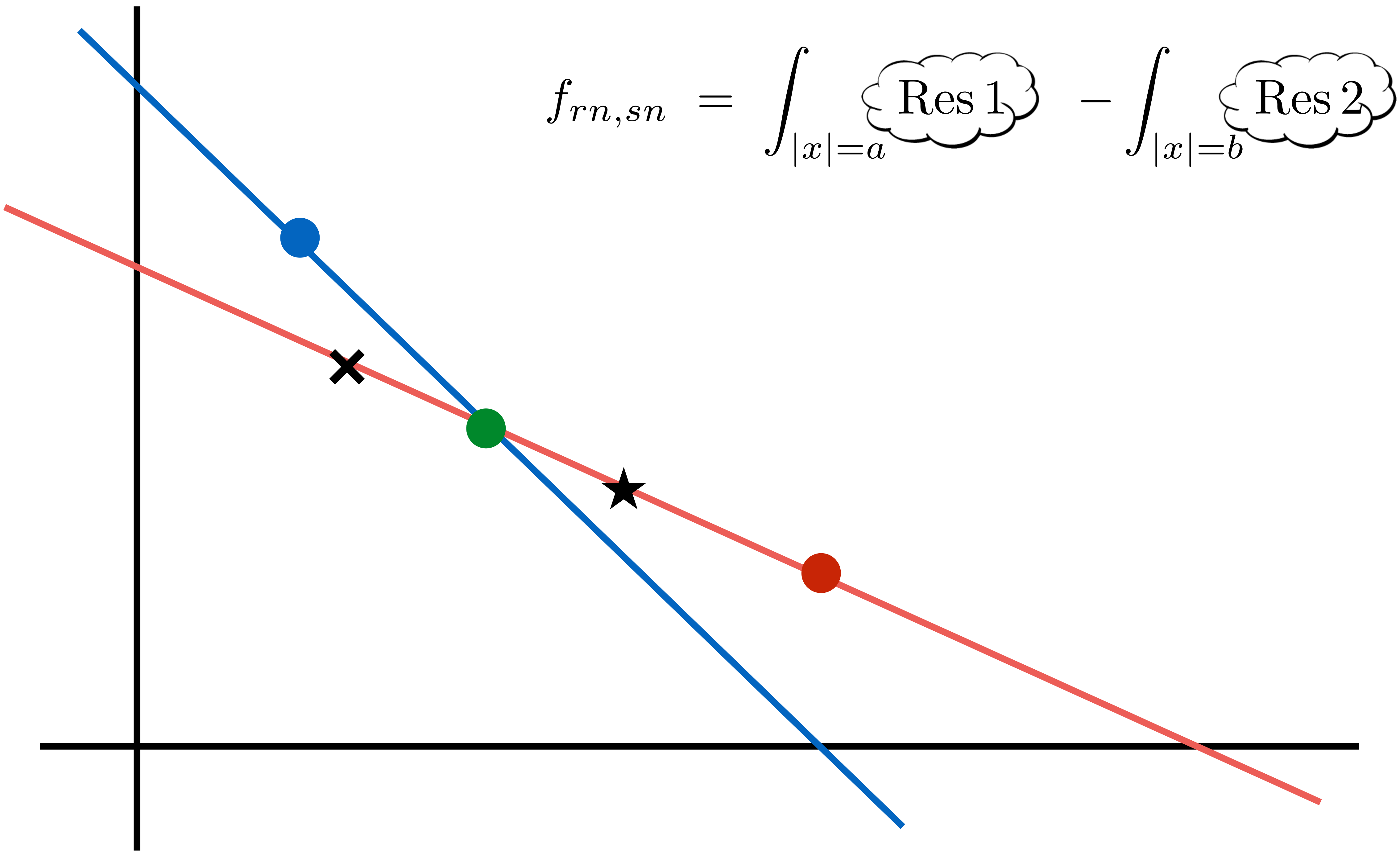

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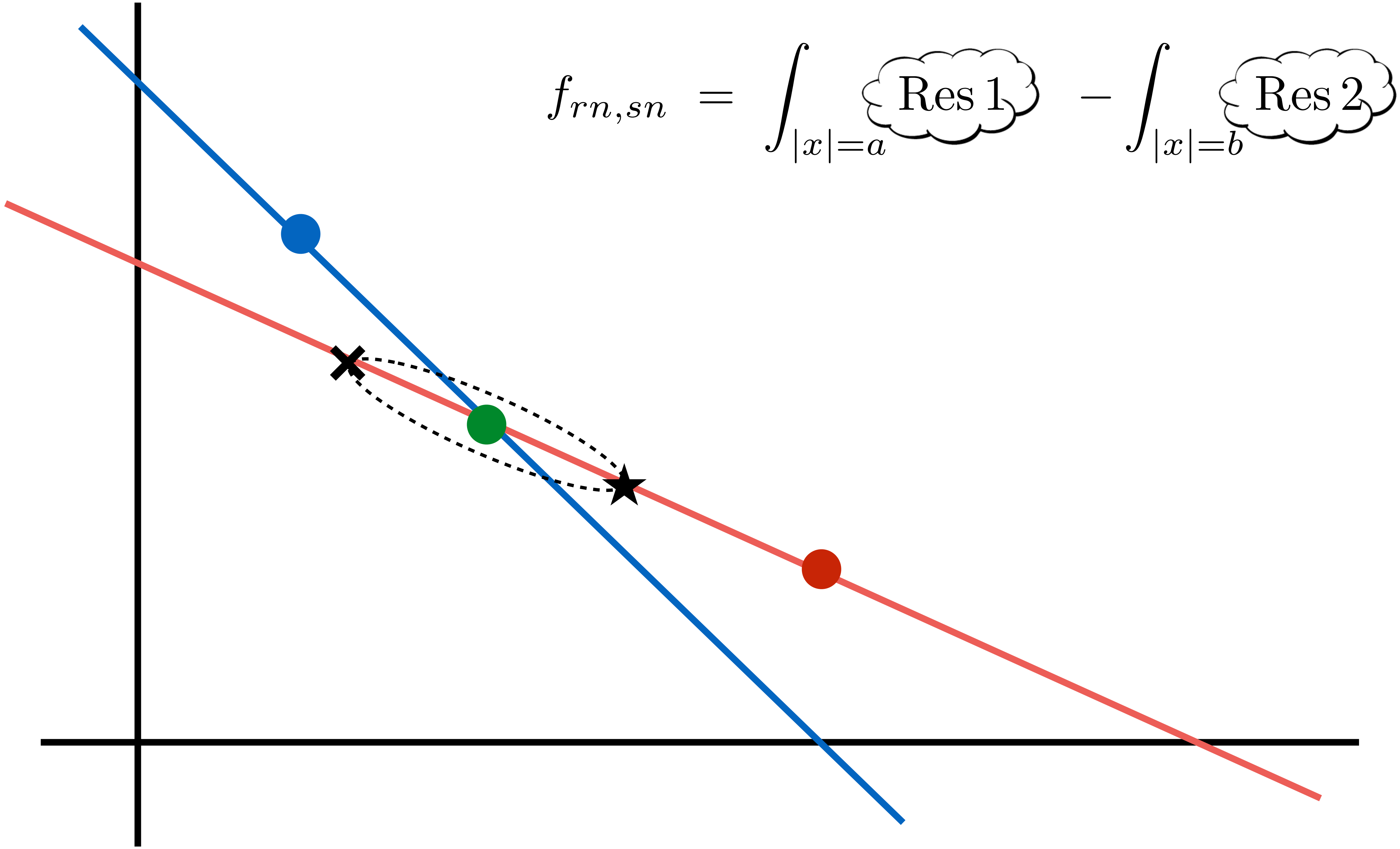

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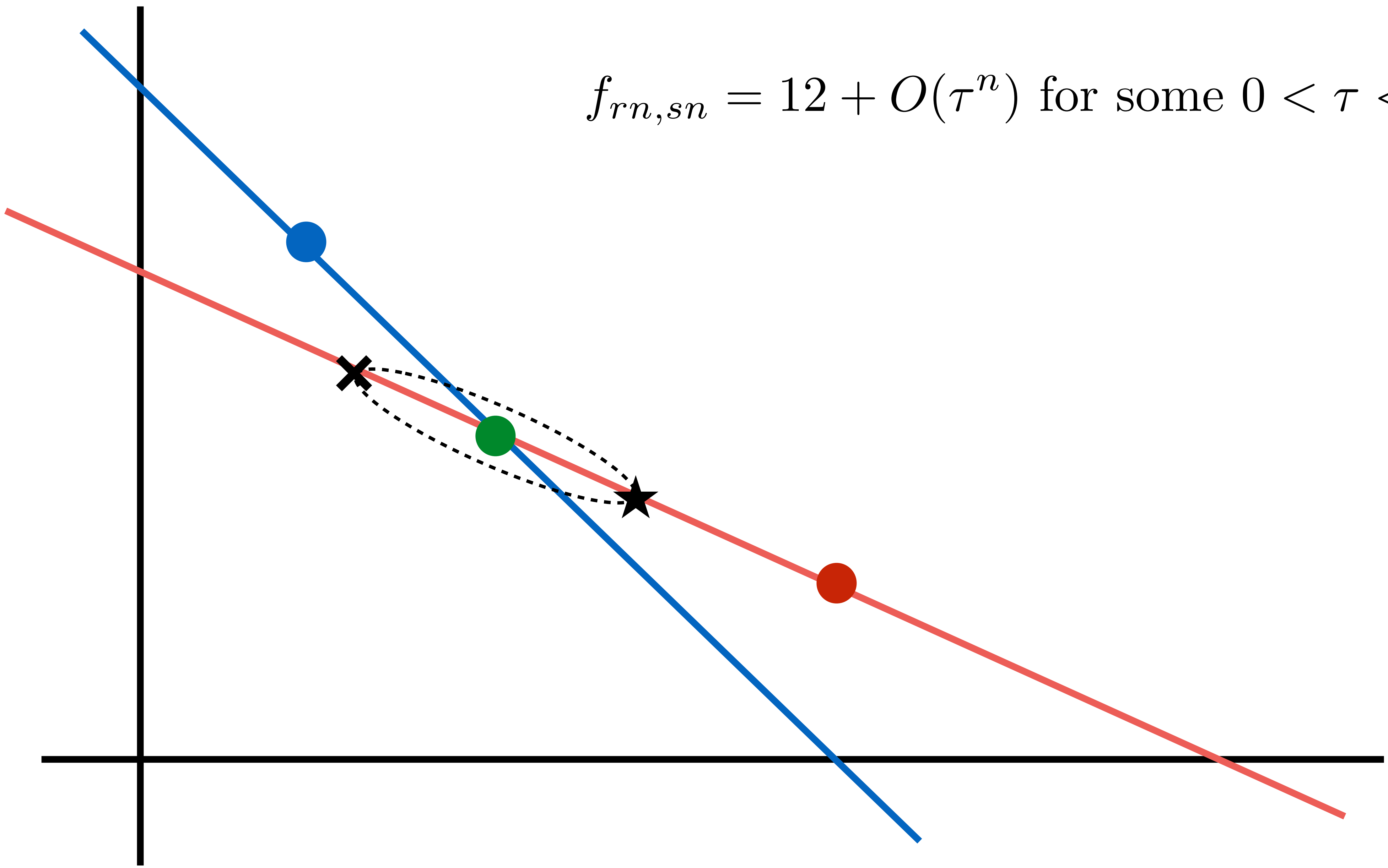
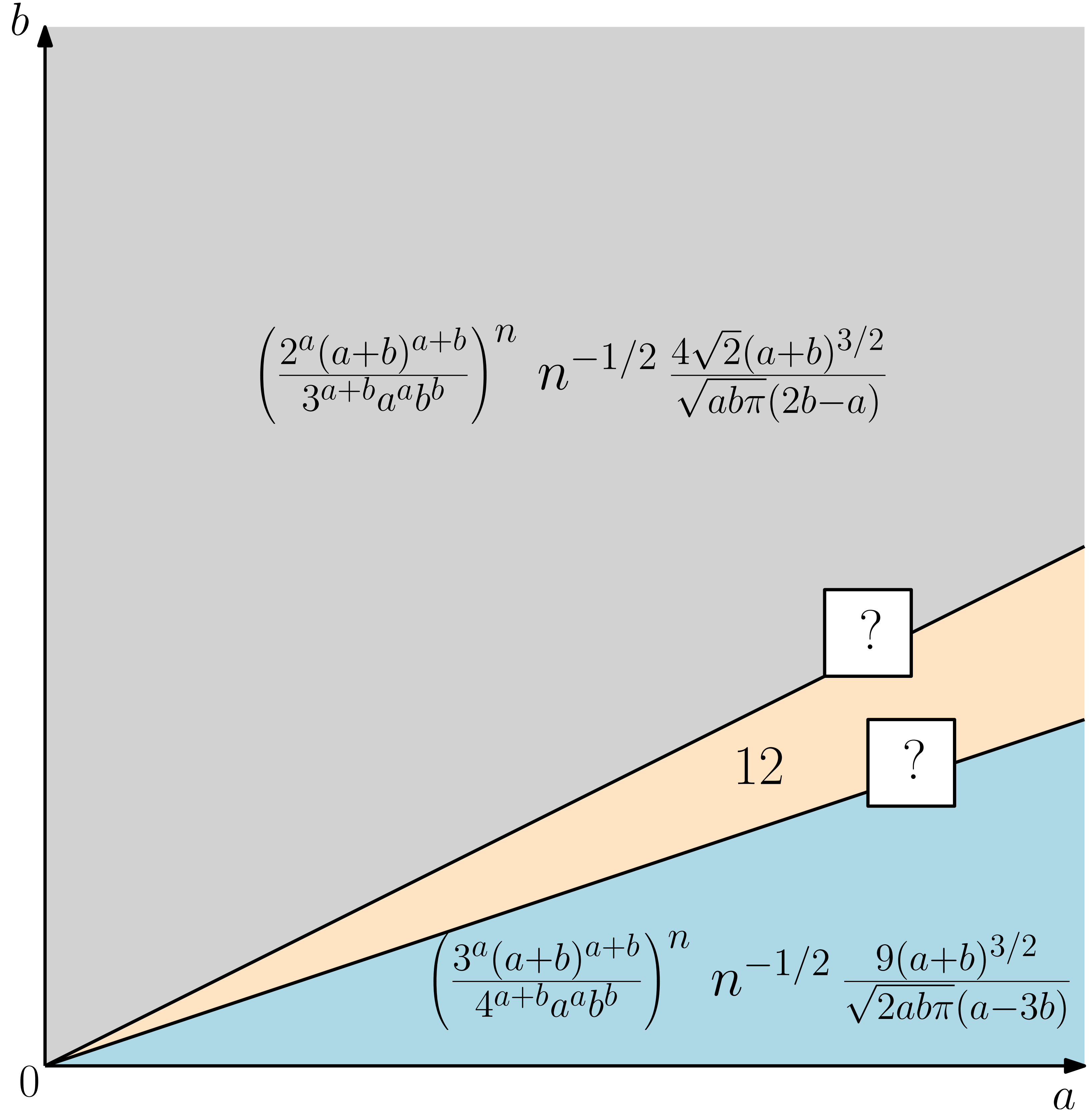
$$f_{rn,sn} = 12 + O(\tau^n) \text{ for some } 0 < \tau < 1$$



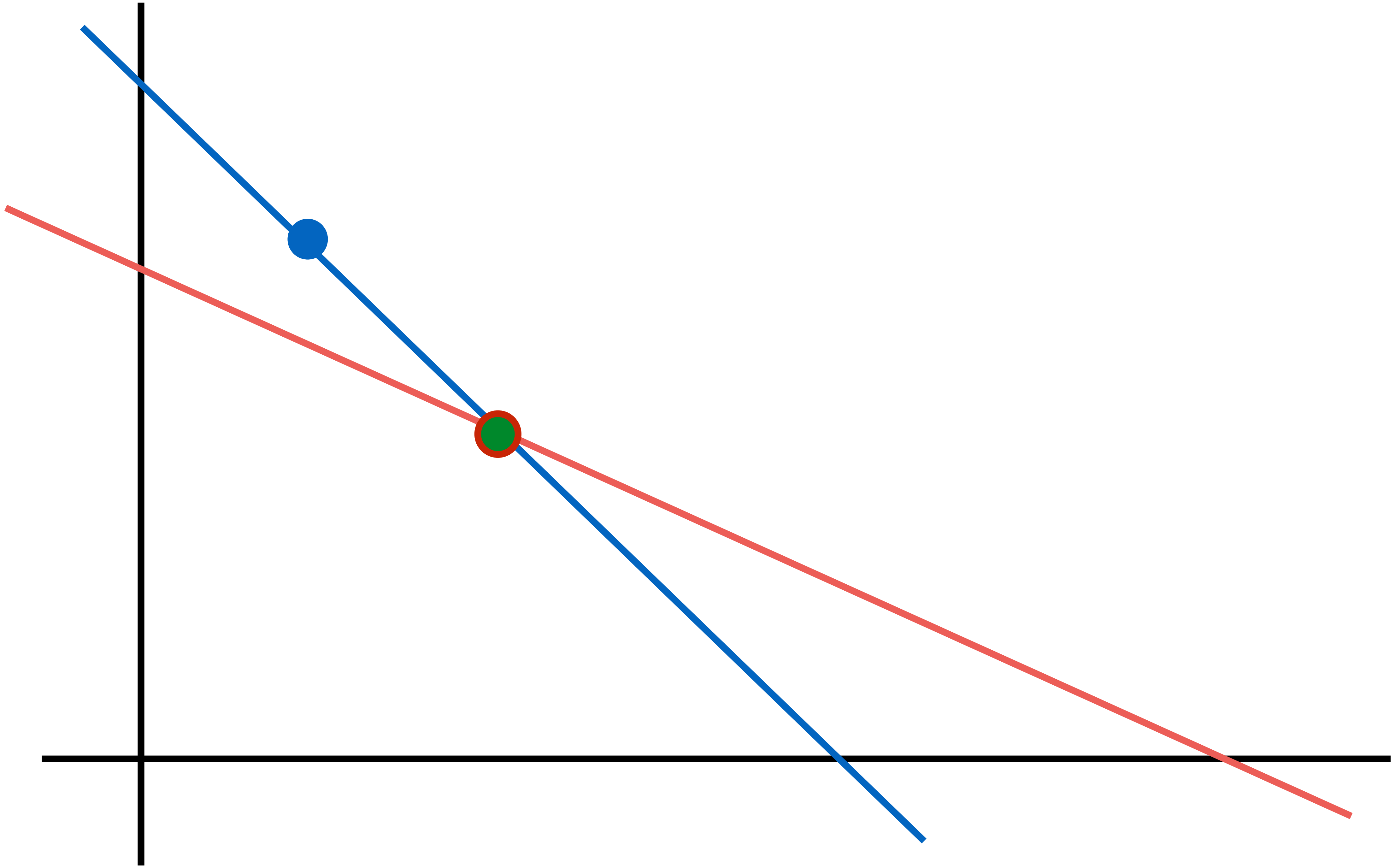


Asymptotics in direction $\mathbf{r} = (a,b)$

Non-Generic Directions

If $r = 2s$ then $\sigma_1 = \sigma_{1,2}$

We can take a residue over one, but not both, lines



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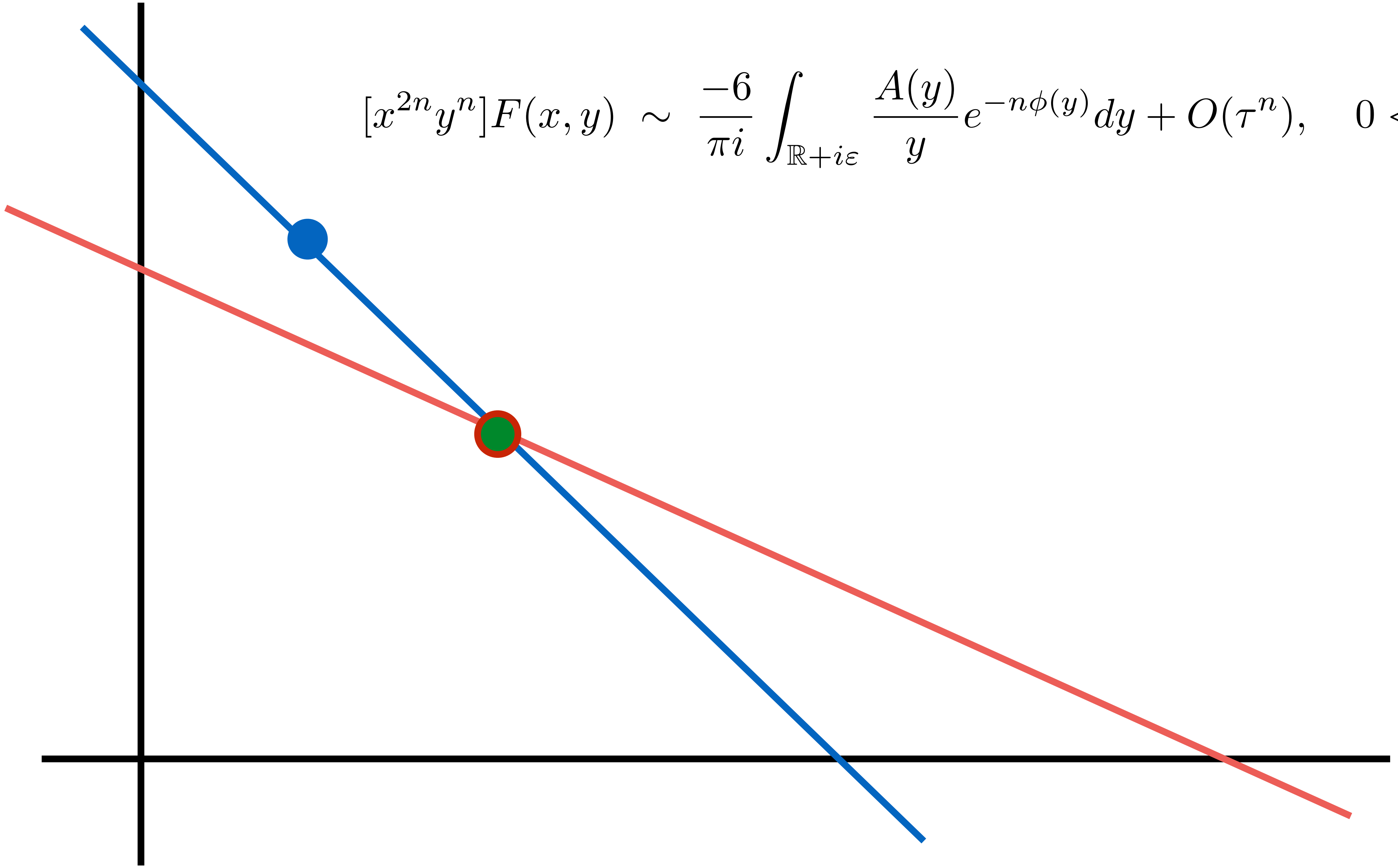
$$[x^{2n}y^n]F(x,y) \sim \frac{-6}{\pi i}\int_{\mathbb{R}+i\varepsilon} \frac{A(y)}{y} e^{-n\phi(y)} dy + O(\tau^n), \quad 0 < \tau < 1$$



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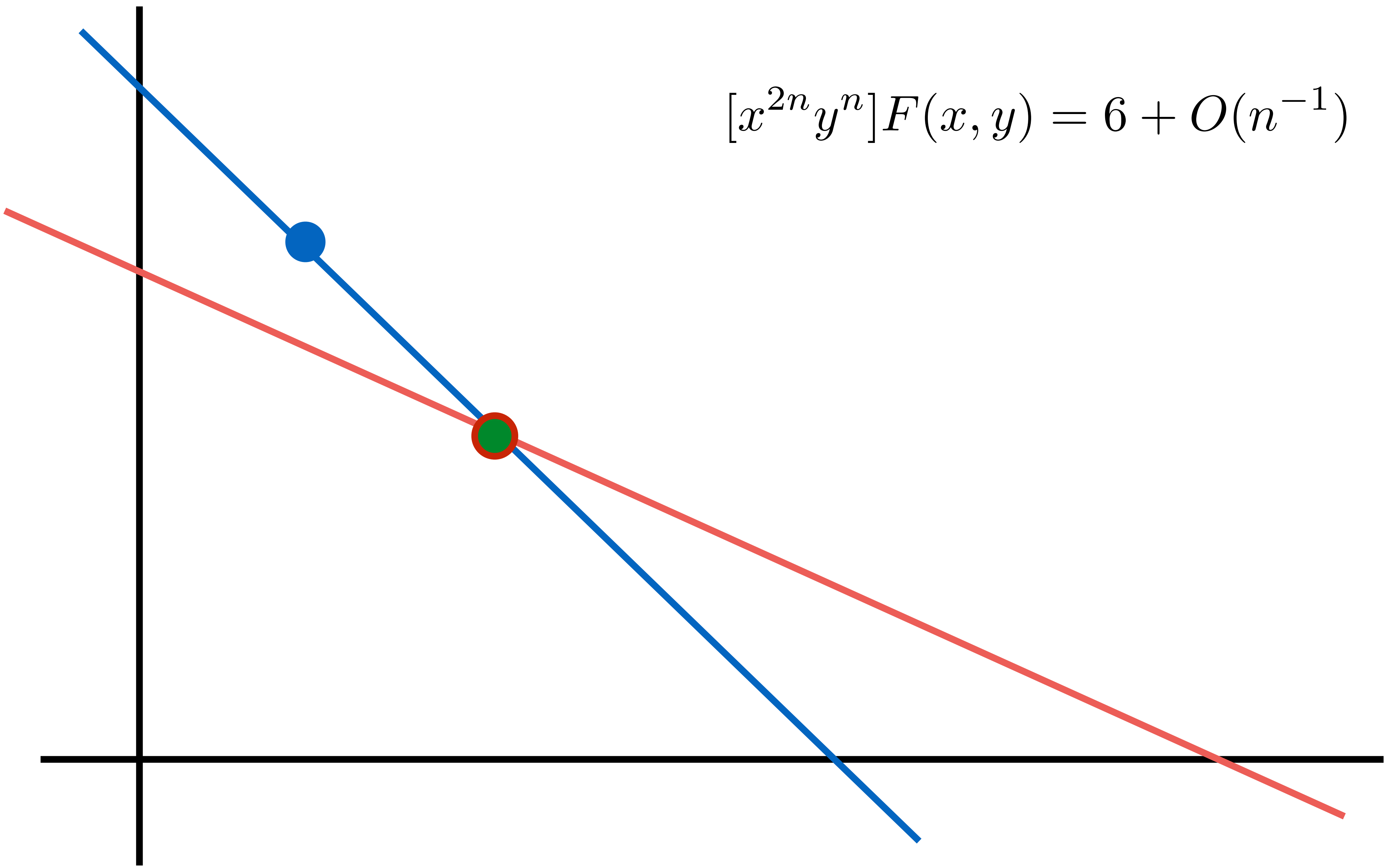
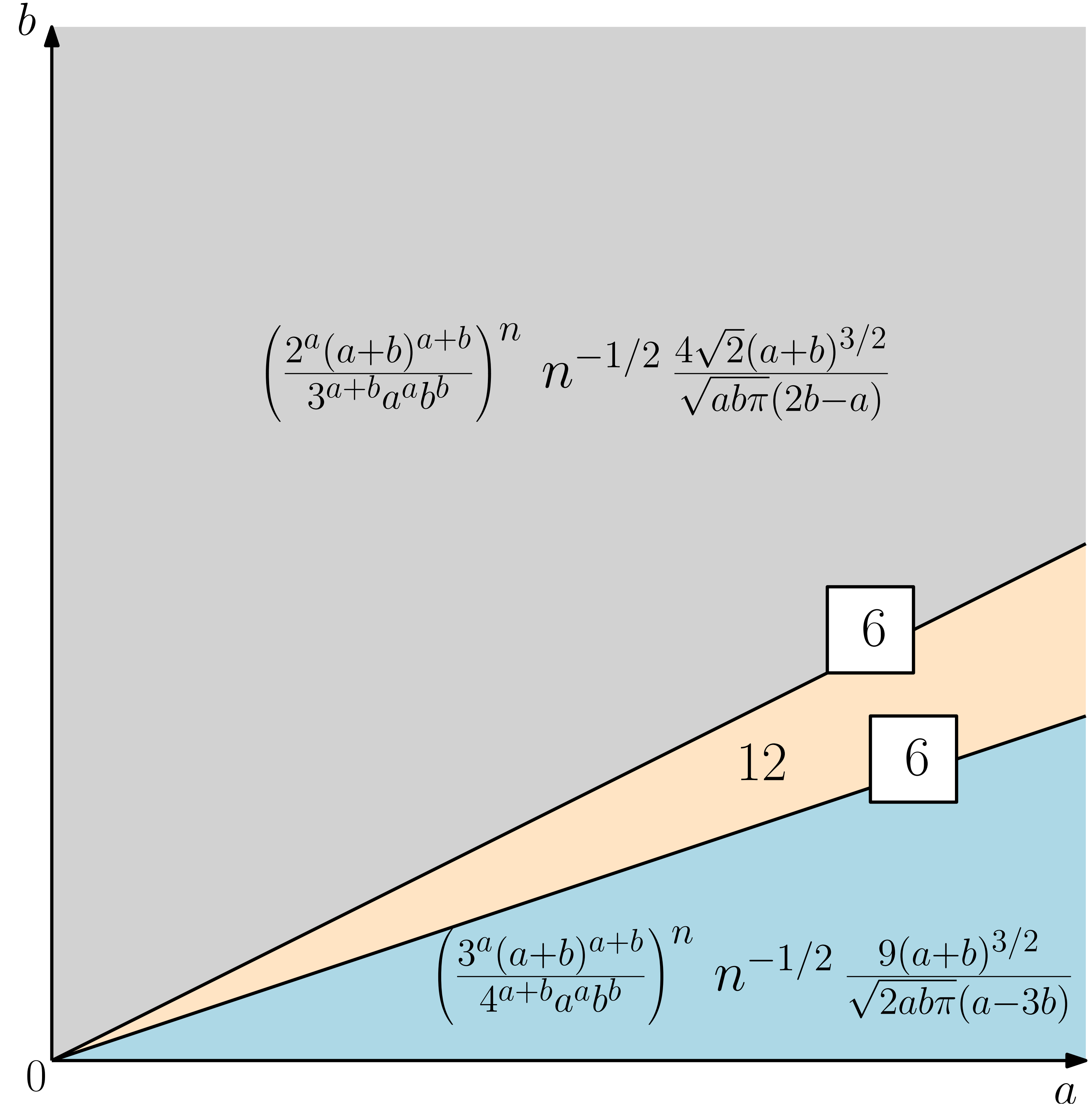


$$[x^{2n}y^n]F(x,y) = 6 + O(n^{-1})$$



Asymptotics in direction $\mathbf{r} = (a,b)$

Beyond Smoothness

How do we generalize?

Need to properly **define critical points**

Their smooth definition is constructed to give saddle-point integrals

Alternative definitions

Assume $H(\mathbf{w}) = 0$ but $(\nabla H)(\mathbf{w}) \neq \mathbf{0}$. The following are equivalent

- $\mathbf{w}$ is critical in the direction $\mathbf{r}$
- $(\nabla \phi)(\mathbf{w}) = \lambda (\nabla H)(\mathbf{w})$ where $\phi(\mathbf{z}) = \mathbf{z}^{\mathbf{r}}$
- $\mathbf{r} = \lambda (\nabla_{\log} H)(\mathbf{w})$ where $(\nabla_{\log} H)(\mathbf{w}) = \Big(w_1 H_{z_1}(\mathbf{w}), \dots, w_d H_{z_d}(\mathbf{w}) \Big)$
- the differential of ϕ restricted to the manifold $\mathcal{V}$ vanishes at $\mathbf{w}$

Multiple Points

In general, **partition** $\mathcal{V}$ into a finite collection of **smooth strata**
Need the strata to *fit together nicely*

Simplest Case

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H_1(\mathbf{z}) \cdots H_m(\mathbf{z})}$$

where

- $(\nabla H_k)(\mathbf{w}) \neq \mathbf{0}$ if $\mathbf{w}$ in $\mathcal{V}_k = \{\mathbf{z} : H_k(\mathbf{z}) = 0\}$
- $\nabla H_{k_1}(\mathbf{w}), \ldots, \nabla H_{k_s}(\mathbf{w})$ linearly independent if $\mathbf{w} \in \mathcal{V}_{k_1} \cap \cdots \cap \mathcal{V}_{k_s}$

This is an example of the **transverse multiple point** case

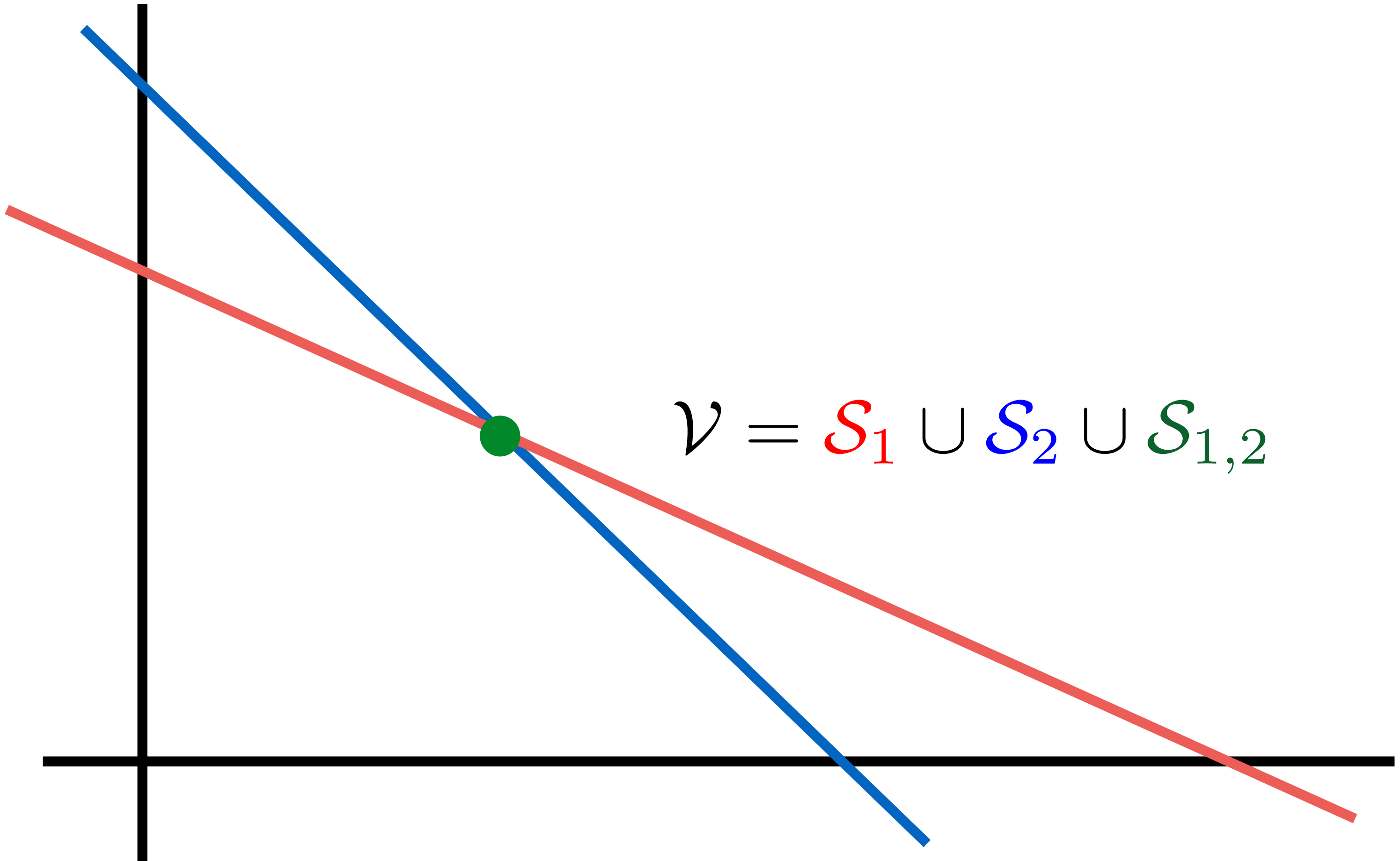
Strata

Under these assumptions, for any $S = \{k_1, \ldots, k_s\}$ we define the **flat**

$$\mathcal{V}_S = \mathcal{V}_{k_1,\ldots,k_s} = \mathcal{V}_{k_1} \cap \cdots \cap \mathcal{V}_{k_s}$$

and the **stratum**

$$\mathcal{S}_S = \mathcal{V}_S \setminus \bigcup_{\mathcal{V}_T \subsetneq \mathcal{V}_S} \mathcal{V}_T$$



$$\mathcal{V} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_{1,2}$$

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Definition

$\mathbf{w} \in \mathcal{S}_S$ is a **critical point** in the direction $\mathbf{r}$ if

$$\mathbf{r} = \lambda_1 (\nabla_{\log} H_1)(\mathbf{w}) + \cdots + \lambda_s (\nabla_{\log} H_s)(\mathbf{w})$$

Strata

Under these assumptions, for any $S = \{k_1, \ldots, k_s\}$ we define the **flat**

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The critical points on $\mathcal{S}_S$ are defined by the vanishing of $H_{k_1}, \ldots, H_{k_s}$ and the $(s+1) \times (s+1)$ minors of

$$\begin{pmatrix} (\nabla_{\log} H_{k_1})(\mathbf{w}) \\ \vdots \\ (\nabla_{\log} H_{k_s})(\mathbf{w}) \\ \mathbf{r} \end{pmatrix}$$

Contributing Points

Suppose $\mathbf{w}$ is a minimal critical point on the stratum $\mathcal{S}_{1,\ldots,s}$
If $(\partial H_j/\partial z_{k_j})(\mathbf{w}) \neq 0$ then define

$$\mathbf{v}_j = \frac{(\nabla_{\log} H_j)(\mathbf{w})}{w_{k_j}(\partial H_j/\partial z_{k_j})(\mathbf{w})} = \left(\frac{w_1(\partial H_j/\partial z_1)(\mathbf{w})}{w_{k_j}(\partial H_j/\partial z_{k_j})(\mathbf{w})}, \ldots, \frac{w_d(\partial H_j/\partial z_d)(\mathbf{w})}{w_{k_j}(\partial H_j/\partial z_{k_j})(\mathbf{w})} \right)$$

It turns out $\mathbf{v}_j \in \mathbb{R}^d$

Definition
If $\mathbf{r} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_s \mathbf{v}_s$ where each $\lambda_k > 0$ then $\mathbf{w}$ is called a **contributing point**.

Note: Smooth minimal critical points are always contributing!

Multiple Point Asymptotics

Suppose $\mathbf{w}$ is a minimal contributing singularity of

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H_1(\mathbf{z}) \cdots H_m(\mathbf{z})}$$

with no other singularity with the same coordinate-wise modulus.

If w lies on a stratum of codimension s then there exist explicit matrices $\mathcal{M}_{\mathbf{w}}$ and $\Gamma_{\mathbf{w}}$ such that

$$f_{n\mathbf{r}} = \mathbf{w}^{-n\mathbf{r}}\, n^{(s-d)/2}\, (2\pi r_d)^{(s-d)/2} \det(\mathcal{M}_{\mathbf{w}})^{-1/2} \left(\frac{G(\mathbf{w})}{\det \Gamma_{\mathbf{w}}} + O\left(\frac{1}{n}\right)\right)$$

when $\det \mathcal{M}_{\mathbf{w}} \neq 0$

Proof Idea

Locally, $\mathcal{V}$ looks like a union of s hyperplanes near $\mathbf{w}$

1) Introduce asymptotically negligible integrals

2) Take residues in s dimensions

3) Approximate a $(d-s)-$dimensional saddle-point integral

Proof Idea

Locally, $\mathcal{V}$ looks like a union of s hyperplanes near $\mathbf{w}$

1) Introduce asymptotically negligible integrals

2) Take residues in s dimensions

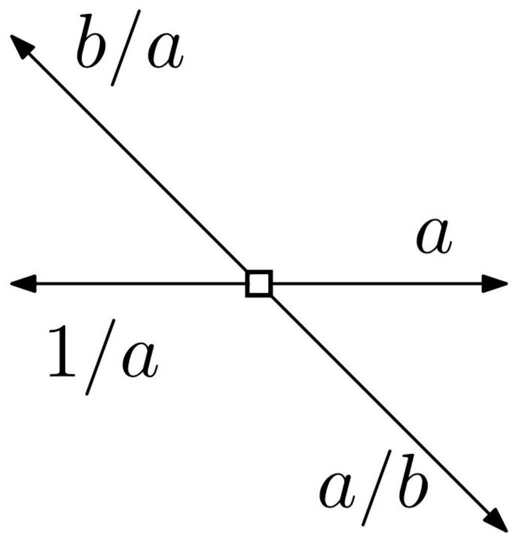
3) Approximate a $(d-s)-$dimensional saddle-point integral

If $s = d$ then

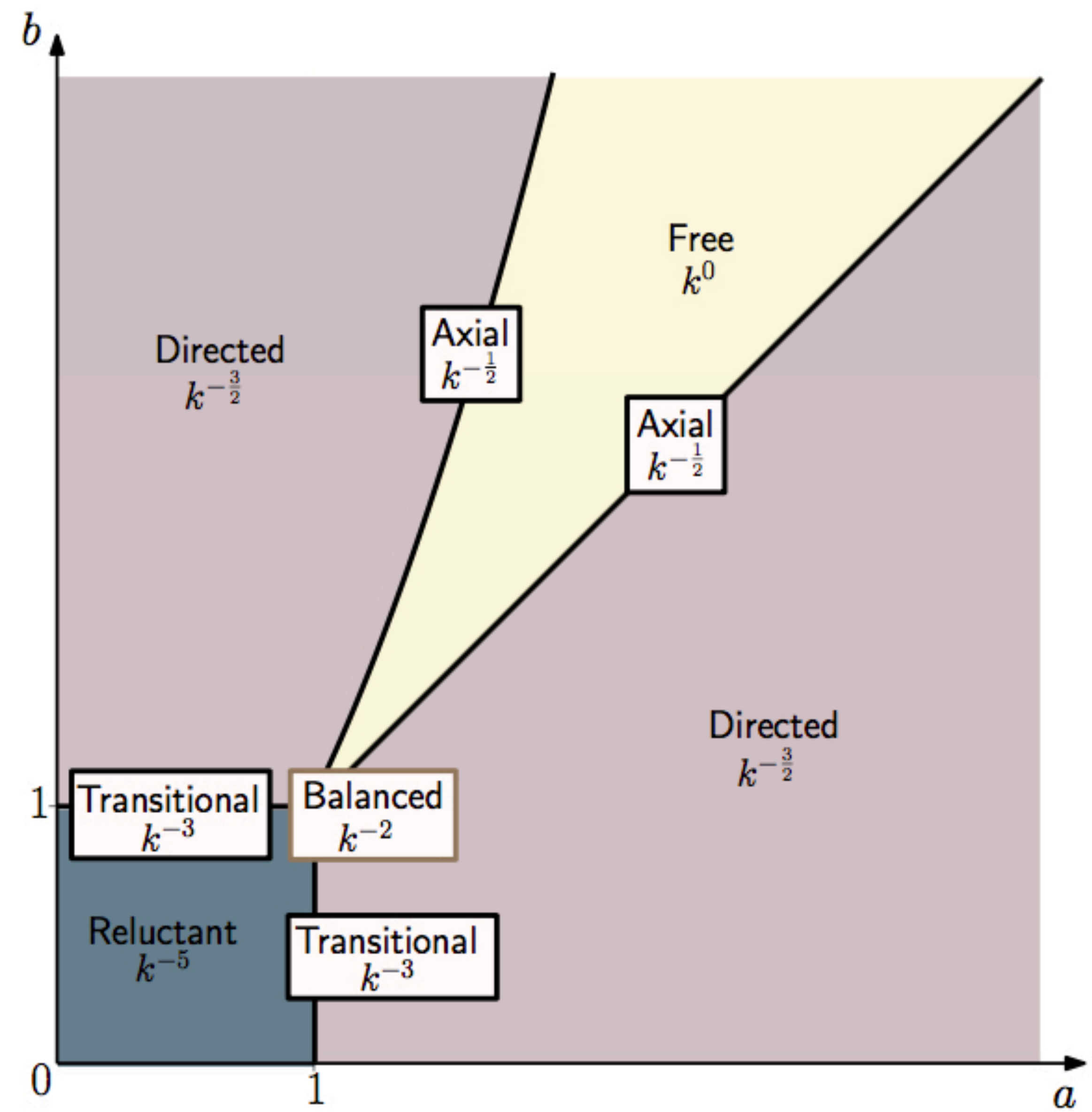
$$f_{n\mathbf{r}} = \mathbf{w}^{-n\mathbf{r}} \cdot \frac{G(\mathbf{w})}{|\det \Gamma_{\mathbf{w}}|} + O\left(\tau^{n}\right)$$

where $0 < \tau < |\mathbf{w}^{-\mathbf{r}}|$ and $\Gamma_{\mathbf{w}} = \begin{pmatrix} (\nabla_{\log} H_1)(\mathbf{w}) \\ \vdots \\ (\nabla_{\log} H_s)(\mathbf{w}) \end{pmatrix}$

Quadrant Walks on Weighed Stepsets



$$\frac{yt^2(y-b)(a-x)(a+x)(a^2y-bx^2)(ay-bx)(ay+bx)}{a^4b^3\left(1-txy\left(\frac{1}{ax}+ax+\frac{ax}{by}+\frac{by}{ax}\right)\right)(1-x)(1-y)}$$



Courtiel, M., Mishna, and Raschel (2017)

Topic 6
Geometric Approach to ACSV

melczer.ca/ALEA22

Whitney Stratifications

In general, we partition $\mathcal{V}$ into a finite collection of **strata** that are manifolds satisfying the *Whitney conditions*

The **critical points** of F in direction $\mathbf{r}$ are the critical points of $\phi(\mathbf{z}) = \mathbf{z}^{\mathbf{r}}$ restricted to each of these manifolds

Fact: The exist effectively computable algebraic sets

$$\mathcal{V} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_m = \varnothing$$

such that the connected components of $\mathcal{F}_k \setminus \mathcal{F}_{k+1}$ form a Whitney stratification.

Corollary: All critical points defined by algebraic (in)equations

Height Functions

For large n the modulus of

$$f_{n\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_{\mathcal{C}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{n\mathbf{r}+\mathbf{1}}}$$

is captured by

$$h(\mathbf{z}) = -r_1 \log|z_1| - \cdots - r_d \log|z_d|$$

Idea 1: We start with points of C with high height
Try to push them down as far as possible while avoiding $\mathcal{V}$

Idea 2: Use *Leray residues* to reduce to integral 'on' $\mathcal{V}$
Try to push down resulting *intersection cycles*

Pushing Down Cycle

Start with the expression

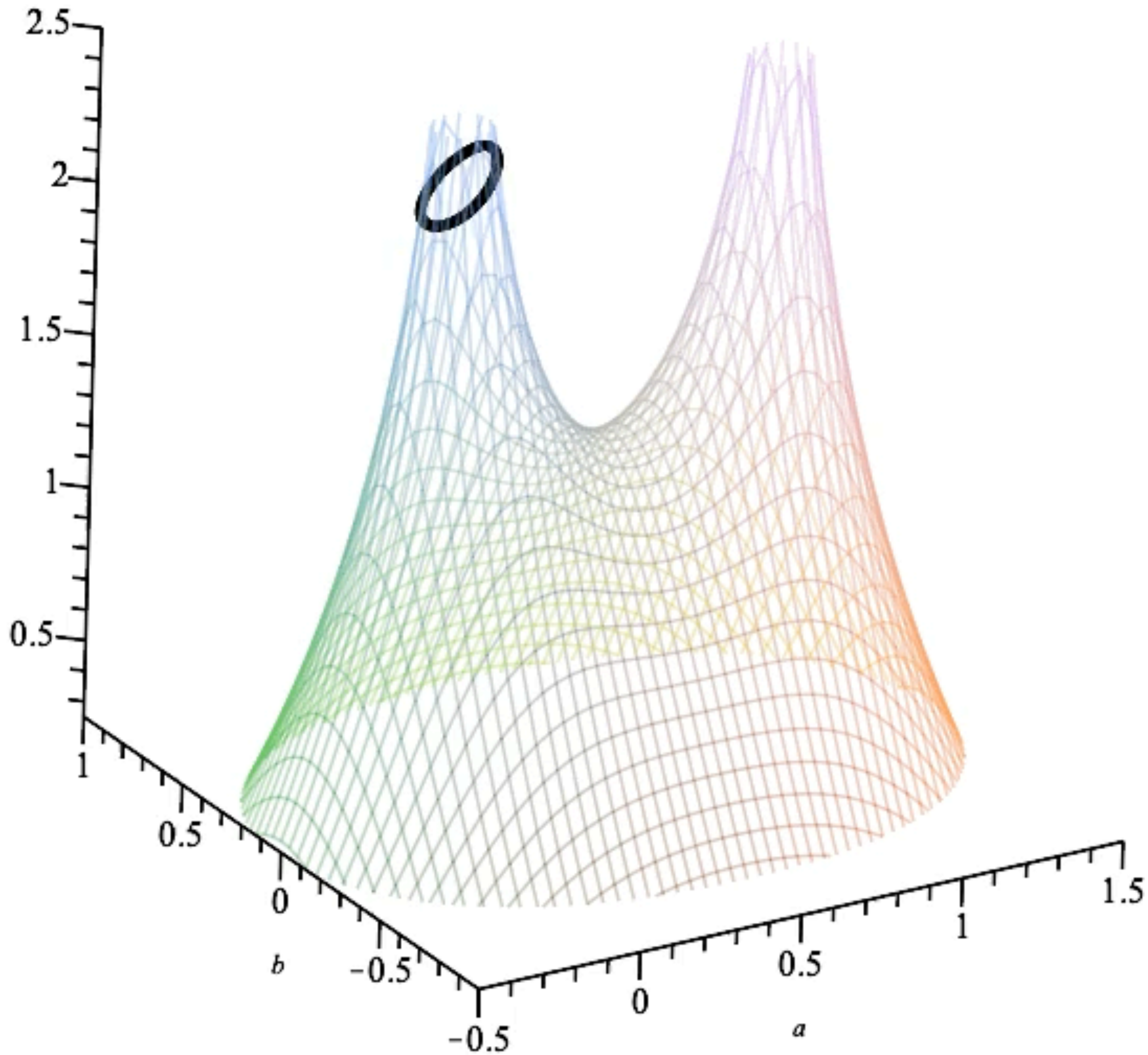
$$\binom{2n}{n} = \frac{1}{(2\pi i)^2} \int_{|x|=\varepsilon, |y|=\varepsilon} \frac{1}{1-x-y} \frac{dxdy}{x^{n+1}y^{n+1}}$$

Expand $|y|$ until you hit $\mathcal{V}$ and take a residue

$$\binom{2n}{n} = \frac{-1}{2\pi i} \int_{|x|=\varepsilon} \frac{dx}{x^{n+1}(1-x)^{n+1}}$$

Then flow the cycle $|x| = \varepsilon$ to points of lower height

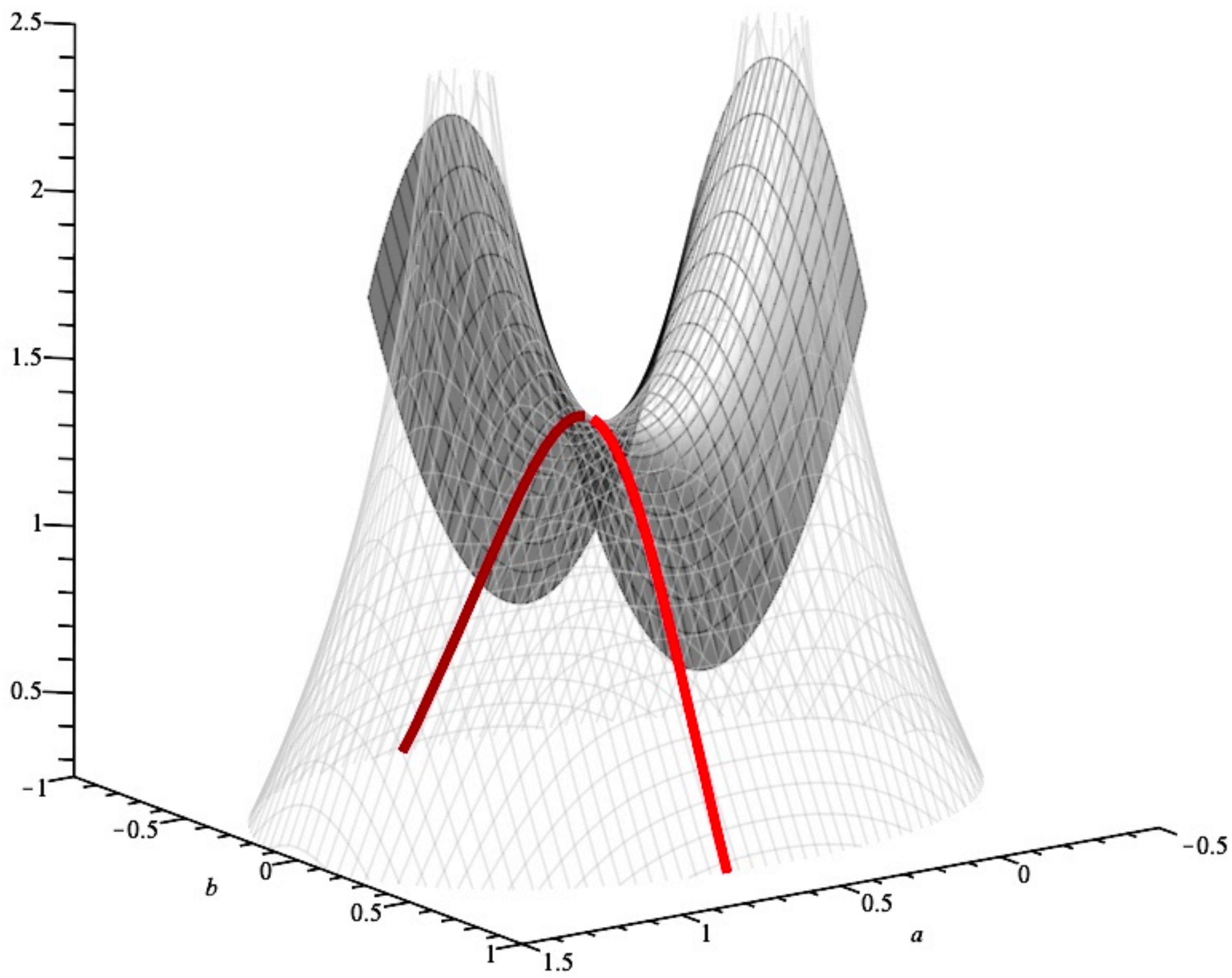
Maximum height = 2.408



Flow of $x = a + ib$ starting with $|x| = 1/10$

with $h(a,b) = -\log|x| - \log|1-x| = -\dfrac{1}{2}\log(a^2+b^2) - \dfrac{1}{2}\log((1-a)^2+b^2)$

We get stuck at saddle-point



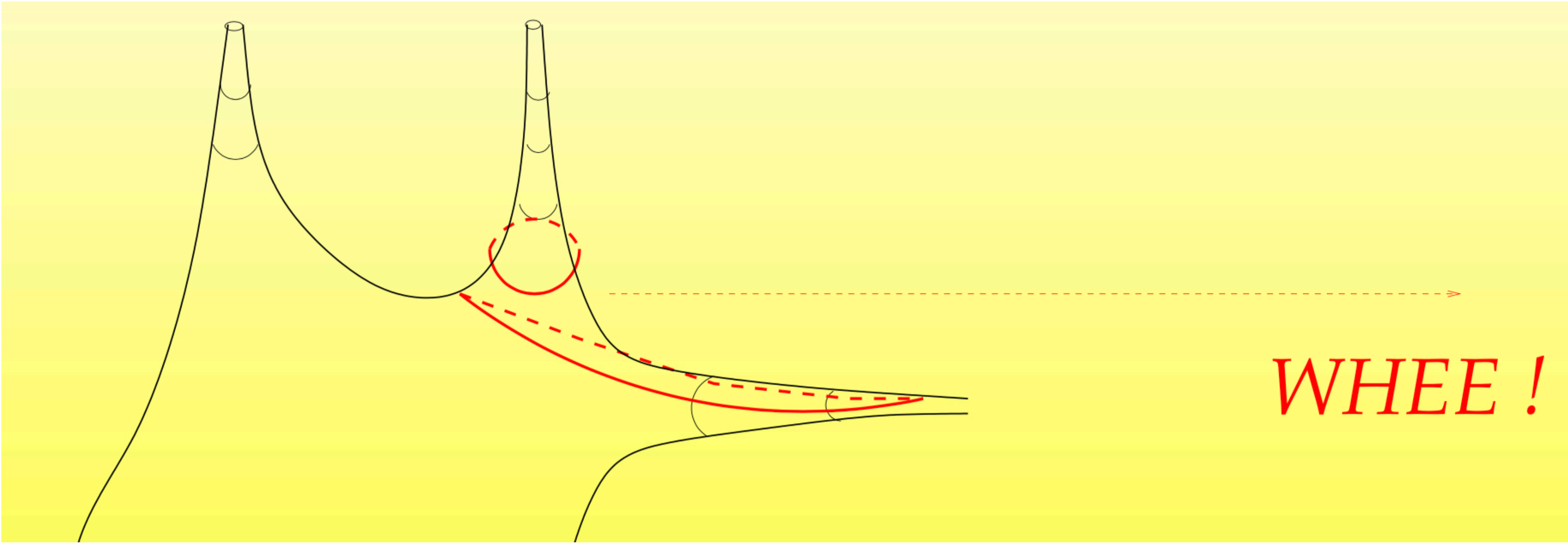
$$\binom{2n}{n} = \frac{-1}{2\pi i}\int_{\mathcal{C}'} \frac{dx}{x^{n+1}(1-x)^{n+1}} = \frac{4^n}{\sqrt{\pi n}}\left(1 + O\left(n^{-1}\right)\right)$$

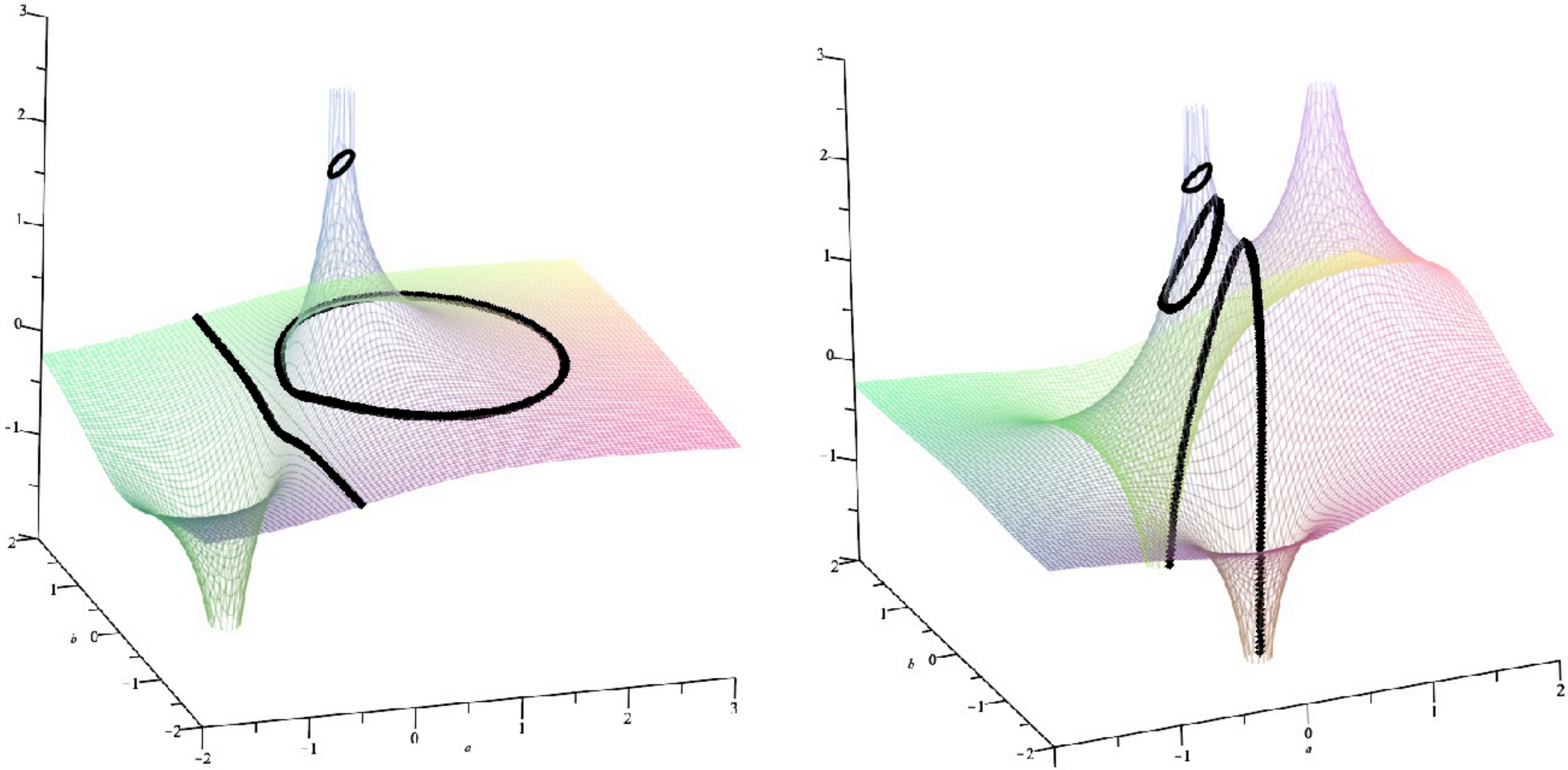
Morse theory

Actually performing these flows is inefficient!
But we can **predict** where they get stuck — at **critical points**!
And we will get **saddle-point integrals**!

Morse theory: Topological study of manifolds via height functions
Stratified Morse theory: Study of varieties (and more)

Problem: Our height functions are not *proper*
Solution: Ignore parts not contributing to dominant asymptotics





Flows of $|x| = 1/10$ on $\mathcal{V}(1 - x - xy)$ and $\mathcal{V}(1 - x - y - x^2 y)$

Morse Theory For Asymptotics

(Baryshnikov, M. and Pemantle 2021)

There exist checkable algebraic conditions, under which the results of Morse theory hold and we can write

$$f_{n\mathbf{r}} = \sum_{\mathbf{w} \in \mathrm{crit}} \kappa_{\mathbf{w}} \Psi_{\mathbf{w}}(n) + \text{asymptotically negligible terms}$$

where $\kappa_{\mathbf{w}} \in \mathbb{Z}$

$\Psi_{\mathbf{w}}(n)$ is a local integral

Cannot always determine $\kappa_{\mathbf{w}}$ directly, but knowing they are integers we can sometimes use **rigorous numerical analytic continuation**

Two Complementary Approaches: Approach #1

Consider $F(w,x,y,z) = \dfrac{1}{1-(w+x+y+z)+27wxyz}$

There are **three critical points**, with asymptotic contributions

$$\Phi_1(n) = 81^n \cdot \Theta(n) \qquad (w=x=y=z=1/3)$$

$$\Phi_2(n) = \left(\frac{11621 - i\sqrt{30803599}}{20420}\right)\frac{\left(-7-4i\sqrt{2}\right)^n}{n^{3/2}\pi^{3/2}} \qquad (w=x=y=z=-1/3+i\sqrt{2}/3)$$

$$\Phi_3(n) = \left(\frac{11621 + i\sqrt{30803599}}{20420}\right)\frac{\left(-7+4i\sqrt{2}\right)^n}{n^{3/2}\pi^{3/2}} \qquad (w=x=y=z=-1/3-i\sqrt{2}/3)$$

Our Morse result immediately implies

$$\boxed{f_{n,n,n,n} = \kappa_1\Phi_1(n) + \kappa_2\Phi_2(n) + \kappa_3\Phi_3(n)}$$

Two Complementary Approaches: Approach #2

Consider $F(w,x,y,z) = \dfrac{1}{1-(w+x+y+z)+27wxyz}$

The diagonal coefficients satisfy a **linear recurrence**

$$(n^3+6n^2+12n+8)c_{n+2}+(14n^3+63n^2+97n+51)c_{n+1}+(81n^3+243n^2+243n+81)c_n=0$$

whose solutions form a **complex vector-space** with basis

$$\Psi_1(n)=\frac{(94i\sqrt{2}-7)^n}{n^{3/2}}\left(1+O\left(\frac{1}{n}\right)\right) \qquad \Psi_2(n)=\frac{(-94i\sqrt{2}-7)^n}{n^{3/2}}\left(1+O\left(\frac{1}{n}\right)\right)$$

so

$$\boxed{f_{n,n,n,n}=\sigma_1\Psi_1(n)+\sigma_2\Psi_2(n)}$$

Now the $\sigma_j \in \mathbb{C}$ but **we can rigorously approximate them to any desired accuracy using *numeric analytic continuation***

Two Complementary Approaches: Combined

The sequence in question is

Unknown integers

$$f_{n,n,n,n} = \kappa_1 \Phi_1(n) + \kappa_2 \Phi_2(n) + \kappa_3 \Phi_3(n)$$

and

$$f_{n,n,n,n} = \sigma_1 \Psi_1(n) + \sigma_2 \Psi_2(n)$$

Approximated Complex Numbers

By **combining methods** we can **exactly determine asymptotics**.

Arbitrary approximation cannot prove equalities without bounds!

Two Complementary Approaches: Combined

The sequence in question is

Unknown integers

$$f_{n,n,n,n} = \kappa_1 \Phi_1(n) + \kappa_2 \Phi_2(n) + \kappa_3 \Phi_3(n)$$

and

$$f_{n,n,n,n} = \sigma_1 \Psi_1(n) + \sigma_2 \Psi_2(n)$$

Approximated Complex Numbers

This process shows

$$\kappa_1 = 0.000\ldots$$

$$\kappa_2 = 2.999\ldots$$

$$\kappa_3 = 2.999\ldots$$

to hundreds of decimal places in seconds (on my laptop)

Two Complementary Approaches: Combined

The sequence in question is

Unknown integers

$$f_{n,n,n,n} = \kappa_1 \Phi_1(n) + \kappa_2 \Phi_2(n) + \kappa_3 \Phi_3(n)$$

and

$$f_{n,n,n,n} = \sigma_1 \Psi_1(n) + \sigma_2 \Psi_2(n)$$

Approximated Complex Numbers

Theorem 2.5. *The diagonal coefficients $a_{n,n,n,n}$ of the function in Example 2.4 have an asymptotic expansion in decreasing powers of n, beginning as follows.*

$$a_{n,n,n,n} = 3 \cdot \left(\frac{\left(4i\sqrt{2}-7\right)^n}{n^{3/2}} \frac{\left(5i-\sqrt{2}\right)\sqrt{-2i\sqrt{2}-8}}{24\pi^{3/2}} + \frac{\left(-4i\sqrt{2}-7\right)^n}{n^{3/2}} \frac{\left(-5i-\sqrt{2}\right)\sqrt{2i\sqrt{2}-8}}{24\pi^{3/2}} \right) \quad (2.5)$$
$$+O\left(9^n \, n^{-5/2}\right)$$

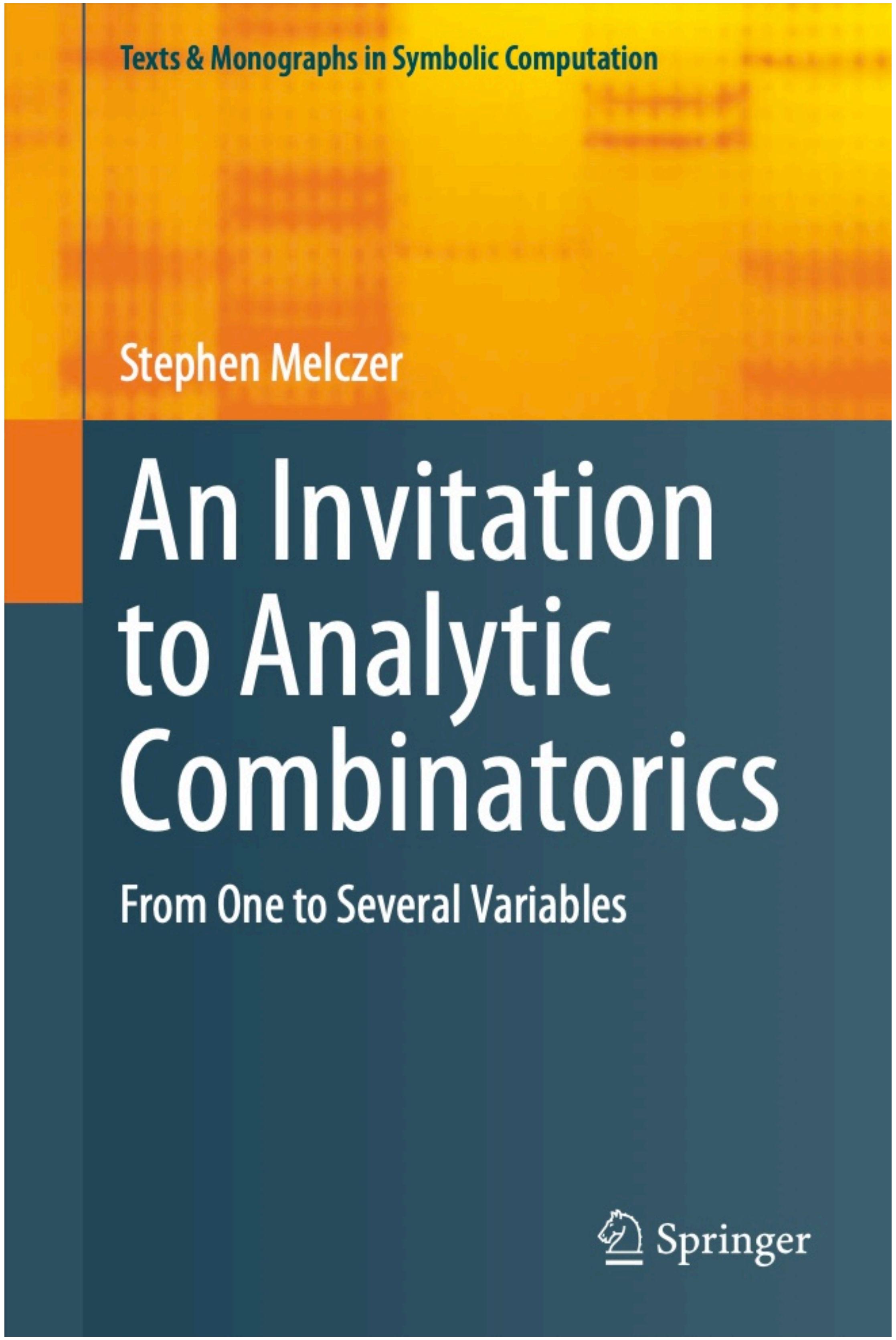
(Baryshnikov, M. and Pemantle 2021)

General Picture

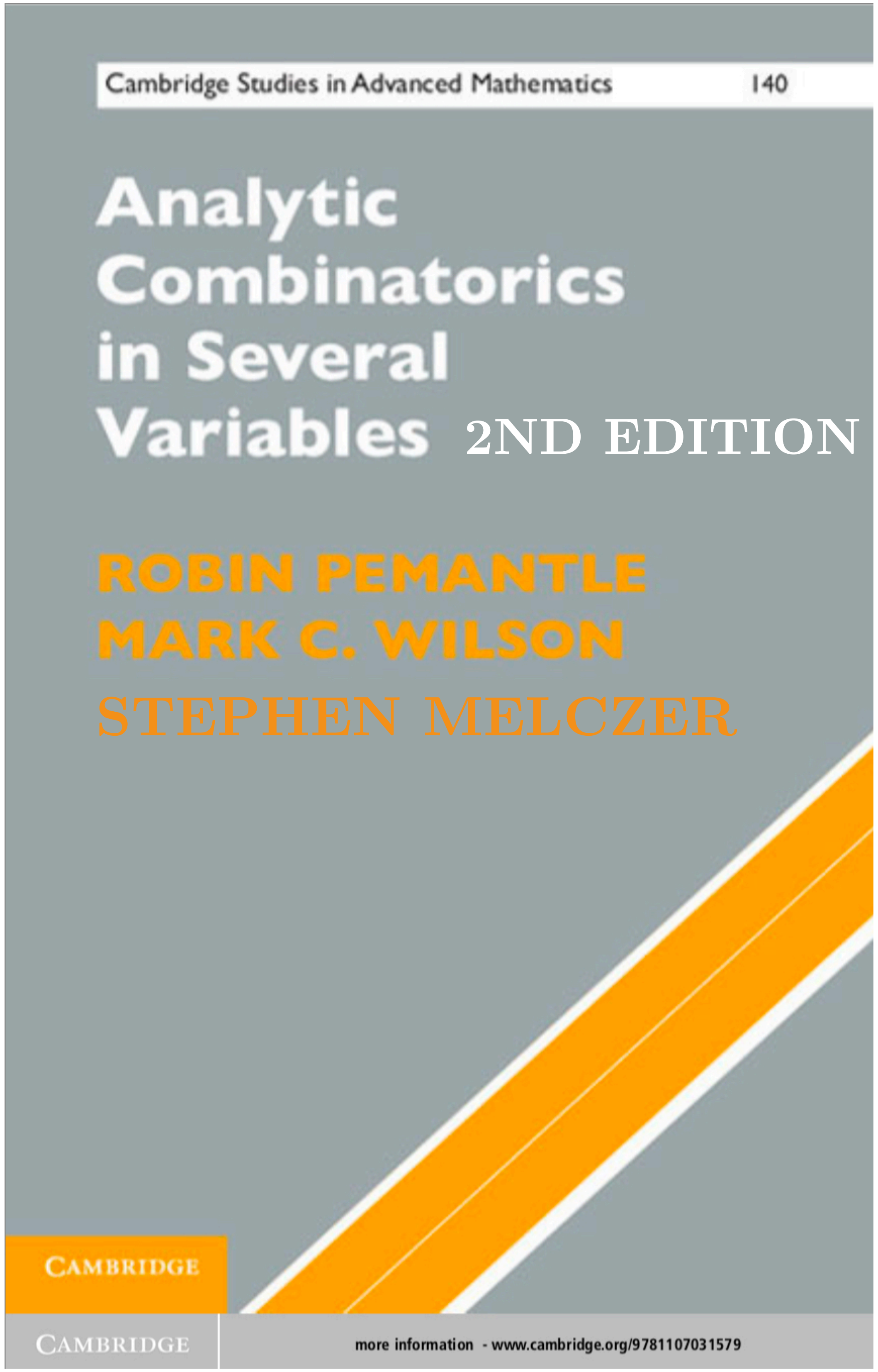
- Compute critical points and sort by height
- Verify *critical points at infinity* don't interfere
- Determine the integer coefficients $\kappa_{\mathbf{w}}$ of highest crit pts
- Keep going until you get non-zero coefficients
- Try to approximate local integrals $\Psi_{\mathbf{w}}(n)$

Hardest part: Finding integer coefficients (and checking non-zero)

Currently we can only find the coefficients for **minimal critical points**, or in **dimension two**, or when H has only **linear factors**



For new researchers — focus more on explicit results and computation



Most general theory, covering topological approaches

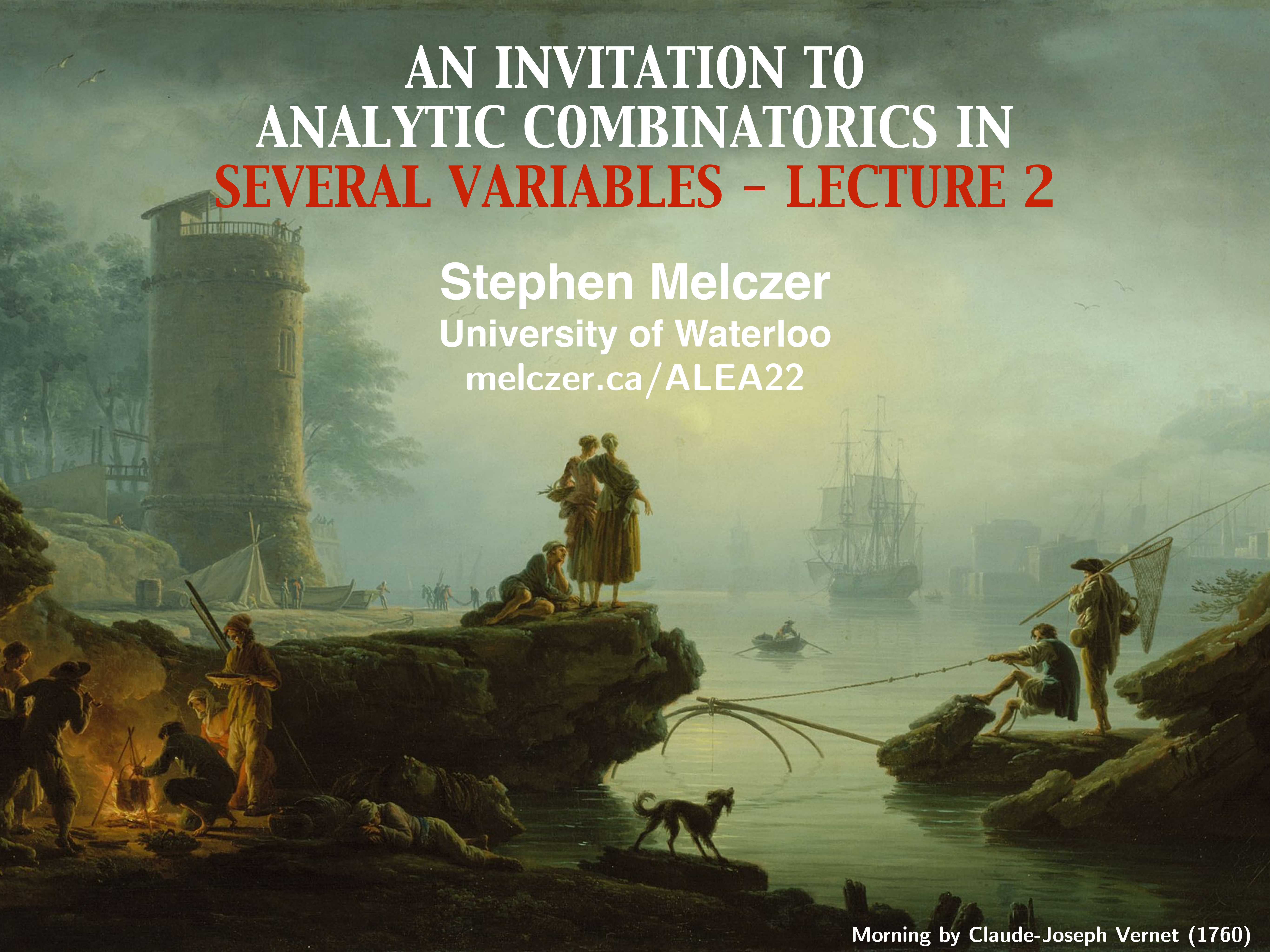
THANK YOU!

An Invitation to Analytic Combinatorics
melczer.ca/textbook

melczer.ca/ALEA22

AN INVITATION TO ANALYTIC COMBINATORICS IN SEVERAL VARIABLES – LECTURE 2

Stephen Melczer
University of Waterloo
melczer.ca/ALEA22



Morning by Claude-Joseph Vernet (1760)

Analytic Combinatorics in Several Variables

We study the **r**-diagonals of

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})} = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$$

Assume $H(\mathbf{z}) = H_{z_1}(\mathbf{z}) = \cdots = H_{z_d}(\mathbf{z}) = 0$ has no solution

Then the **singular variety** $\mathcal{V} = \{\mathbf{z} \in \mathbb{C}^d : H(\mathbf{z}) = 0\}$ is **smooth**

Analytic Combinatorics in Several Variables

We study the **r**-diagonals of

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Then the **singular variety** $\mathcal{V} = \{\mathbf{z} \in \mathbb{C}^d : H(\mathbf{z}) = 0\}$ is **smooth**

Minimal Points: Coordinate-wise smallest singularities

Critical Points: Solve $H = 0, \quad r_j z_1 H_{z_1} = r_1 z_j H_{z_j} \quad (2 \leq j \leq d)$

Main Theorem of Smooth ACSV

Suppose that

$$H = 0, \qquad r_j z_1 H_{z_1} = r_1 z_j H_{z_j} \quad (2 \leq j \leq d)$$

admits a finite number of solutions. If

- there is exactly one minimal solution, $\mathbf{w} \in \mathbb{C}_*^d$
- $H_{z_d}(\mathbf{w})$ and $\det \mathcal{M}$ are non-zero,

then

$$[\mathbf{z}^{n\mathbf{r}}] \frac{G(\mathbf{z})}{H(\mathbf{z})} = \mathbf{w}^{-n\mathbf{r}} \, (n r_d)^{(1-d)/2} \, (2\pi)^{(1-d)/2} \det(\mathcal{M})^{-1/2} \left(\frac{-G(\mathbf{w})}{w_d \, H_{z_d}(\mathbf{w})} + O\left(\frac{1}{n}\right) \right)$$

If there are a finite number of **critical points** with the same coordinate-wise modulus as $\mathbf{w}$, all satisfying these conditions, then we can add their asymptotic contributions.

Topic 3
Higher-Order Terms

melczer.ca/ALEA22

Higher-Order Terms

Under the assumptions of the Main Theorem of Smooth ACSV, for any $M \in \mathbb{N}$ there is an **expansion**

$$[\mathbf{z}^{n\mathbf{r}}]F(\mathbf{z}) = \mathbf{w}^{-n\mathbf{r}}(2\pi n r_d)^{(1-d)/2}\det(\mathcal{M})^{-1/2}\left(\sum_{j=0}^{M} C_j (r_d n)^{-j} + O\left(n^{-M-1}\right)\right)$$

with each C_j **explicitly computable** from the derivatives of G and H up to order $2(j+2)$ evaluated at $\mathbf{w}$

Higher-Order Terms

Under the assumptions of the Main Theorem of Smooth ACSV, for any $M \in \mathbb{N}$ there is an **expansion**

$$[\mathbf{z}^{n\mathbf{r}}]F(\mathbf{z}) = \mathbf{w}^{-n\mathbf{r}}(2\pi n r_d)^{(1-d)/2}\det(\mathcal{M})^{-1/2}\left(\sum_{j=0}^{M} C_j (r_d n)^{-j} + O\left(n^{-M-1}\right)\right)$$

$$C_j = (-1)^j \sum_{0\le \ell \le 2j} \frac{\mathcal{E}^{\ell+j}\left(P(\theta)\,\psi(\theta)^{\ell}\right)}{2^{\ell+j}\ell!(\ell+j)!}\Bigg|_{\theta=\mathbf{0}} \quad \text{for} \quad \mathcal{E} = -\sum_{1\le i,j\le k}\left(\mathcal{M}^{-1}\right)_{ij}\partial_i\partial_j$$

and

$$P(\theta) = \frac{-G\left(\widehat{\mathbf{w}}e^{i\theta}, g\left(\widehat{\mathbf{w}}e^{i\theta}\right)\right)}{g\left(\widehat{\mathbf{w}}e^{i\theta}\right)H_{z_d}\left(\widehat{\mathbf{w}}e^{i\theta}, g\left(\widehat{\mathbf{w}}e^{i\theta}\right)\right)}$$

$$\psi(\theta) = \log\left(\frac{g\left(\widehat{\mathbf{w}}e^{i\theta}\right)}{g(\widehat{\mathbf{w}})}\right) + i(\hat{\mathbf{r}}\cdot\theta)/r_d - (1/2)\theta\cdot\mathcal{M}\cdot\theta^T$$

```
def smoothContrib(G,H,r,vars,CP,M,g):
    # Preliminary definitions
    dd = len(vars)
    field = SR
    tvars = list(var('t%d'%i) for i in range(dd-1))
    dvars = list(var('dt%d'%i) for i in range(dd-1))

    # Define differential Weyl algebra and set variable names
    W = DifferentialWeylAlgebra(PolynomialRing(field,tvars))
    WR = W.base_ring()
    T = PolynomialRing(field,tvars).gens()
    D = list(W.differentials())

    # Compute Hessian matrix and differential operator Epsilon
    HES = getHes(H,r,vars,CP)
    HESinv = HES.inverse()
    v = matrix(W,[D[k] for k in range(dd-1)])
    Epsilon = -(v * HESinv.change_ring(W) * v.transpose())[0,0]

    # Define quantities for calculating asymptotics
    tsubs = [v == v.subs(CP)*exp(I*t) for [v,t] in zip(vars,tvars)]
    tsubs += [vars[-1]==g.subs(tsubs)]
    P = (-G/g/diff(H,vars[-1])).subs(tsubs)
    psi = log(g.subs(tsubs)/g.subs(CP)) + I * add([r[k]*tvars[k] for k in range(dd-1)])/r[-1]
    v = matrix(SR,[tvars[k] for k in range(dd-1)])
    psiTilde = psi - (v * HES * v.transpose())[0,0]/2

    # Recursive function to convert symbolic expression to polynomial in t variables
    def to_poly(p,k):
        if k == 0:
            return add([a*T[k]^int(b) for [a,b] in p.coefficients(tvars[k])])
        return add([to_poly(a,k-1)*T[k]^int(b) for [a,b] in p.coefficients(tvars[k])])

    # Compute Taylor expansions to sufficient orders
    N = 2*M
    PsiSeries = to_poly(taylor(psiTilde,*((v,0) for v in tvars), N),dd-2)
    PSeries = to_poly(taylor(P,*((v,0) for v in tvars), N),dd-2)

    # Precompute products used for asymptotics
    EE = [Epsilon^k for k in range(3*M-2)]
    PP = [PSeries] + [0 for k in range(2*M-2)]
    for k in range(1,2*M-1):
        PP[k] = PP[k-1]*PsiSeries

    # Function to compute constants appearing in asymptotic expansion
    def Clj(l,j):
        return (-1)^j*SR(eval_op(EE[l+j],PP[l]))/(2^(l+j)*factorial(l)*factorial(l+j))

    # Compute different parts of asymptotic expansion
    var('n')
    ex = (prod([1/v^k for (v,k) in zip(vars,r)]).subs(CP).canonicalize_radical())^n
    pw = (r[-1]*n)^((1-dd)/2)
    se = sqrt((2*pi)^(1-dd)/HES.det()) * add([add([Clj(l,j) for l in range(2*j+1)])/(r[-1]*n)^j for j in range(M)])

    return ex, pw, se.canonicalize_radical()
```

Sage code available on melczer.ca/textbook/

Note: Requires g explicitly

Vanishing Terms

If $G(\mathbf{w}) \neq 0$ then higher order terms give more accuracy

$$[x^n y^n]\frac{1}{1-x-y} = 4^n \left(\frac{1}{\sqrt{\pi} n^{1/2}} - \frac{1}{8\sqrt{\pi} n^{3/2}} + \frac{1}{128\sqrt{\pi} n^{5/2}} + O\left(n^{-7/2}\right) \right)$$

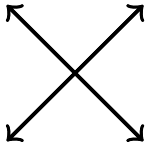
If $G(\mathbf{w}) = 0$ then higher order terms may give dominant asymptotics

$$[x^n y^n]\frac{x-2y^2}{1-x-y} = 4^n \left(\frac{1}{4\sqrt{\pi} n^{3/2}} + \frac{3}{32\sqrt{\pi} n^{5/2}} + O\left(n^{-7/2}\right) \right)$$

In the (rare) **worse case**, all terms may be zero!

$$[x^n y^n]\frac{x-y}{1-x-y} = O\left(\frac{4^n}{n^M}\right) \quad \text{for all } M > 0$$

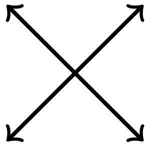
Lattice Path Enumeration

The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\} =$  is

$$\left[(xyt)^n\right] \frac{(1+x)(1+y)}{1 - txyS(x,y)} \;\sim\; \frac{2}{\pi} \cdot \frac{4^n}{n}$$

where $S(x,y) = xy + \frac{x}{y} + \frac{y}{x} + \frac{1}{xy}$.

Lattice Path Enumeration

The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$ =  is

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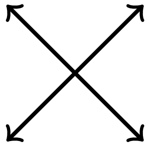
where $S(x,y) = xy + \frac{x}{y} + \frac{y}{x} + \frac{1}{xy}$.

The critical points are

$$\left(1, 1, \tfrac{1}{4}\right) \quad \left(-1, 1, \tfrac{1}{4}\right) \quad \left(1, -1, \tfrac{1}{4}\right) \quad \left(-1, -1, \tfrac{1}{4}\right)$$

Lattice Path Enumeration

The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$ =  is

$$\left[(xyt)^n\right] \frac{(1+x)(1+y)}{1 - txyS(x,y)} \sim \frac{2}{\pi} \cdot \frac{4^n}{n}$$

where $S(x,y) = xy + \frac{x}{y} + \frac{y}{x} + \frac{1}{xy}$.

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Numerator Vanishes

Lattice Path Enumeration

The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$ and **ending on the x-axis** is

$$\left[(xyt)^n\right] \frac{(1-x^2)(1+y)}{1-txyS(x,y)} \sim \frac{2\,(1+(-1)^n)}{\pi} \cdot \frac{4^n}{n^2}$$

The critical points are still

$$\left(1, 1, \tfrac{1}{4}\right) \quad \left(-1, 1, \tfrac{1}{4}\right) \quad \left(1, -1, \tfrac{1}{4}\right) \quad \left(-1, -1, \tfrac{1}{4}\right)$$

Lattice Path Enumeration

The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$ and **ending on the $\boldsymbol{x}$-axis** is

$$\left[(xyt)^n\right] \frac{(1-x^2)(1+y)}{1-txyS(x,y)} \sim \frac{2\left(1+(-1)^n\right)}{\pi} \cdot \frac{4^n}{n^2}$$

The critical points are still

$$\left(1, 1, \tfrac{1}{4}\right) \quad \left(-1, 1, \tfrac{1}{4}\right) \qquad \left(1, -1, \tfrac{1}{4}\right) \quad \left(-1, -1, \tfrac{1}{4}\right)$$

Numerator Vanishes to First Order

Numerator Vanishes to Second Order

Lattice Path Enumeration

The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$ and **ending at the origin** is

$$\left[(xyt)^n\right] \frac{(1-x^2)(1-y^2)}{1-txyS(x,y)} \sim \frac{4\left(1+(-1)^n\right)}{\pi} \cdot \frac{4^n}{n^3}$$

The critical points are still

$$\left(1, 1, \tfrac{1}{4}\right) \quad \left(-1, 1, \tfrac{1}{4}\right) \quad \left(1, -1, \tfrac{1}{4}\right) \quad \left(-1, -1, \tfrac{1}{4}\right)$$

Lattice Path Enumeration

The number of walks in $\mathbb{N}^2$ starting at the origin and taking n steps in $\{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$ and **ending at the origin** is

$$\left[(xyt)^n\right] \frac{(1-x^2)(1-y^2)}{1-txyS(x,y)} \sim \frac{4\,(1+(-1)^n)}{\pi} \cdot \frac{4^n}{n^3}$$

The critical points are still

$$\left(1,1,\tfrac{1}{4}\right) \quad \left(-1,1,\tfrac{1}{4}\right) \quad \left(1,-1,\tfrac{1}{4}\right) \quad \left(-1,-1,\tfrac{1}{4}\right)$$

Numerator Vanishes to Second Order

Topic 4
Perturbing Direction and Central Limit Theorems

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Main Theorem of Smooth ACSV

Suppose that

$$H = 0, \qquad r_j z_1 H_{z_1} = r_1 z_j H_{z_j} \quad (2 \leq j \leq d)$$

admits a finite number of solutions. If

- there is exactly one minimal solution, $\mathbf{w} \in \mathbb{C}_*^d$
- $H_{z_d}(\mathbf{w})$ and $\det \mathcal{M}$ are non-zero,

then

$$[\mathbf{z}^{n\mathbf{r}}] \frac{G(\mathbf{z})}{H(\mathbf{z})} = \mathbf{w}^{-n\mathbf{r}} \, (n r_d)^{(1-d)/2} \, (2\pi)^{(1-d)/2} \det(\mathcal{M})^{-1/2} \left(\frac{-G(\mathbf{w})}{w_d \, H_{z_d}(\mathbf{w})} + O\left(\frac{1}{n}\right)\right)$$

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$$[\mathbf{z}^{n\mathbf{r}}]\frac{G(\mathbf{z})}{H(\mathbf{z})} = \mathbf{w}^{-n\mathbf{r}}\ (nr_d)^{(1-d)/2}\ (2\pi)^{(1-d)/2} \det(\mathcal{M})^{-1/2} \left(\frac{-G(\mathbf{w})}{w_d\, H_{z_d}(\mathbf{w})} + O\left(\frac{1}{n}\right)\right)$$

If $\mathbf{r}$ varies in some compact set where the above conditions are still satisfied then the **error term** in this asymptotic estimate can be uniformly bounded.

Irrational Directions

Recall from Lecture 1 that

$$[x^{rn}y^{sn}]\frac{1}{1-x-y} \sim \left(\frac{r+s}{r}\right)^{rn}\left(\frac{r+s}{s}\right)^{sn}\frac{\sqrt{r+s}}{\sqrt{2rs\pi\, n}}$$

Irrational Directions

Recall from Lecture 1 that

$$[x^{rn}y^{sn}]\frac{1}{1-x-y} \sim \left(\frac{r+s}{r}\right)^{rn}\left(\frac{r+s}{s}\right)^{sn}\frac{\sqrt{r+s}}{\sqrt{2rs\pi\, n}}$$

Only makes sense if $rn, sn \in \mathbb{N}$

Valid for any $r, s > 0$

Irrational Directions

Recall from Lecture 1 that

$$[x^{rn}y^{sn}]\frac{1}{1-x-y} \sim \left(\frac{r+s}{r}\right)^{rn}\left(\frac{r+s}{s}\right)^{sn}\frac{\sqrt{r+s}}{\sqrt{2rs\pi\, n}}$$

What if we put $(r,s)=(\pi,1)$ into the approximation?
What does this correspond to?

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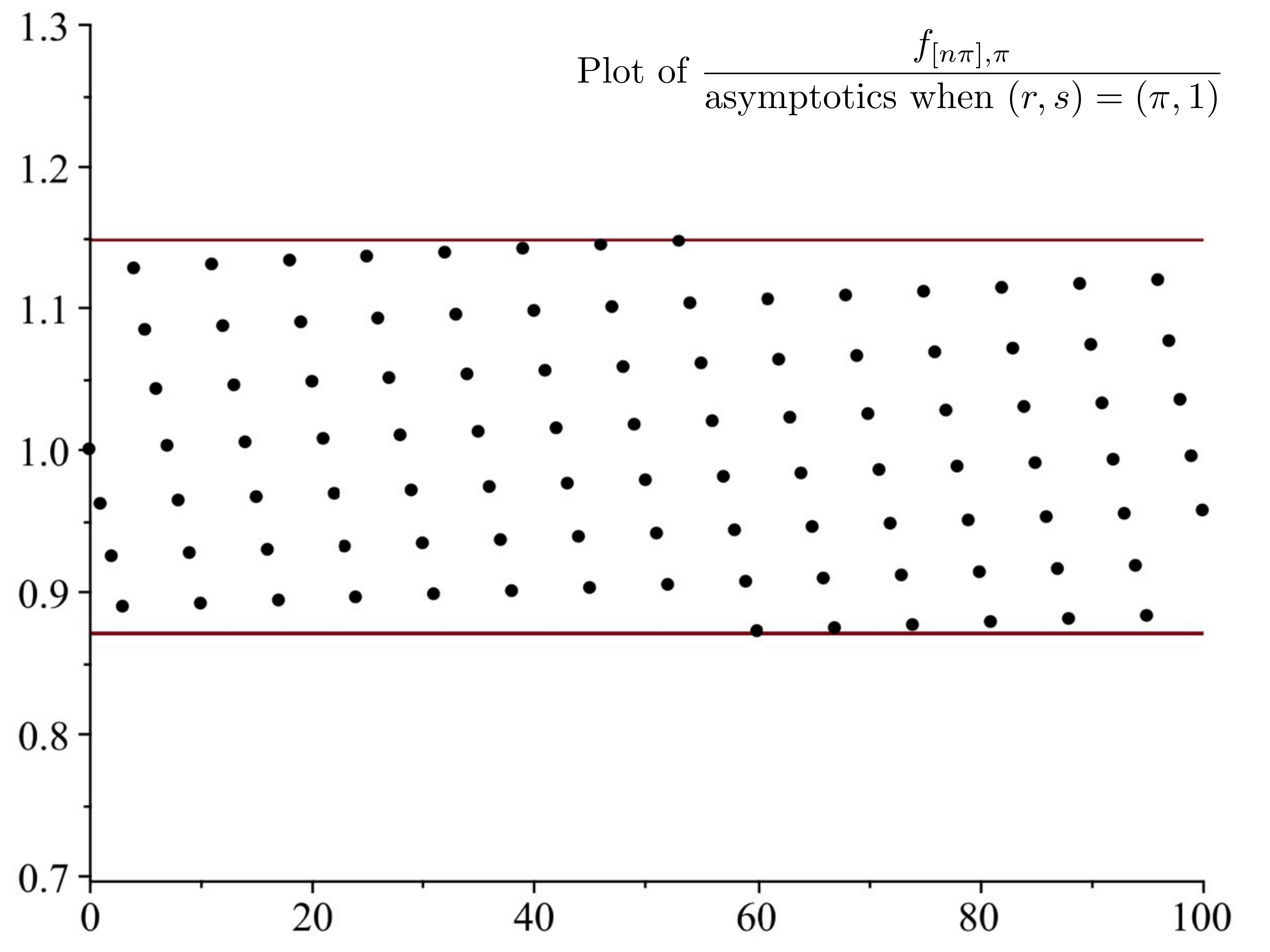
What if we put $(r,s)=(\pi,1)$ into the approximation?
What does this correspond to?

Compare to $f_{[n\pi],n}$ where $[z]=$ closest integer to $\boldsymbol{z}$


Plot of $\frac{f_{[n\pi],\pi}}{\text{asymptotics when } (r,s)=(\pi,1)}$
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0
20
40
60
80
100

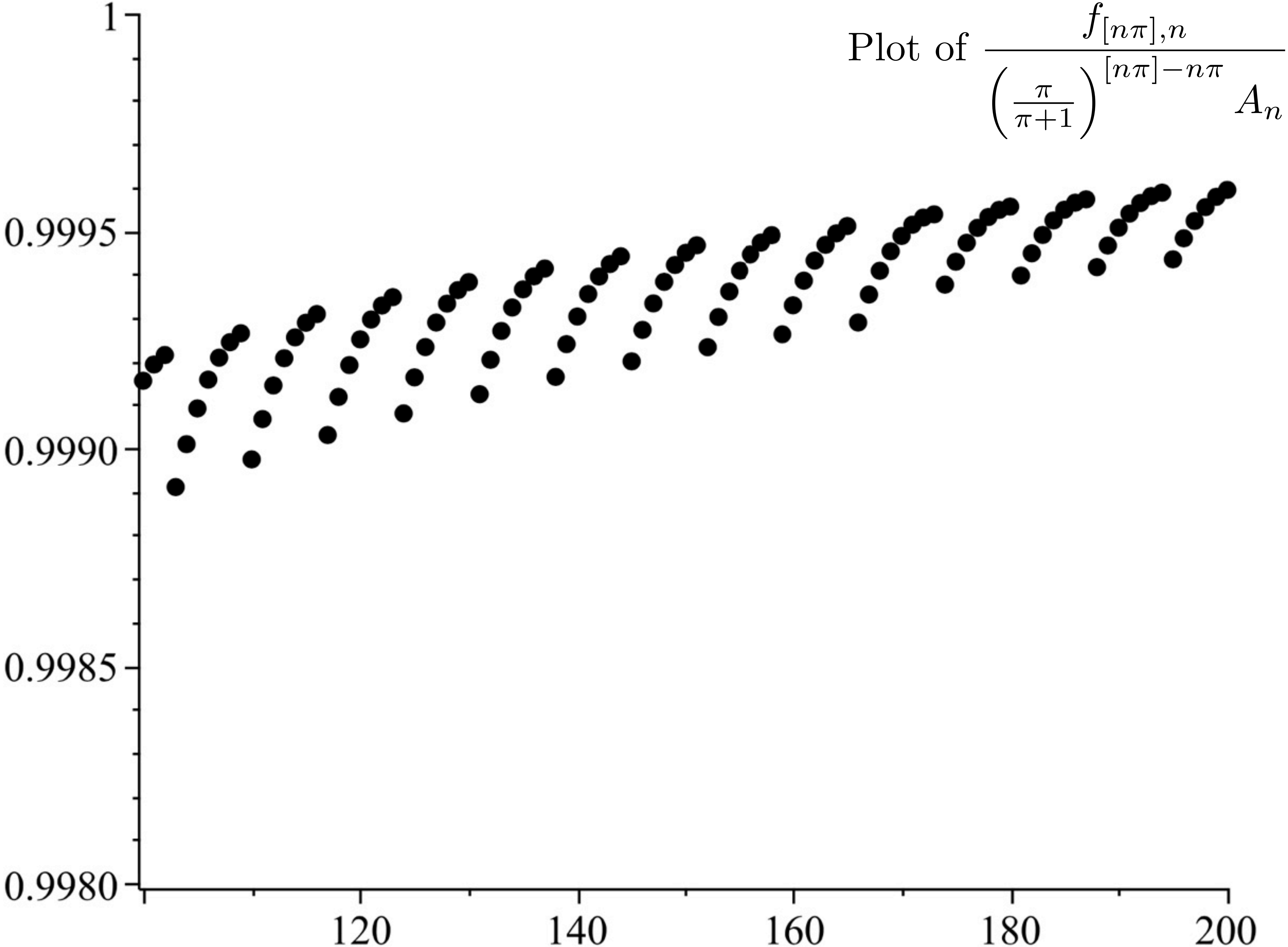
Irrational Directions

Recall from Lecture 1 that

$$[x^{rn}y^{sn}]\frac{1}{1-x-y} \sim \left(\frac{r+s}{r}\right)^{rn}\left(\frac{r+s}{s}\right)^{sn}\frac{\sqrt{r+s}}{\sqrt{2rs\pi\, n}}$$

If A_n is this approximation with $(r,s)=(\pi,1)$ then

$$f_{[n\pi],n} \sim \underbrace{\left(\frac{\pi}{\pi+1}\right)^{[n\pi]-n\pi}}_{\text{bounded factor}} A_n$$



Plot of $\frac{f_{[n\pi],n}}{\left(\frac{\pi}{\pi+1}\right)^{[n\pi]-n\pi} A_n}$
1
0.9995
0.9990
0.9985
0.9980
120
140
160
180
200

Smooth Variation of Coefficients

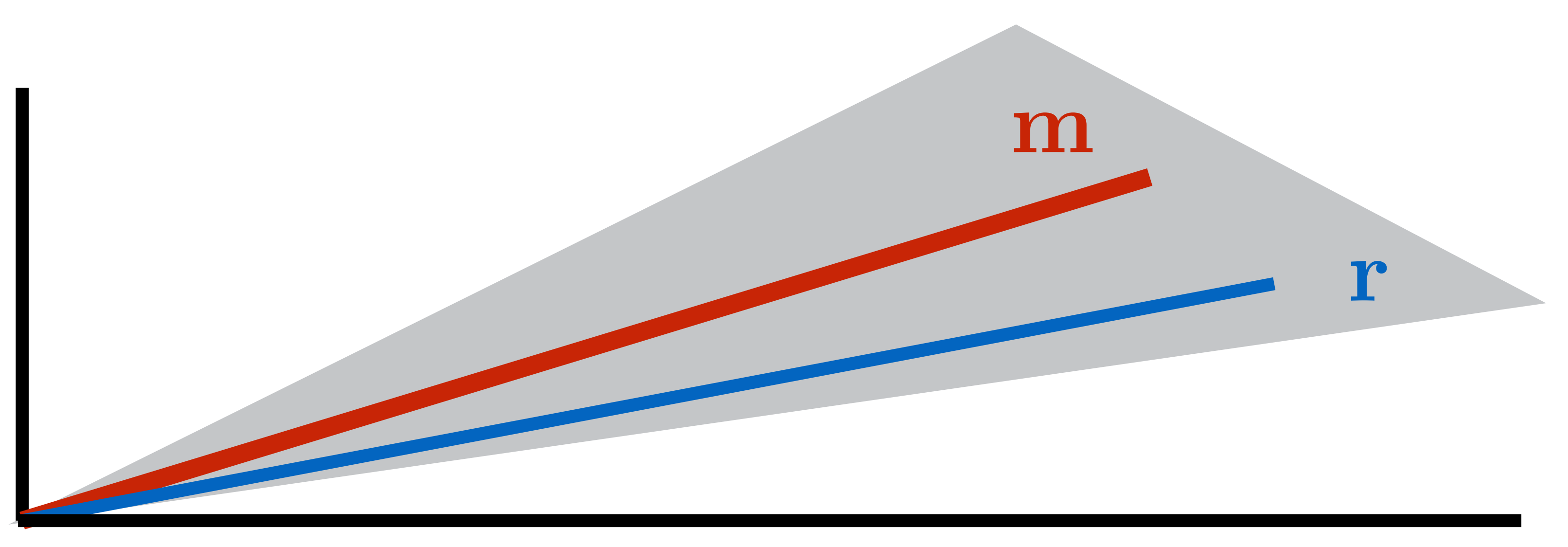
Fix direction $\mathbf{m} = (\widehat{\mathbf{m}}, 1)$ and suppose that for all $\mathbf{r} = (\hat{\mathbf{r}}, 1)$ in a neighbourhood of $\mathbf{m}$ there is a smoothly varying minimal critical point $\mathbf{w}(\mathbf{r})$ such that

1. no other singularity has the same coordinate-wise modulus as $\mathbf{w}(\mathbf{r})$
2. $H_{z_d}(\mathbf{w}(\mathbf{r}))$ and $G(\mathbf{w}(\mathbf{r}))$ are non-zero
3. the matrix $\mathcal{M}_{\mathbf{w}(\mathbf{r})}$ is non-singular

If $\hat{\mathbf{s}} = \hat{\mathbf{s}}(n)$ is a sequence in $\mathbb{N}^{d-1}$ with each coordinate of $|\hat{\mathbf{s}} - n\widehat{\mathbf{m}}|$ in $o(n^{2/3})$ then

$$f_{\hat{\mathbf{s}},n} \sim \mathbf{w}^{-n\mathbf{m}} n^{(1-d)/2} \left(\frac{-G(\mathbf{w})(2\pi)^{(1-d)/2}}{w_d H_{z_d}(\mathbf{w})\sqrt{\det \mathcal{M}}} \right) \widehat{\mathbf{w}}^{-(\hat{\mathbf{s}}-n\widehat{\mathbf{m}})} \exp\left[-\frac{(\hat{\mathbf{s}} - n\widehat{\mathbf{m}})^T \mathcal{M}^{-1} (\hat{\mathbf{s}} - n\widehat{\mathbf{m}})}{2n} \right]$$

where $\mathbf{w} = \mathbf{w}(\mathbf{m})$ and $\mathcal{M} = \mathcal{M}_{\mathbf{w}}$.



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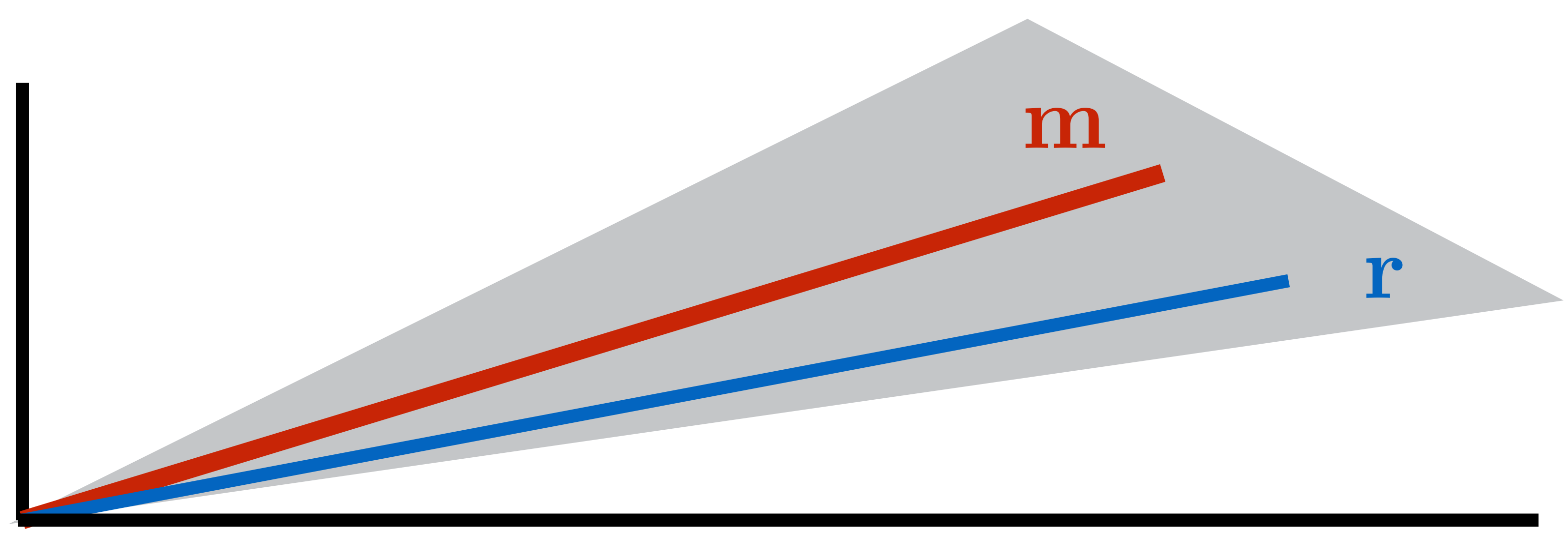

Local Central Limit Theorem

Suppose that in some direction $(\mathbf{m}, 1)$ there is a minimal critical point $\mathbf{w} = (\mathbf{1}, t)$ with $t > 0$ such that

1. no other singularity has the same coordinate-wise modulus as $\mathbf{w}$
2. $H_{z_d}(\mathbf{w})$ and $G(\mathbf{w})$ are non-zero
3. the explicit matrix $\mathcal{M}$ is non-singular

Then as $n \to \infty$ the coefficients of $[z_d^n]F(\mathbf{z})$ approach a multivariate normal distribution with density

$$\nu_n(\mathbf{s}) = -\frac{G(\mathbf{w})}{w_d H_{z_d}(\mathbf{w})} \frac{(2\pi n)^{(1-d)/2}}{\sqrt{\det \mathcal{M}}} \exp\left[-\frac{(\mathbf{s} - n\mathbf{m})^T \mathcal{M}^{-1} (\mathbf{s} - n\mathbf{m})}{2n}\right].$$

$$F(x, y, z) = \square + \cdots + \left(\square + \square x + \square y + \square xy + \cdots\cdots\right) z^n + \cdots$$

Local Central Limit Theorem

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Then

$$\sup_{\mathbf{s}\in\mathbb{Z}^{d-1}} n^{(d-1)/2}\left|t^n f_{\mathbf{s},n} - \nu_n(\mathbf{s})\right| \to 0$$

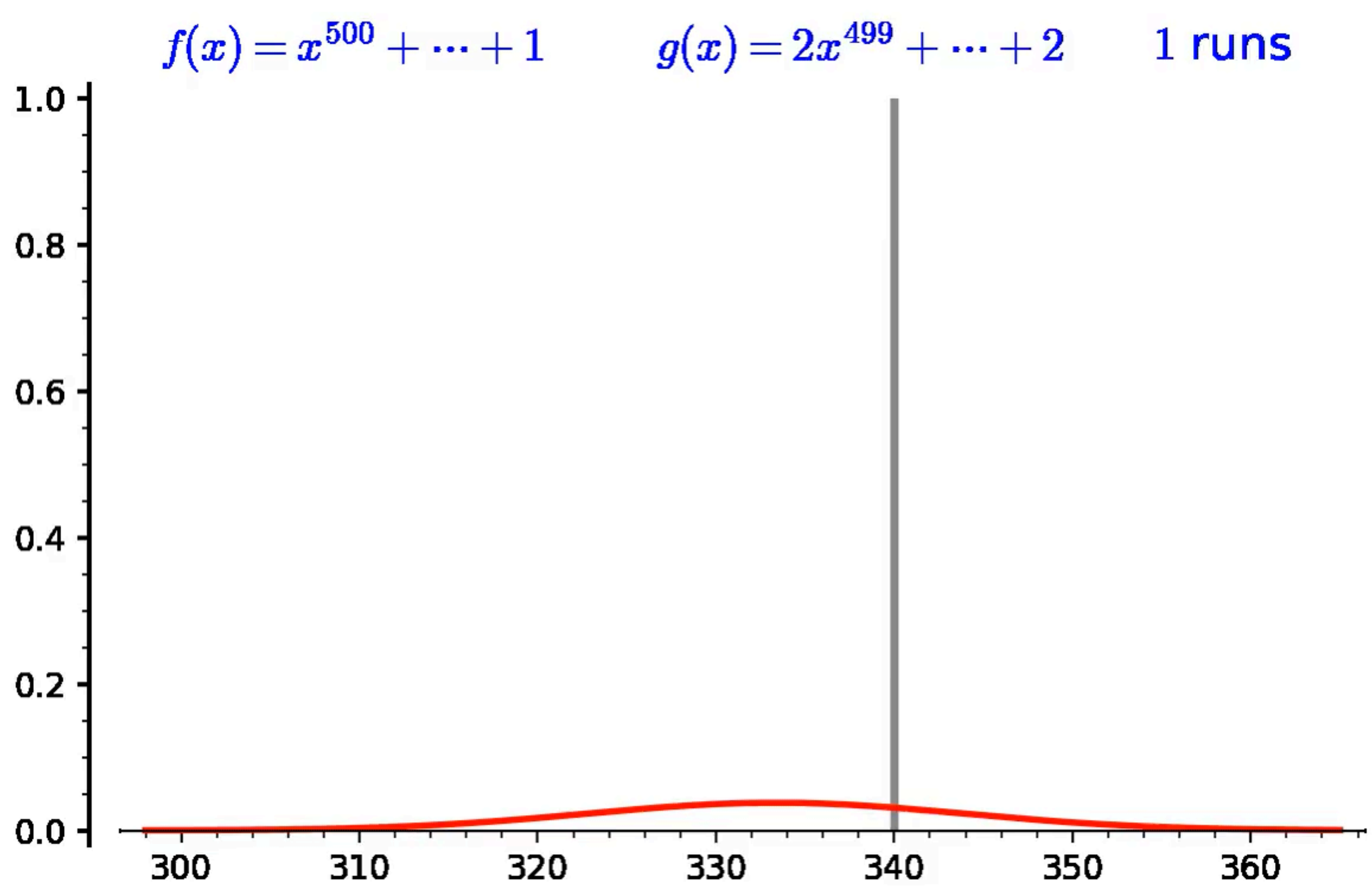
as $n \to \infty$, where

$$\nu_n(\mathbf{s}) = -\frac{G(\mathbf{w})}{w_d H_{z_d}(\mathbf{w})} \frac{(2\pi n)^{(1-d)/2}}{\sqrt{\det \mathcal{M}}} \exp\left[-\frac{(\mathbf{s}-n\mathbf{m})^T \mathcal{M}^{-1}(\mathbf{s}-n\mathbf{m})}{2n}\right].$$

CLT for the Euclidean Algorithm

If $e_{k,n}$ denotes the number of pairs of polynomials $f_0, f_1 \in \mathbb{F}_p[x]$ such that $\deg(f_1) < \deg(f_0) = n$ and the Euclidean algorithm performs k divisions then

$$F(u,z) = \sum_{n,k \geq 0} e_{k,n} u^k z^n = \frac{1}{1 - pz - p(p-1)uz}$$



A running count of the number of divisions performed when running the Euclidean algorithm on pairs of polynomials in $\mathbb{Z}_3[x]$ with the higher degree polynomial monic of degree 500, compared to the expected distribution from the limit curve.

Cycles Lengths in Permutations w/ Restricted Positions

P. Diaconis: "My latest paper has an explicit multivariate rational generating function. I'm pretty sure a CLT holds..."

Permanental generating functions and sequential importance sampling

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A R T I C L E   I N F O

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Dedicated to Joseph Kung

MSC:
62D99

A B S T R A C T

We introduce techniques for deriving closed form generating functions for enumerating permutations with restricted positions keeping track of various statistics. The method involves evaluating permanents with variables as entries. These are applied to determine the sample size required for a novel sequential importance sampling algorithm for generating random perfect matchings in classes of bipartite graphs.

Cycles Lengths in Permutations w/ Restricted Positions

Fix $t \in \mathbb{N}$.
Let $\mathcal{F}_t(n)$ be the set of permutations $\sigma \in S_n$ with $i - t \leq \sigma(i) \leq i + 1$

Theorem 1. If $a_i(\sigma)$ denotes the number of i cycles in σ,

$$F(\mathbf{x}, z) = \sum_{\substack{n \geq 0 \\ \sigma \in \mathcal{F}_t(n)}} \mathbf{x}^{\mathbf{a}(\sigma)} z^n = \frac{1}{1 - x_1 z - x_2 z^2 - \cdots - x_{t+1} z^{t+1}}$$

Cycles Lengths in Permutations w/ Restricted Positions

Conditions for a CLT to hold

1. no other singularity has the same coordinate-wise modulus as $\mathbf{w}$
2. $H_{z_d}(\mathbf{w})$ and $G(\mathbf{w})$ are non-zero
3. the explicit matrix $\mathcal{M}$ is non-singular

$$F(\mathbf{x}, z) = \frac{1}{1 - x_1 z - x_2 z^2 - \cdots - x_{t+1} z^{t+1}}$$

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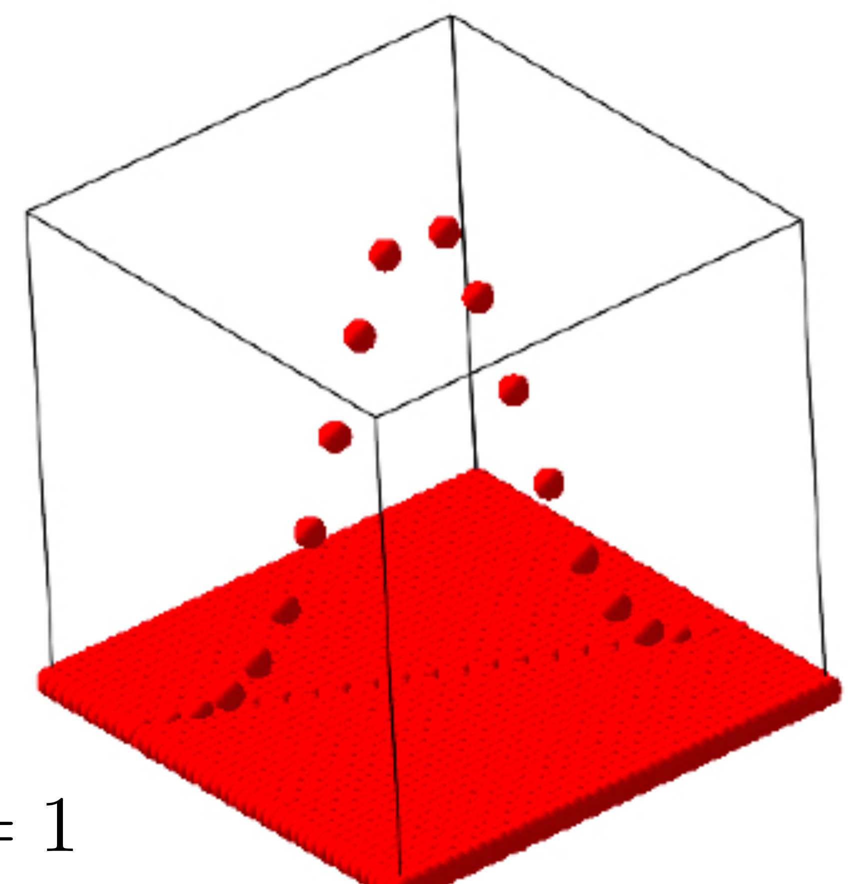


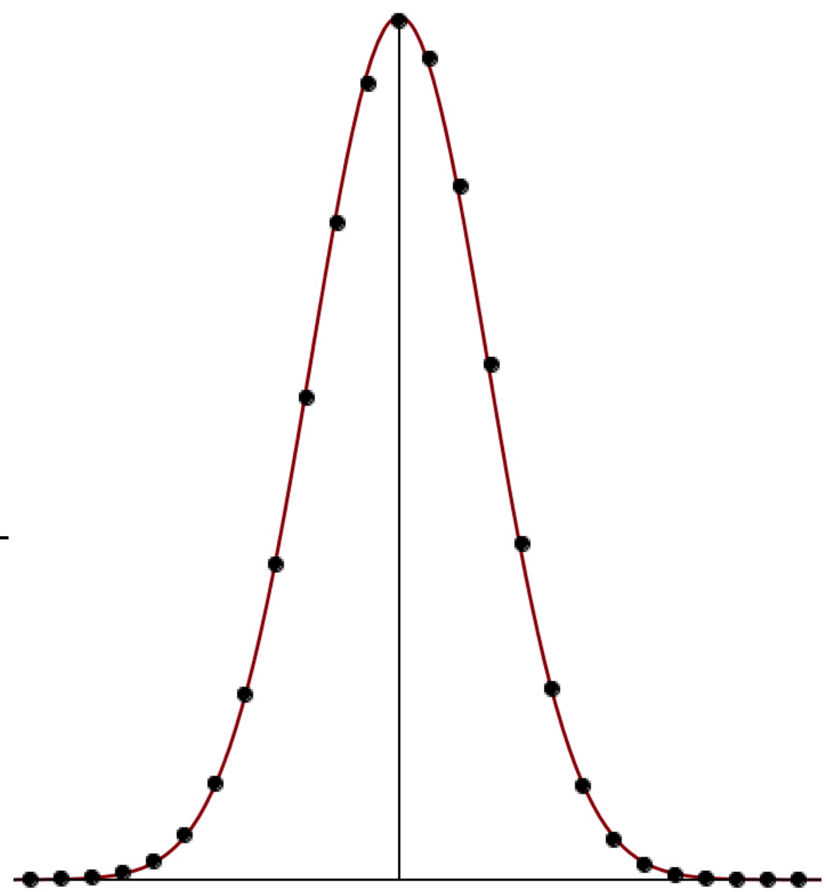
$t = 1$

Cycles Lengths in Permutations w/ Restricted Positions

Conditions for a CLT to hold

1. no other singularity has the same coordinate-wise modulus as $\mathbf{w}$
2. $H_{z_d}(\mathbf{w})$ and $G(\mathbf{w})$ are non-zero
3. the explicit matrix $\mathcal{M}$ is non-singular

$$F(1, x_2, \ldots, x_{t+1}, z) = \frac{1}{1 - z - x_2 z^2 - \cdots - x_{t+1} z^{t+1}}$$



$t = 1$

Cycles Lengths in Permutations w/ Restricted Positions

Conditions for a CLT to hold

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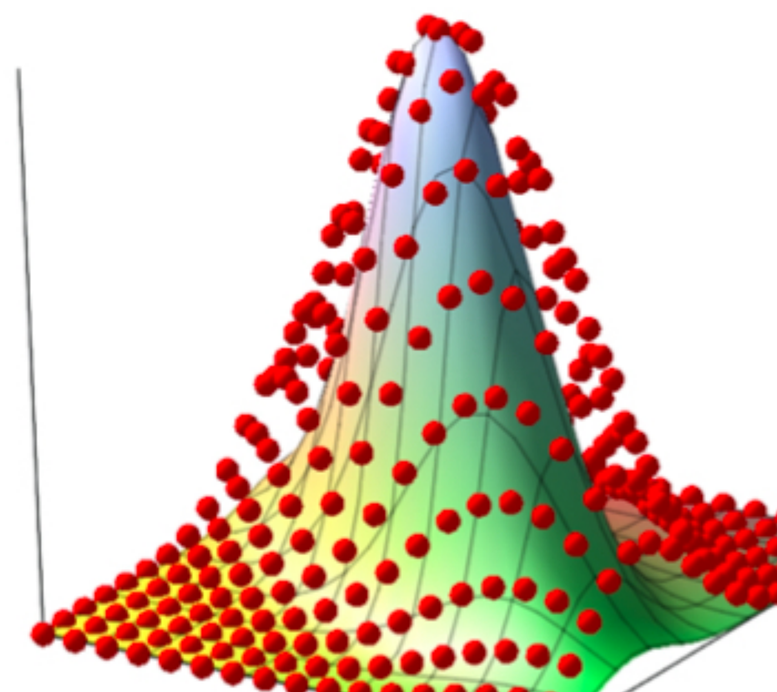
$$F(1, x_2, \ldots, x_{t+1}, z) = \frac{1}{1 - z - x_2 z^2 - \cdots - x_{t+1} z^{t+1}}$$

$t = 2$



Bivariate Multinomial ML-Degree

The **maximum likelihood degree (ML-degree)** is a measure of the complexity of the statistical *maximum likelihood method* for estimating parameters in a multivariate probability model with missing data.

Theorem (Hosten, Sullivant 2010)
If $\mathrm{ML}(n,k)$ denotes the ML-degree for multinomial random variables $X_1 \in \{1, \ldots, n\}$ and $X_2 \in \{1, \ldots, k\}$ then

$$\sum_{n,k \geq 0} \mathrm{ML}(n,k) \frac{x^n y^k}{n!k!} = \frac{e^{-x-y}}{e^{-x} + e^{-y} - 1}$$

Bivariate Multinomial ML-Degree

The **maximum likelihood degree (ML-degree)** is a measure of the complexity of the statistical *maximum likelihood method* for estimating parameters in a multivariate probability model with missing data.

Theorem (Khera, Lundberg, M. 2021)
For all fixed $K > 0$,

$$\sup_{|k-n/2|\leq K\sqrt{n}} \left| \frac{(2\log 2)^n}{n!} \mathrm{ML}(n-k,k) - \frac{2^{-\frac{2(k-n/2)^2}{n(1-\log 2)}}}{(4\log 2)\sqrt{1-\log 2}} \right| \to 0$$

Topic 5
Beyond Smoothness

melczer.ca/ALEA22

Higher Order Poles

What if there are solutions to

$$H(\mathbf{z}) = H_{z_1}(\mathbf{z}) = \cdots = H_{z_d}(\mathbf{z}) = 0$$

Higher Order Poles

What if there are solutions to

$$H(\mathbf{z}) = H_{z_1}(\mathbf{z}) = \cdots = H_{z_d}(\mathbf{z}) = 0$$

Easiest Case: $H(\mathbf{z}) = P(\mathbf{z})^k$ for some $k > 1$ where

$$P(\mathbf{z}) = P_{z_1}(\mathbf{z}) = \cdots = P_{z_d}(\mathbf{z}) = 0$$

has no solutions.

Then $\mathcal{V} = \{\mathbf{z} : H(\mathbf{z}) = 0\} = \{\mathbf{z} : P(\mathbf{z}) = 0\}$ is still a manifold
Minimal points unchanged, and critical points defined by P

The residue computation in the Main Theorem of Smooth ACSV has a minor modification to account for the **higher order pole**

Higher Order Poles

What if there are solutions to

$$H(\mathbf{z}) = H_{z_1}(\mathbf{z}) = \cdots = H_{z_d}(\mathbf{z}) = 0$$

Easiest Case: $H(\mathbf{z}) = P(\mathbf{z})^k$ for some $k > 1$ where

$$P(\mathbf{z}) = P_{z_1}(\mathbf{z}) = \cdots = P_{z_d}(\mathbf{z}) = 0$$

has no solutions.

If the usual assumptions hold with the smooth critical point equations for P then

$$f_{n\mathbf{r}} = \mathbf{w}^{-n\mathbf{r}}\, n^{k-1+(1-d)/2}\, (2\pi r_d)^{(1-d)/2}\, \det(\mathcal{M})^{-1/2} \cdot \frac{(-1)^k G(\mathbf{w})}{(k-1)!\, (w_d P_{z_d}(\mathbf{w}))^k} \left(1 + O\left(n^{-1}\right)\right)$$

Non-Smooth Points

More pathologically, $H(\mathbf{z}) = H_{z_1}(\mathbf{z}) = \cdots = H_{z_d}(\mathbf{z}) = 0$ if $\mathcal{V}$ self-intersects at **non-smooth points**.

This does not happen generically, but **does come up** in combinatorial examples.

I'd *estimate* 75% of naturally occurring combinatorial examples have $\mathcal{V}$ smooth.

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This does not happen generically, but **does come up** in combinatorial examples.

I'd *estimate* 75% of naturally occurring combinatorial examples have $\mathcal{V}$ smooth.

Simplest Non-Smooth Case:

$$F(x,y) = \frac{G(x,y)}{\ell_1(x,y)\ell_2(x,y)}$$

where ℓ_1 and ℓ_2 are linear.

Hyperplane Example

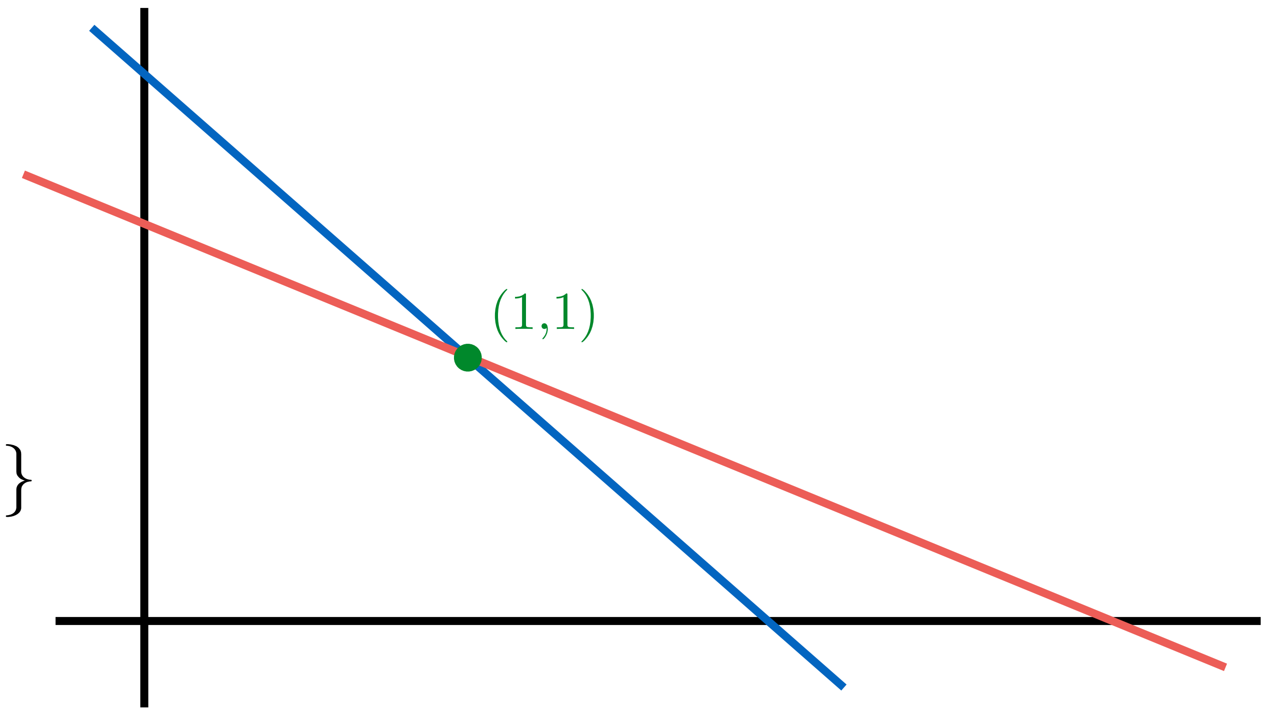
Let

$$F(x,y) = \frac{G(x,y)}{\ell_1(x,y)\ell_2(x,y)}$$

for $\quad \ell_1(x,y) = 1 - \frac{2x+y}{3} \qquad$ and $\qquad \ell_2(x,y) = 1 - \frac{3x+y}{4}$

Then $\mathcal{V} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_{1,2}$ for smooth sets

$$\begin{aligned} \mathcal{S}_1 &= \{\ell_1(x,y) = 0\} \setminus \mathcal{S}_{1,2} \\ \mathcal{S}_2 &= \{\ell_2(x,y) = 0\} \setminus \mathcal{S}_{1,2} \\ \mathcal{S}_{1,2} &= \{\ell_1(x,y) = \ell_2(x,y) = 0\} = \{(1,1)\} \end{aligned}$$



Hyperplane Example

We compute **critical points** on each *stratum*

On $\mathcal{S}_1$ $\quad \ell_1(x,y) = sx(\ell_1)_x(x,y) - ry(\ell_1)_y(x,y) = 0$

has the unique solution $\boldsymbol{\sigma}_1 = \frac{1}{r+s}\left(\frac{3r}{2}, 3s\right)$

$r \neq 2s$

On $\mathcal{S}_2$ $\quad \ell_2(x,y) = sx(\ell_2)_x(x,y) - ry(\ell_2)_y(x,y) = 0$

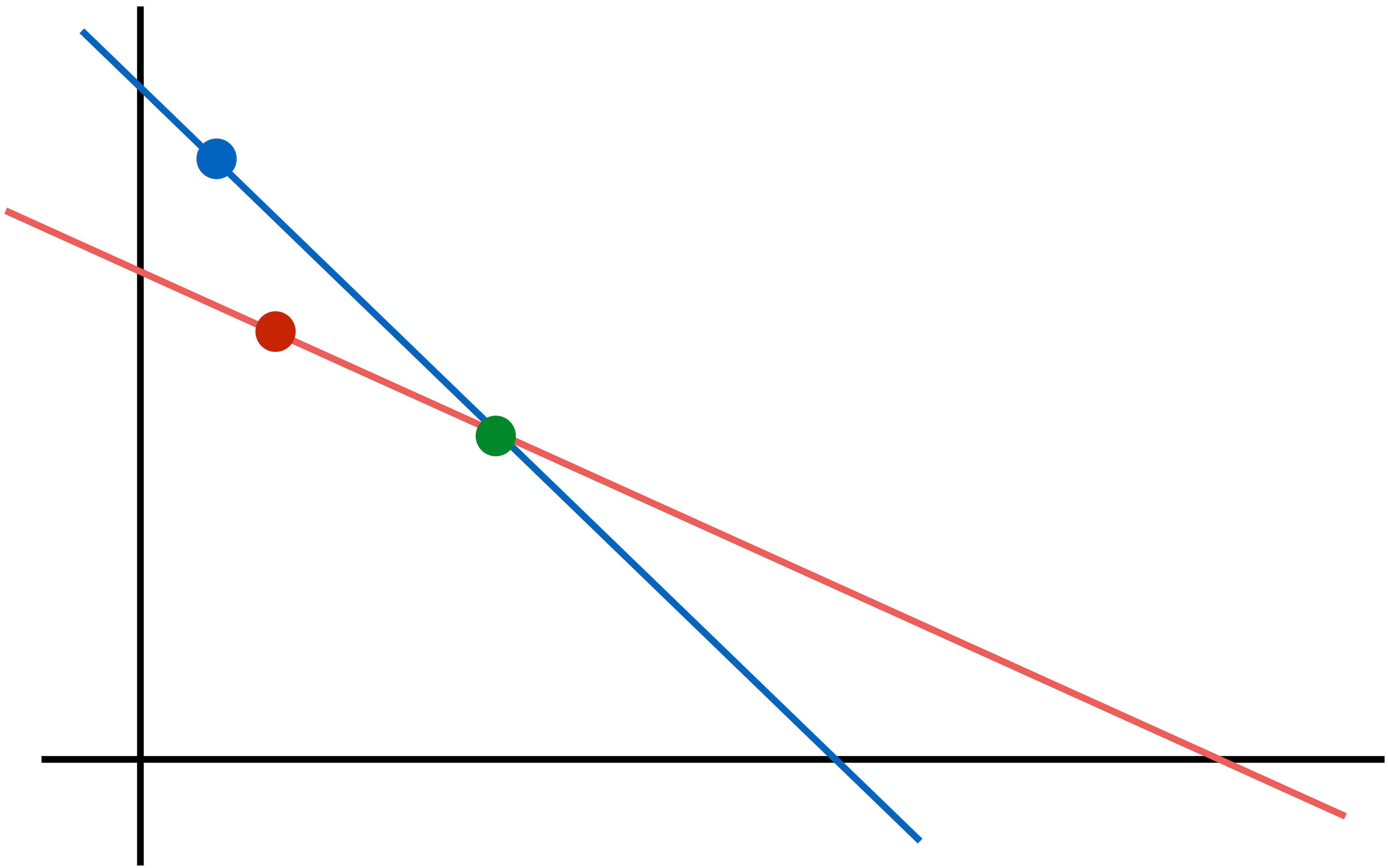
has the unique solution $\boldsymbol{\sigma}_2 = \frac{1}{r+s}\left(\frac{4r}{3}, 4s\right)$

$r \neq 3s$

On $\mathcal{S}_{1,2}$ the only point $\boldsymbol{\sigma}_{1,2} = (1,1)$ is trivially critical

Case 1: $0 < \dfrac{r}{r+s} < 2/3$

The point σ_1 is a smooth minimal critical point
We can compute asymptotics as before



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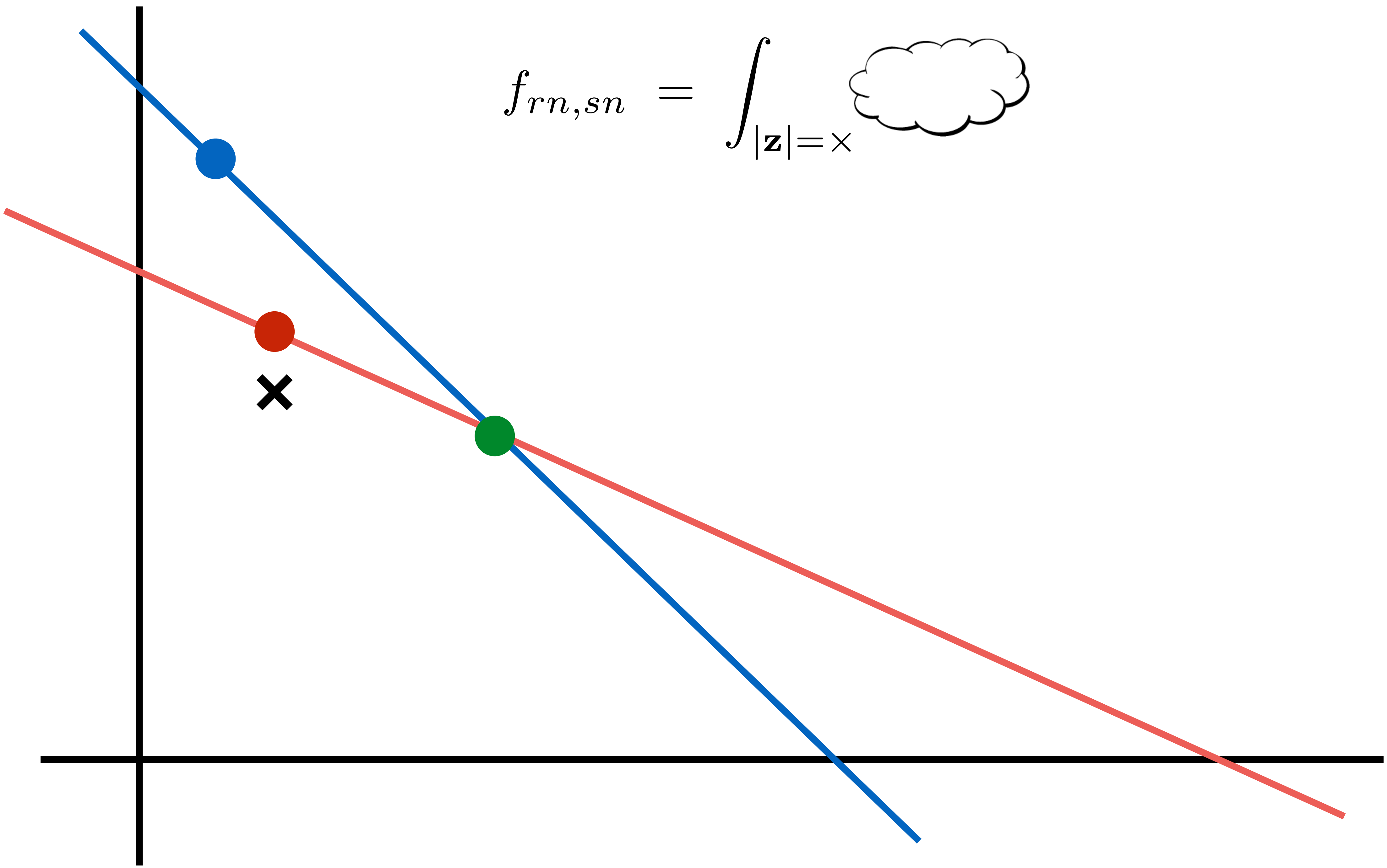

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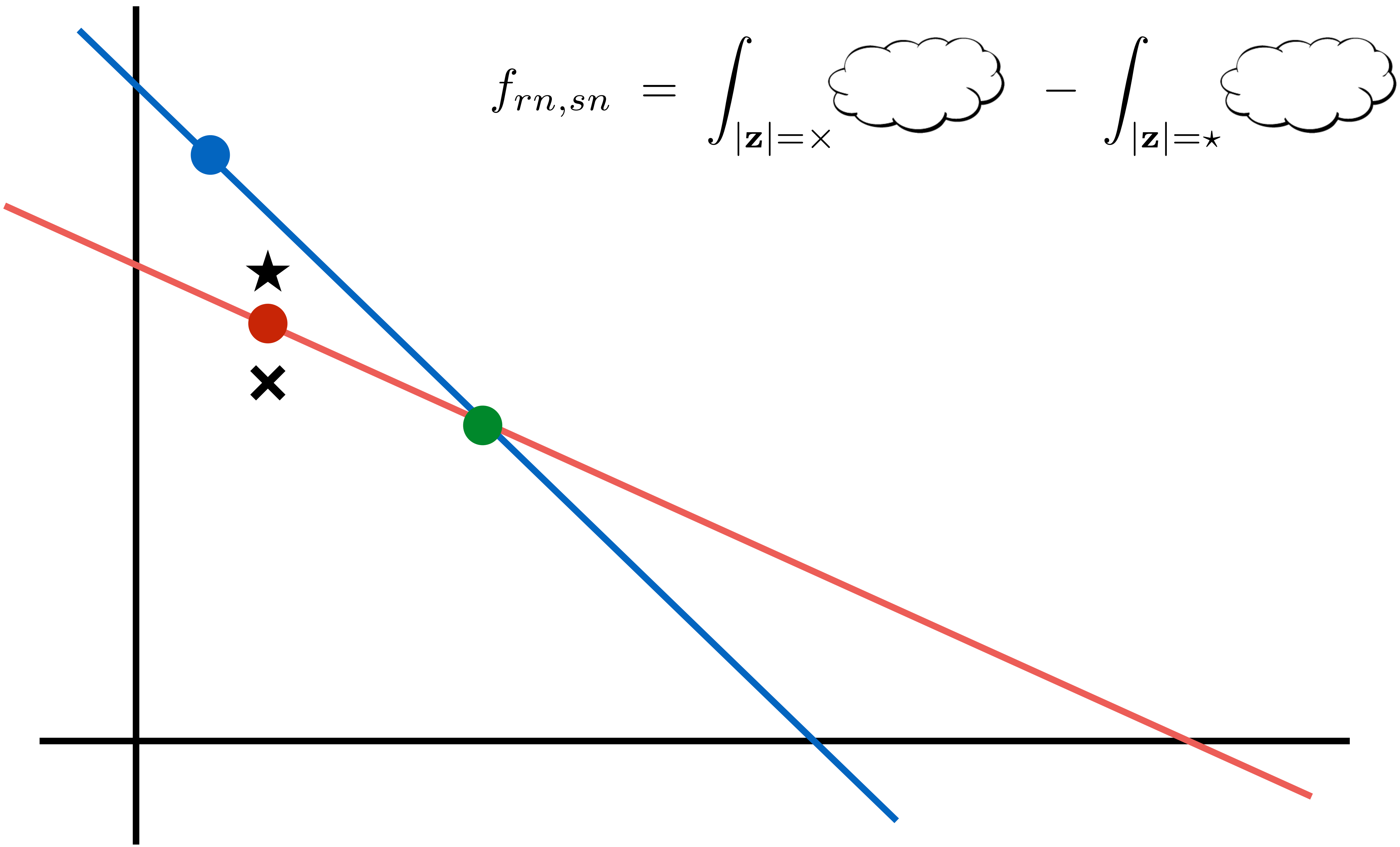

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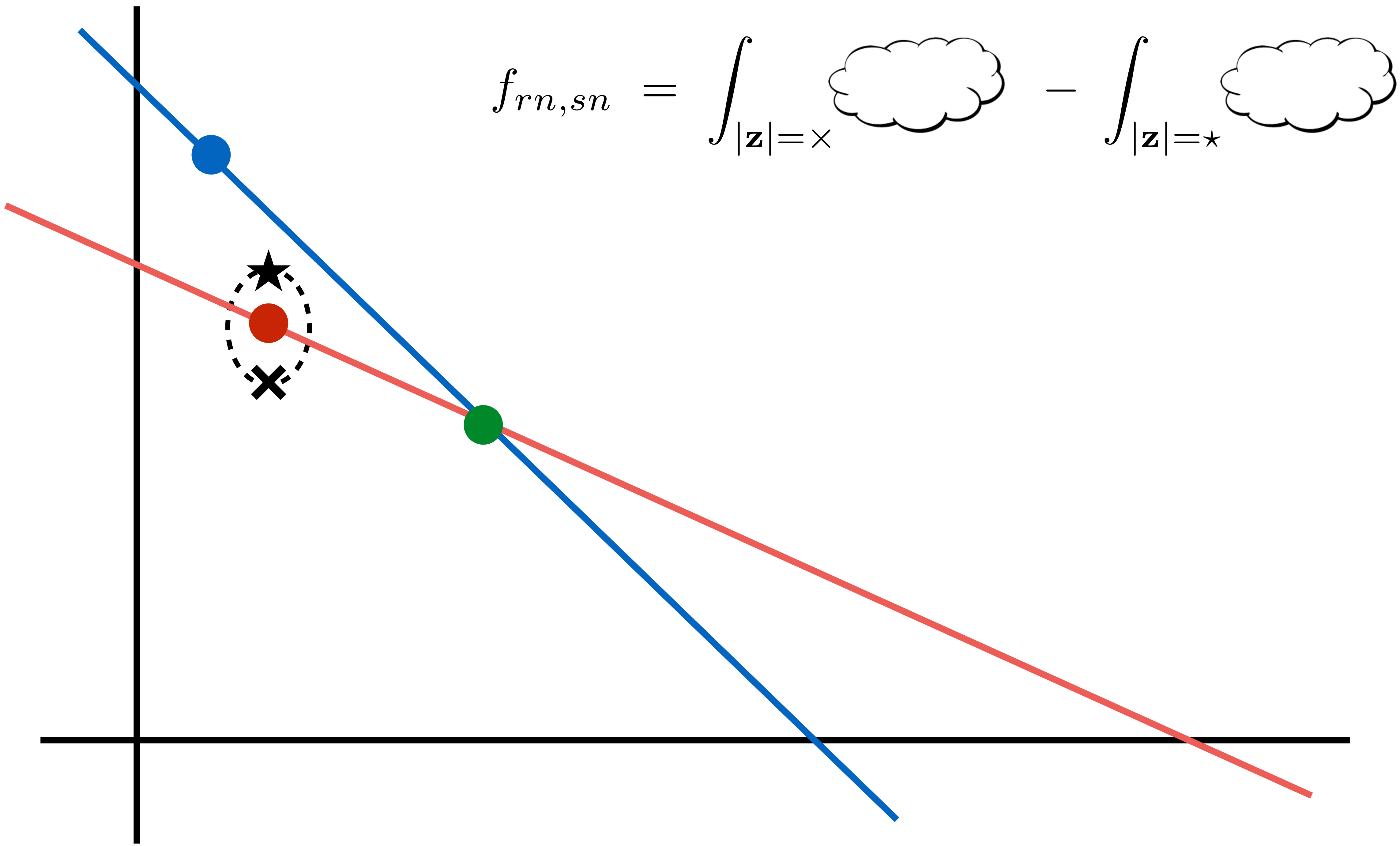

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The point σ_1 is a smooth minimal critical point

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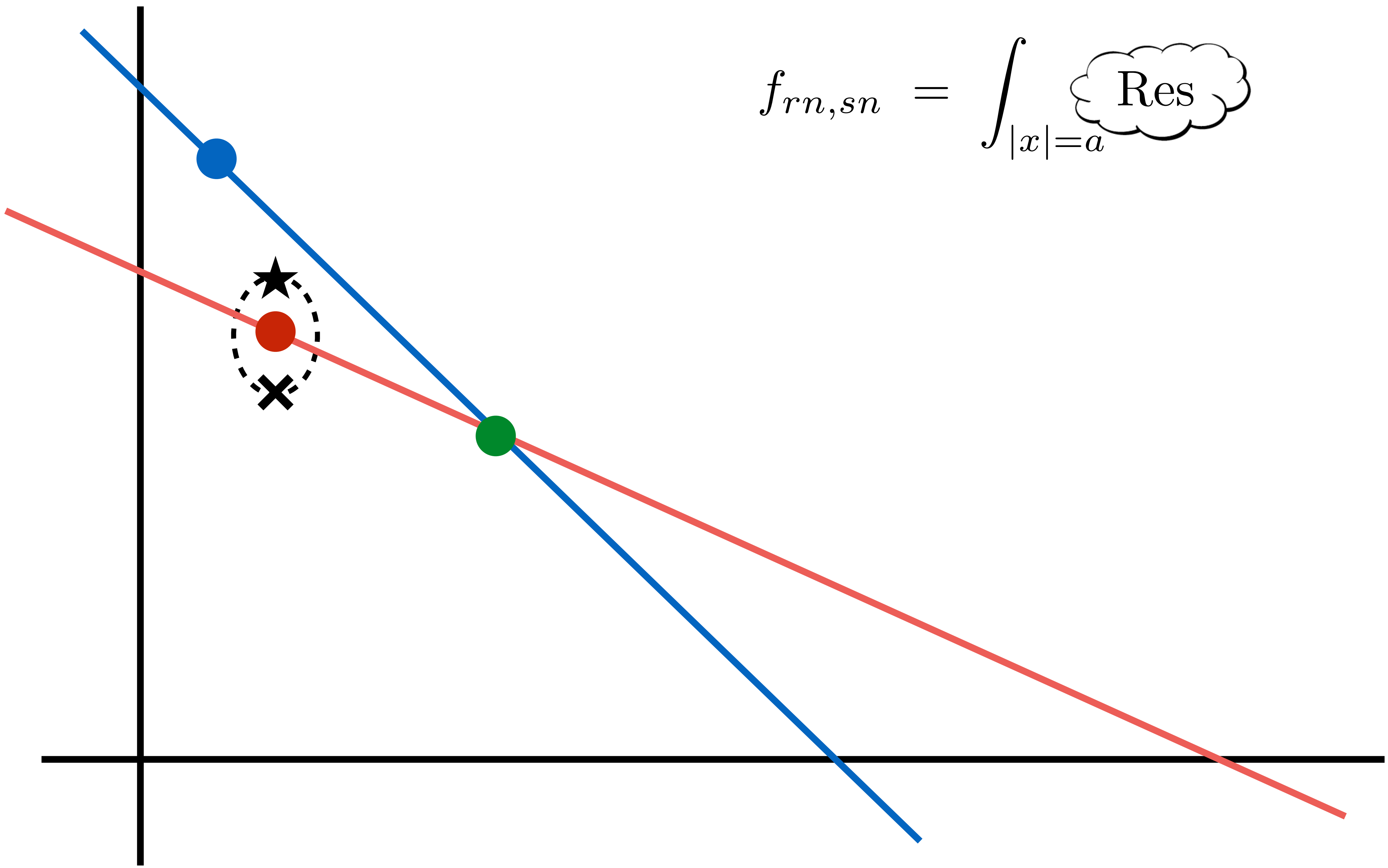
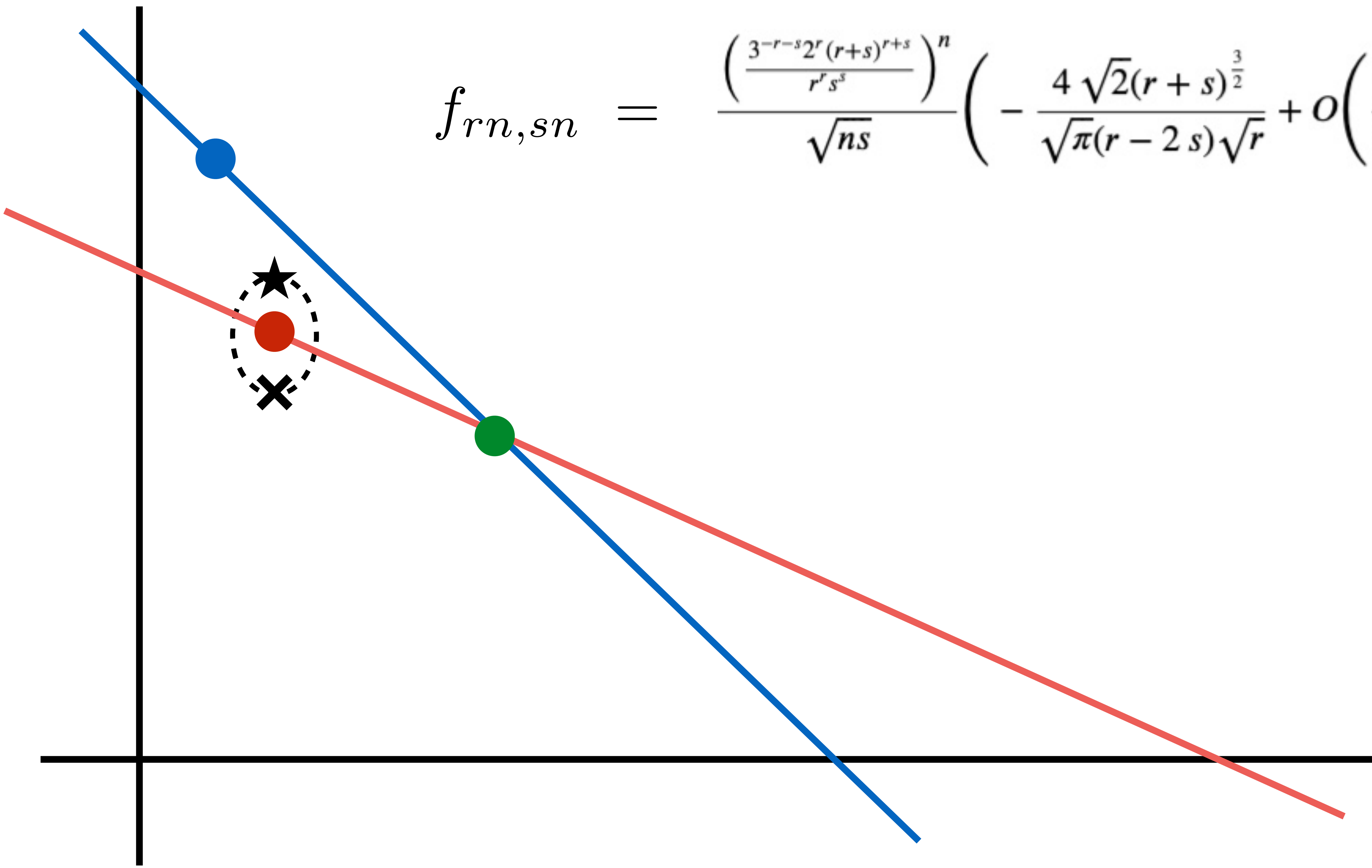


Case 1: $0 < \frac{r}{r+s} < 2/3$

The point σ_1 is a smooth minimal critical point

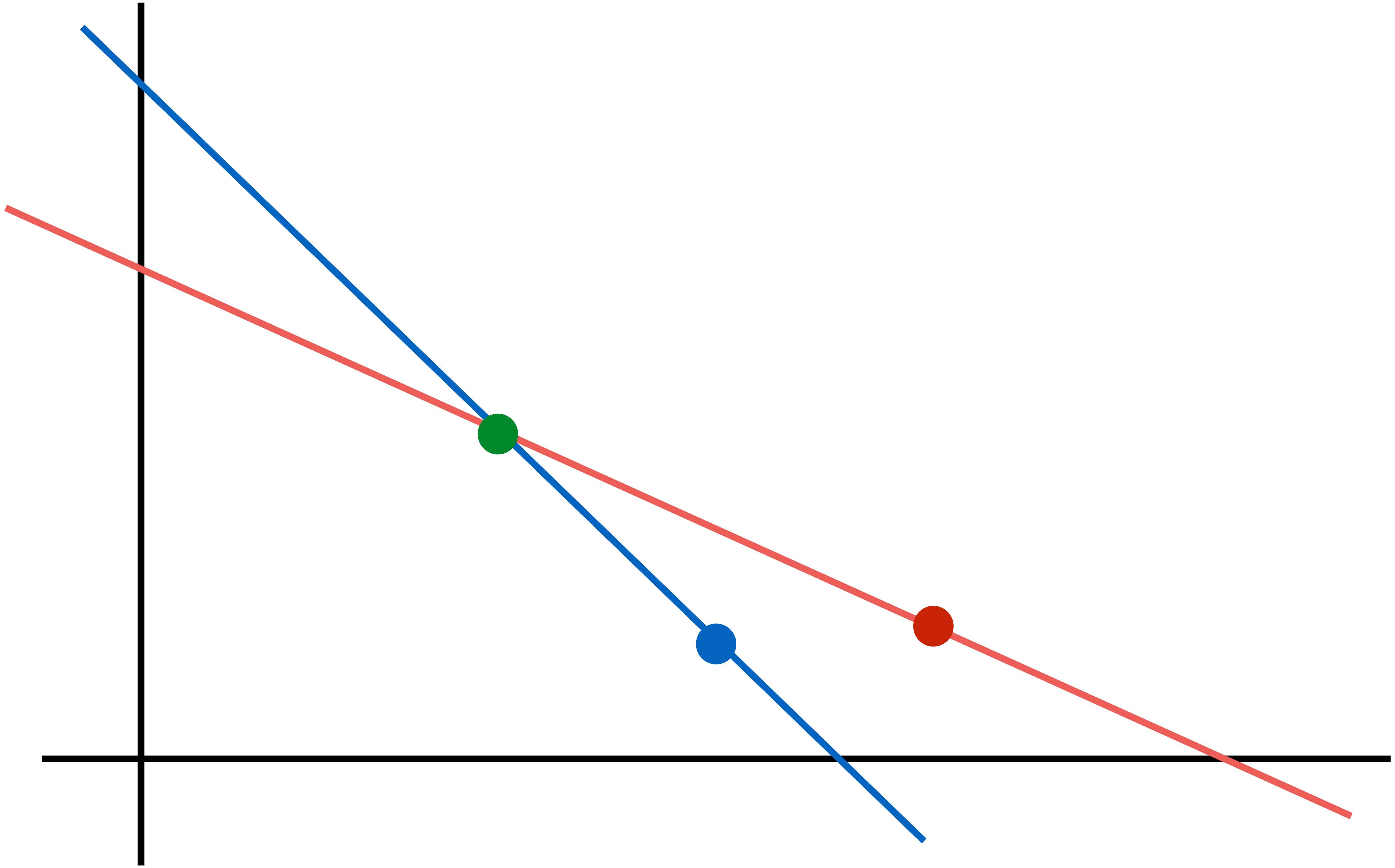
We can compute asymptotics as before

$$f_{rn,sn} = \frac{\left(\frac{3^{-r-s}2^r(r+s)^{r+s}}{r^r s^s}\right)^n}{\sqrt{ns}}\left(-\frac{4\sqrt{2}(r+s)^{\frac{3}{2}}}{\sqrt{\pi}(r-2s)\sqrt{r}} + O\left(\frac{1}{n}\right)\right)$$



Case 2: $1/3 < \dfrac{r}{r+s}$

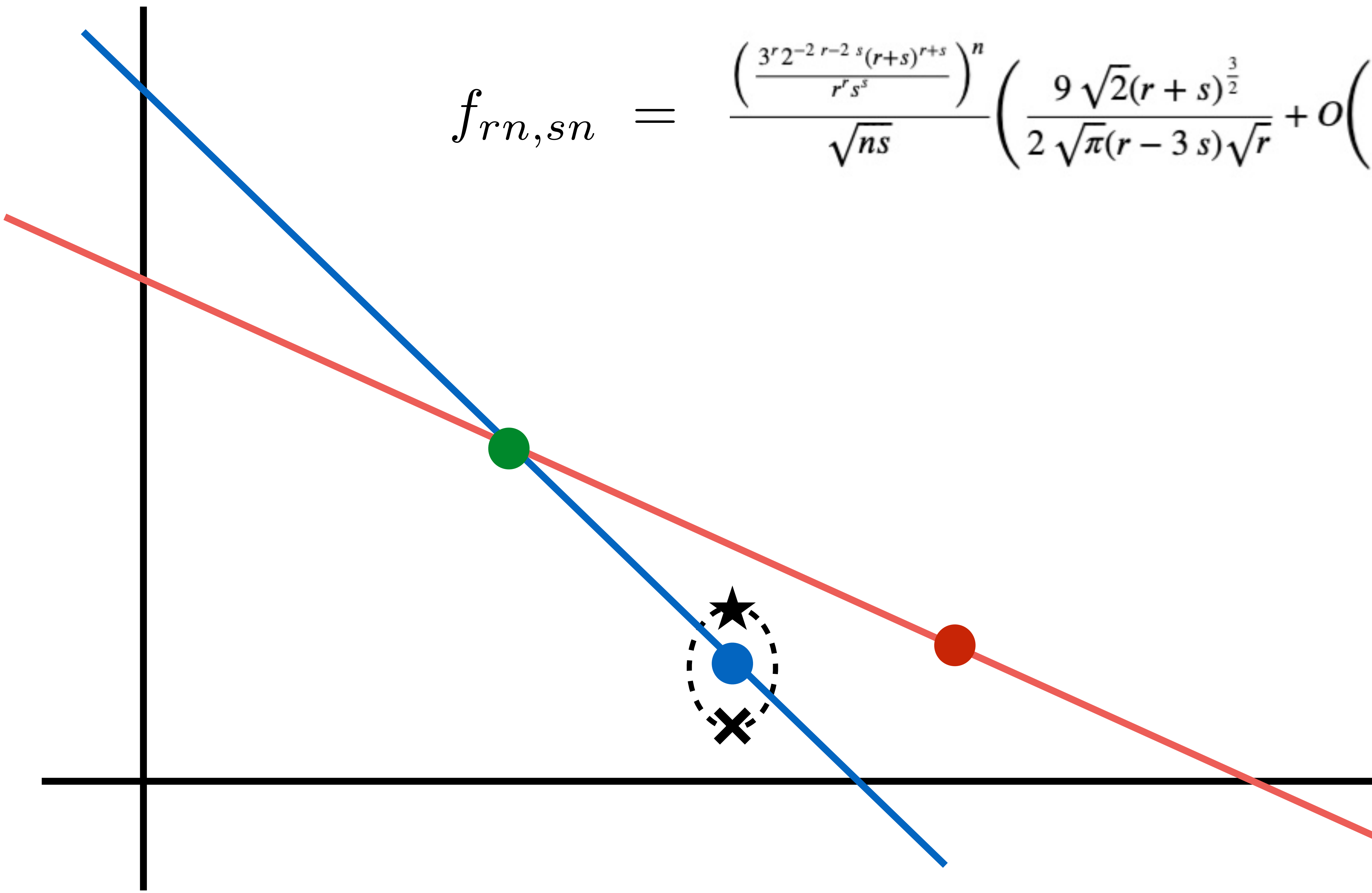
The point σ_2 is a smooth minimal critical point
We can compute asymptotics as before



Case 2: $1/3 < \dfrac{r}{r+s}$

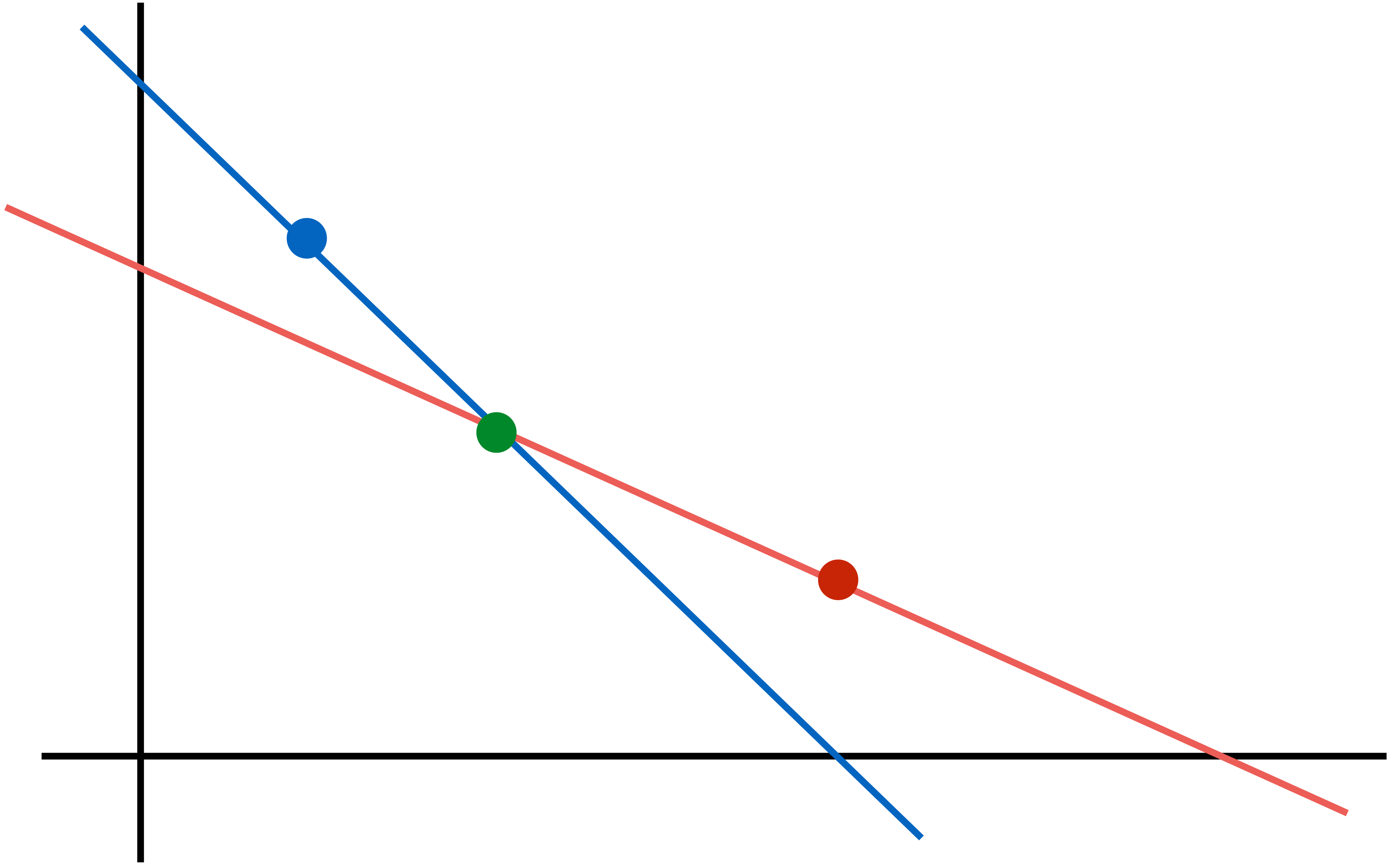
The point σ_2 is a smooth minimal critical point
We can compute asymptotics as before

$$f_{rn,sn} = \frac{\left(\frac{3^r 2^{-2r-2s}(r+s)^{r+s}}{r^r s^s}\right)^n}{\sqrt{ns}} \left(\frac{9\sqrt{2}(r+s)^{\frac{3}{2}}}{2\sqrt{\pi}(r-3s)\sqrt{r}} + O\left(\frac{1}{n}\right)\right)$$



Case 3: $2/3 < \dfrac{r}{r+s} < 3/4$

The **non-smooth** point $\sigma_{1,2}$ is the only minimal critical point
Only now can we introduce **three** new integrals with **small error**



Case 3: $2/3 < \frac{r}{r+s} < 3/4$

The **non-smooth** point $\sigma_{1,2}$ is the only minimal critical point

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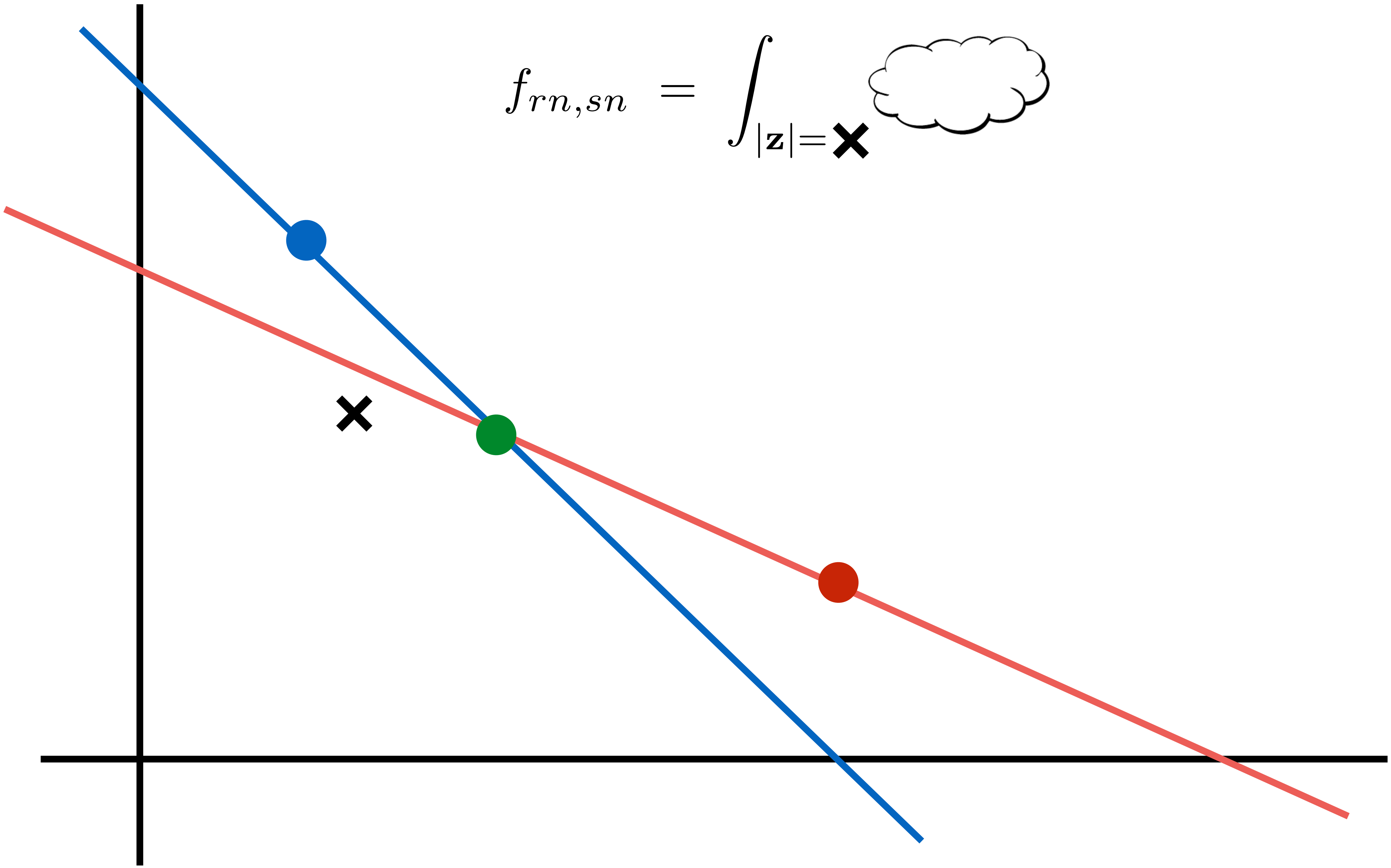
$$f_{rn,sn} = \int_{|\mathbf{z}|=\times}$$



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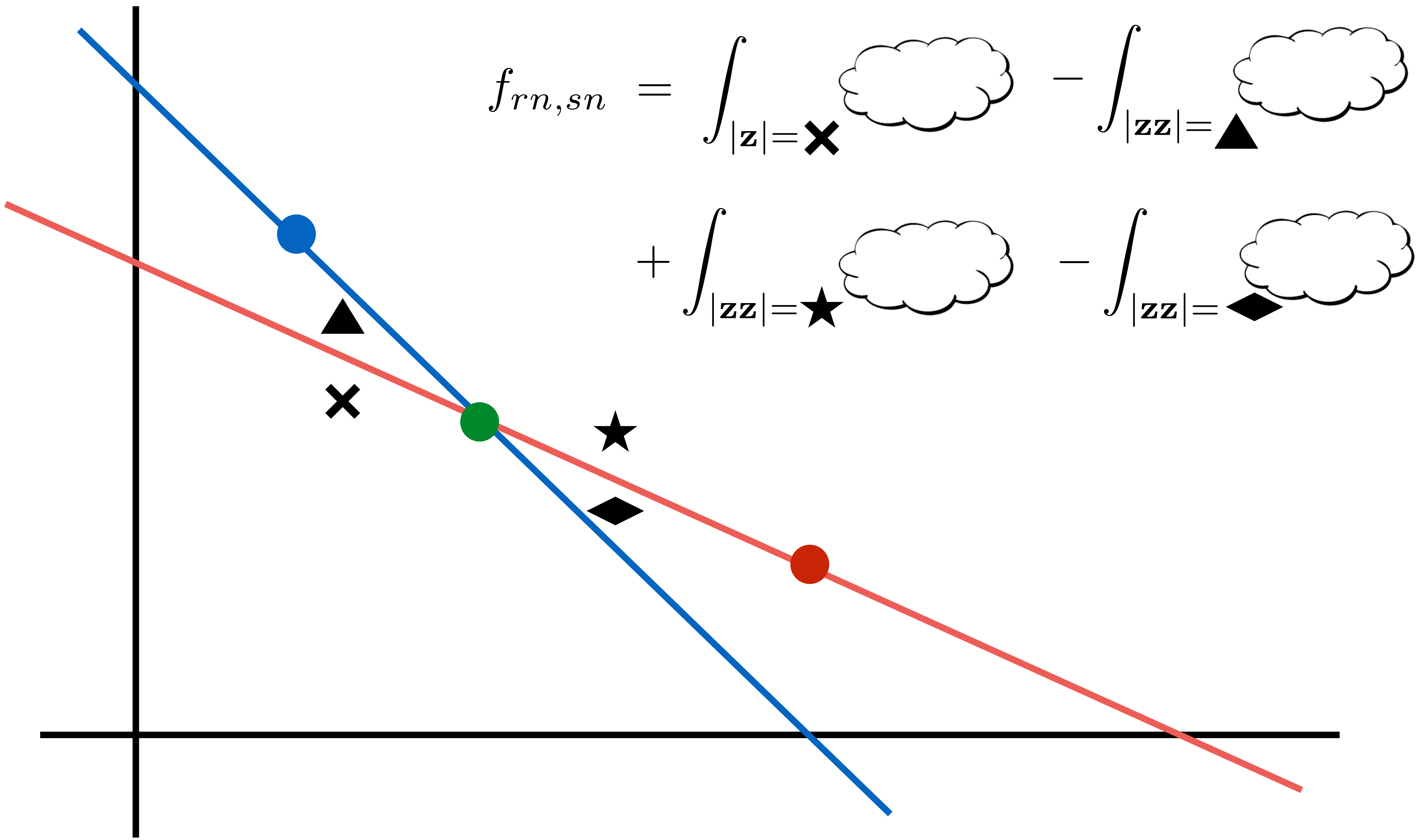

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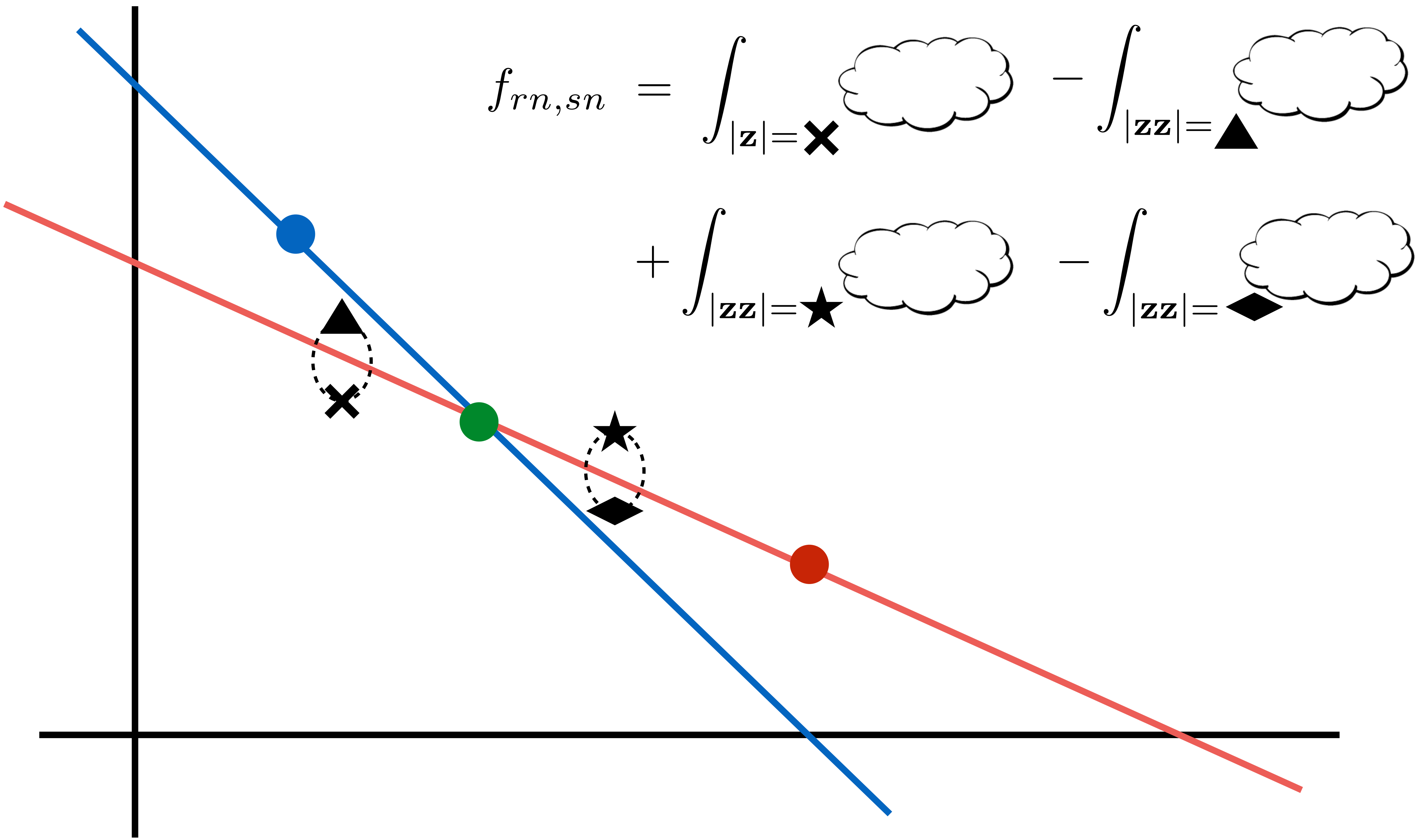
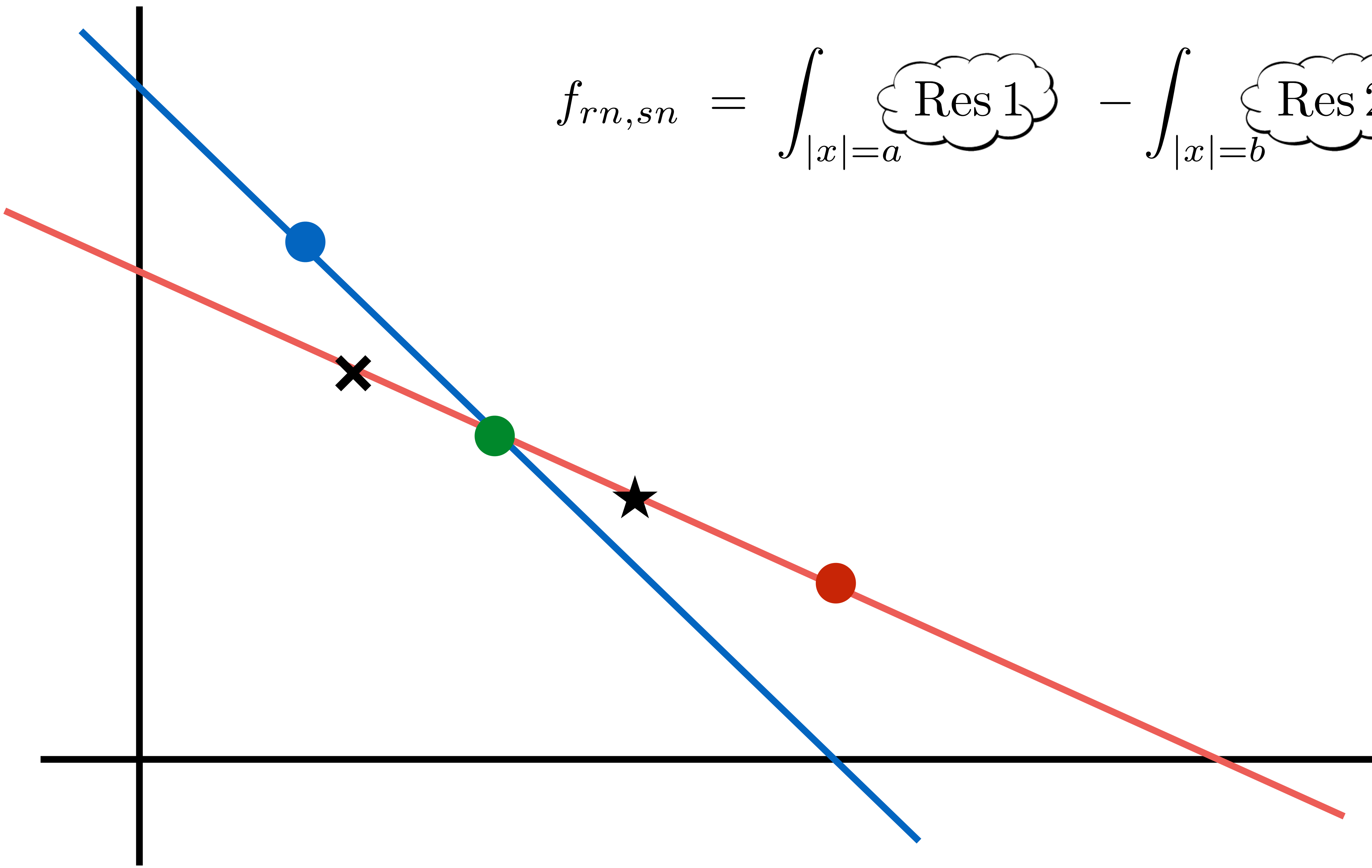


Case 3: $2/3 < \dfrac{r}{r+s} < 3/4$

The **non-smooth** point $\sigma_{1,2}$ is the only minimal critical point
Only now can we introduce **three** new integrals with **small error**

$$f_{rn,sn} = \int_{|x|=a} \text{Res 1} \; - \int_{|x|=b} \text{Res 2}$$



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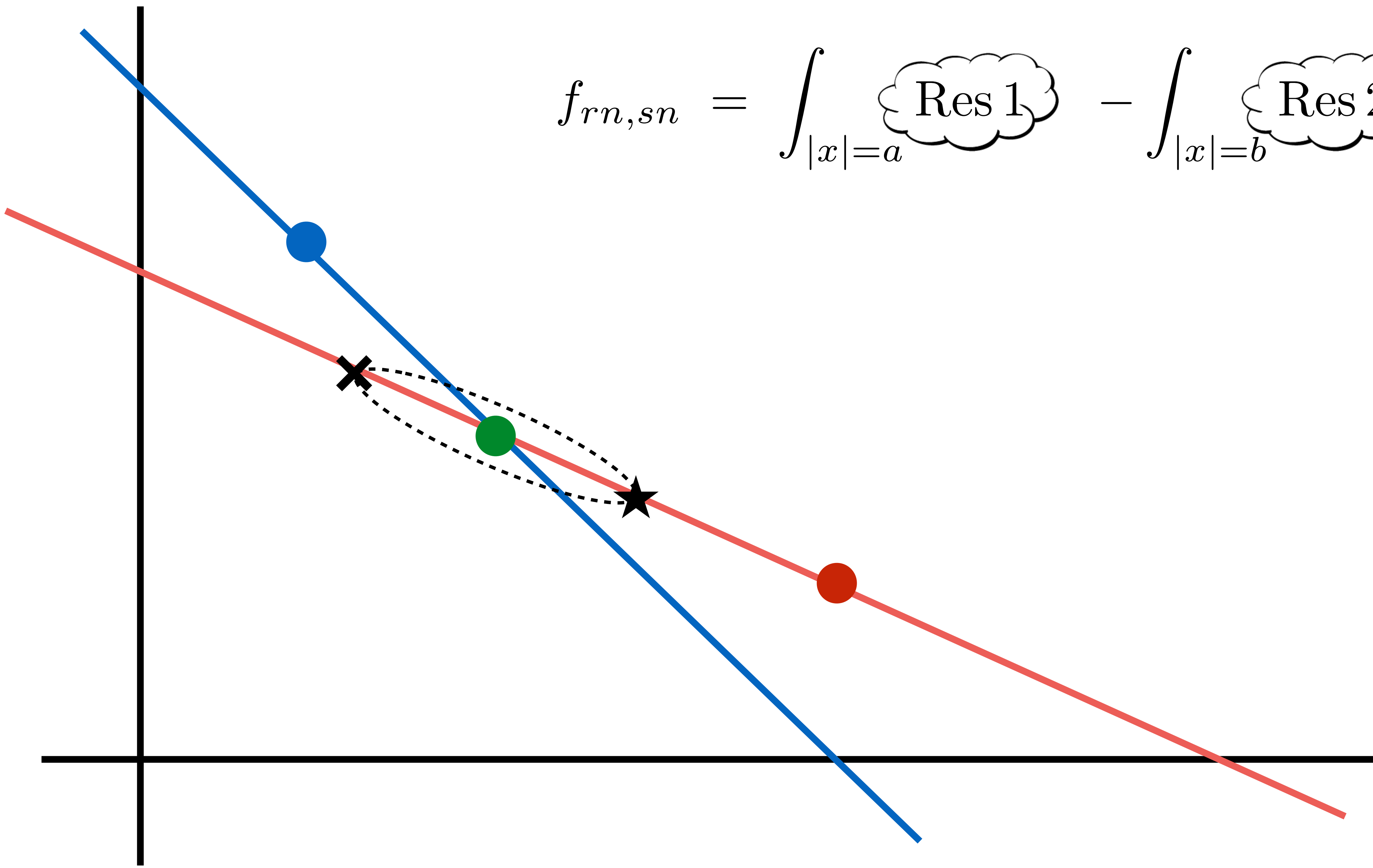
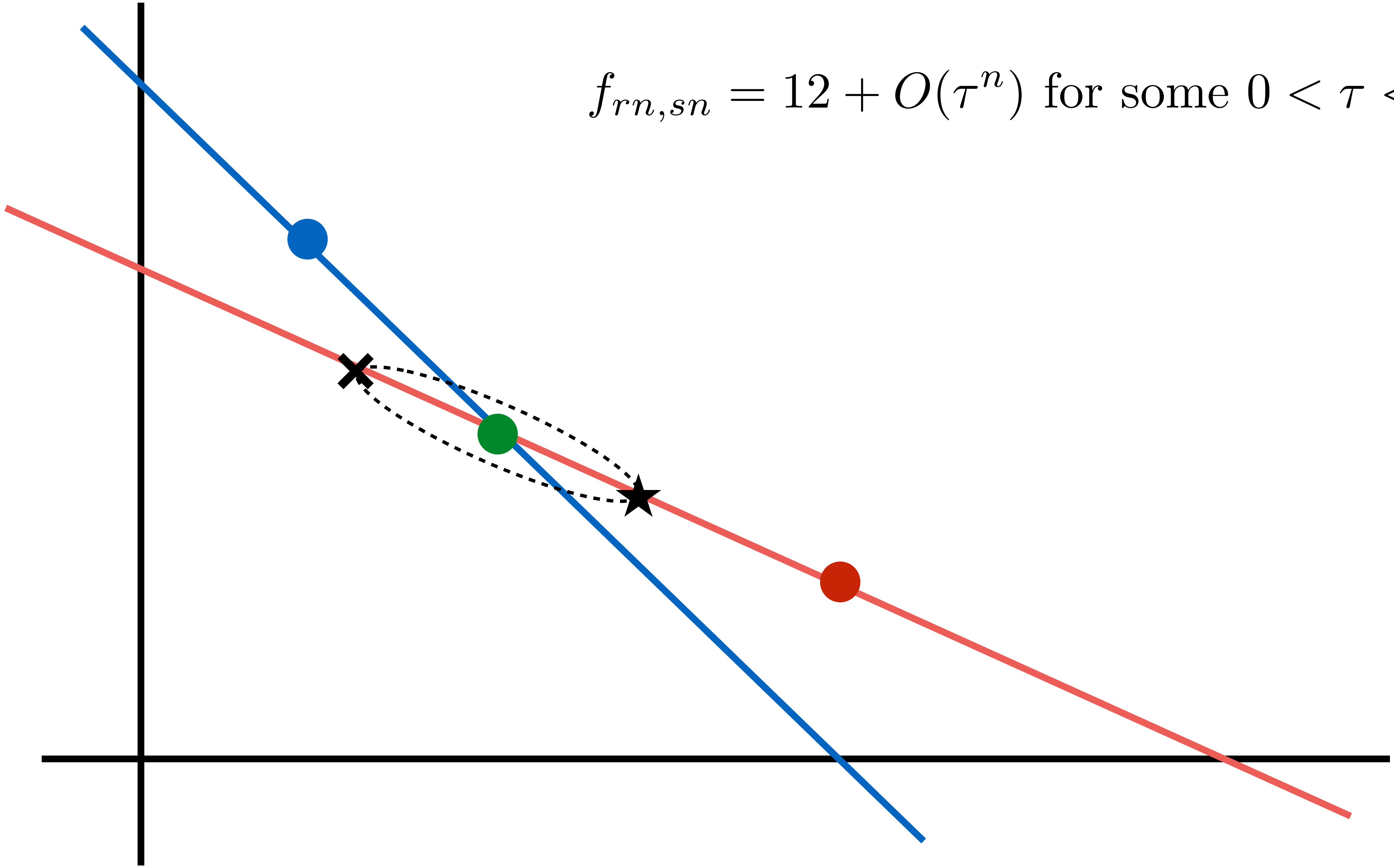
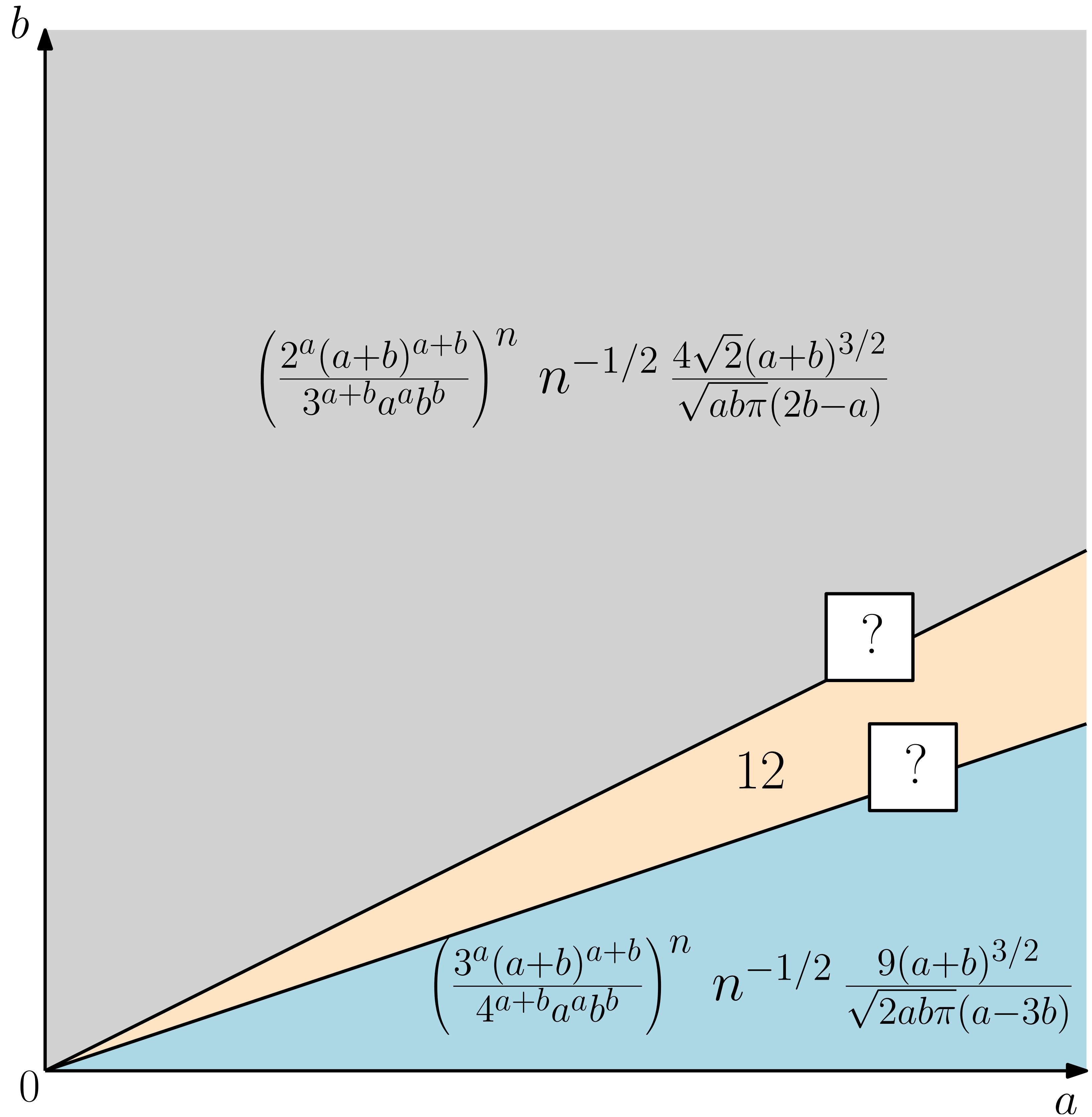


Case 3: $2/3 < \frac{r}{r+s} < 3/4$

The **non-smooth** point $\sigma_{1,2}$ is the only minimal critical point
Only now can we introduce **three** new integrals with **small error**

$f_{rn,sn} = 12 + O(\tau^n)$ for some $0 < \tau < 1$



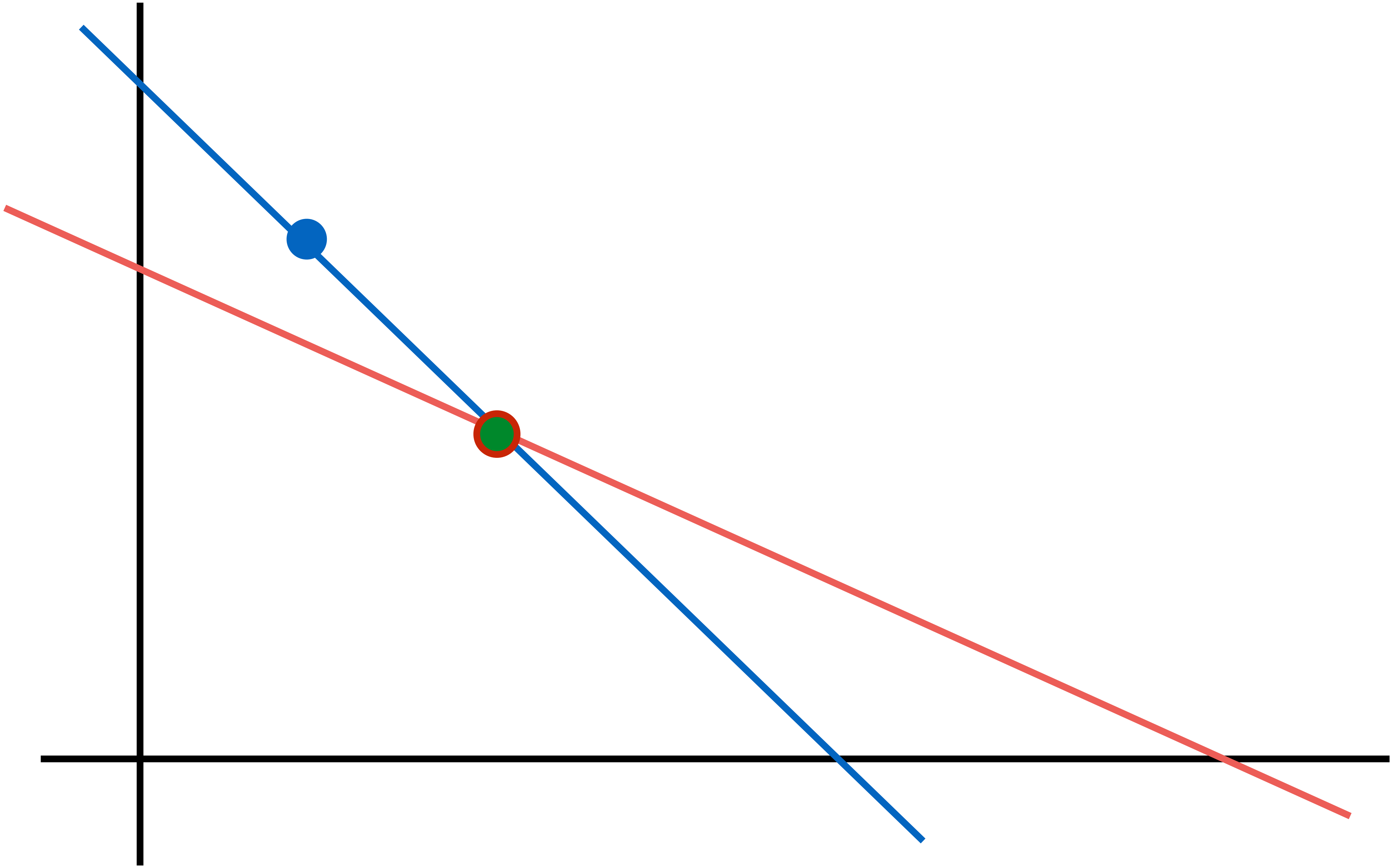


Asymptotics in direction $\mathbf{r} = (a,b)$

Non-Generic Directions

If $r = 2s$ then $\sigma_1 = \sigma_{1,2}$

We can take a residue over one, but not both, lines

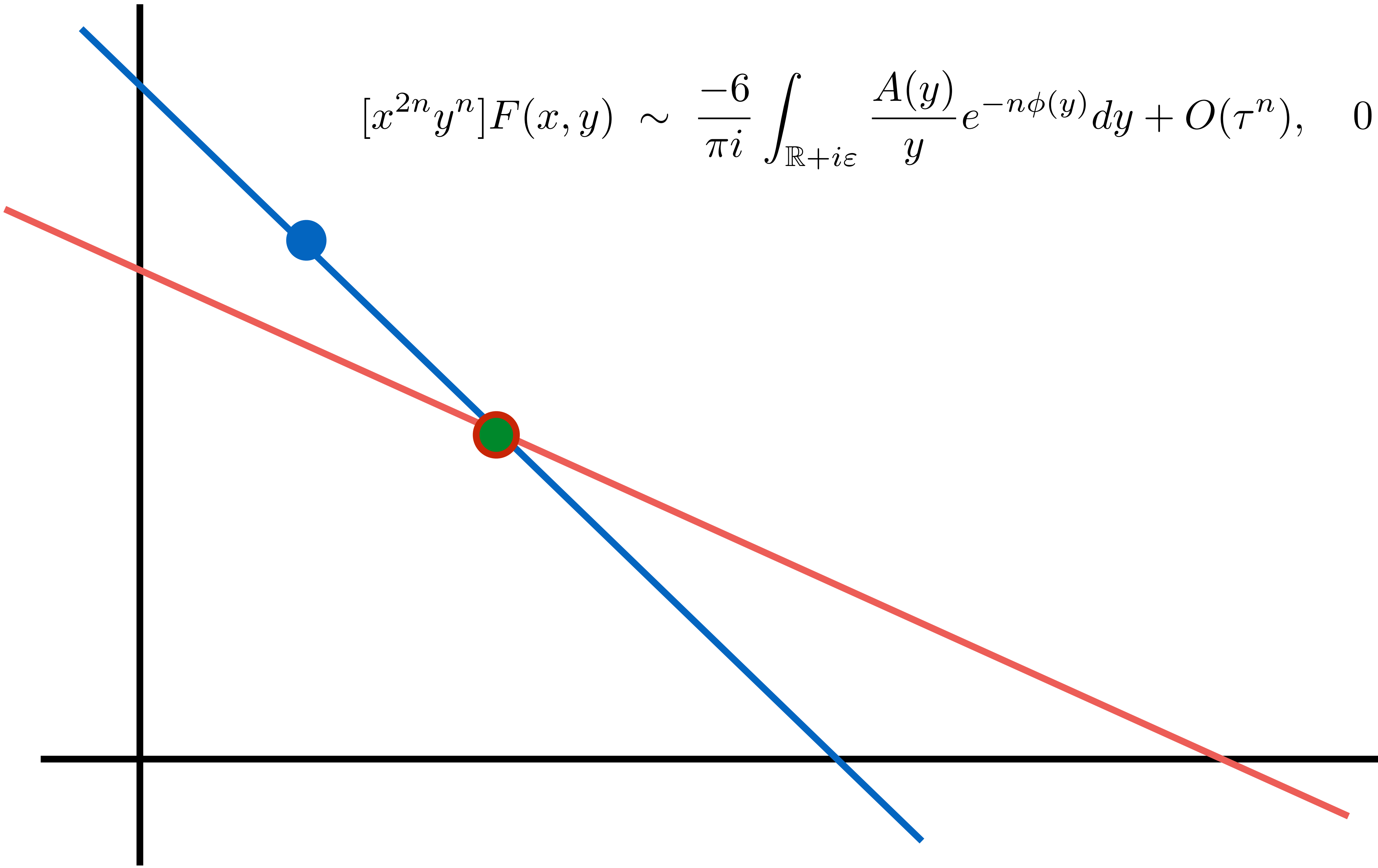


Non-Generic Directions

If $r = 2s$ then $\sigma_1 = \sigma_{1,2}$

We can take a residue over one, but not both, lines

$$[x^{2n}y^n]F(x,y) \sim \frac{-6}{\pi i}\int_{\mathbb{R}+i\varepsilon} \frac{A(y)}{y} e^{-n\phi(y)} dy + O(\tau^n), \quad 0 < \tau < 1$$



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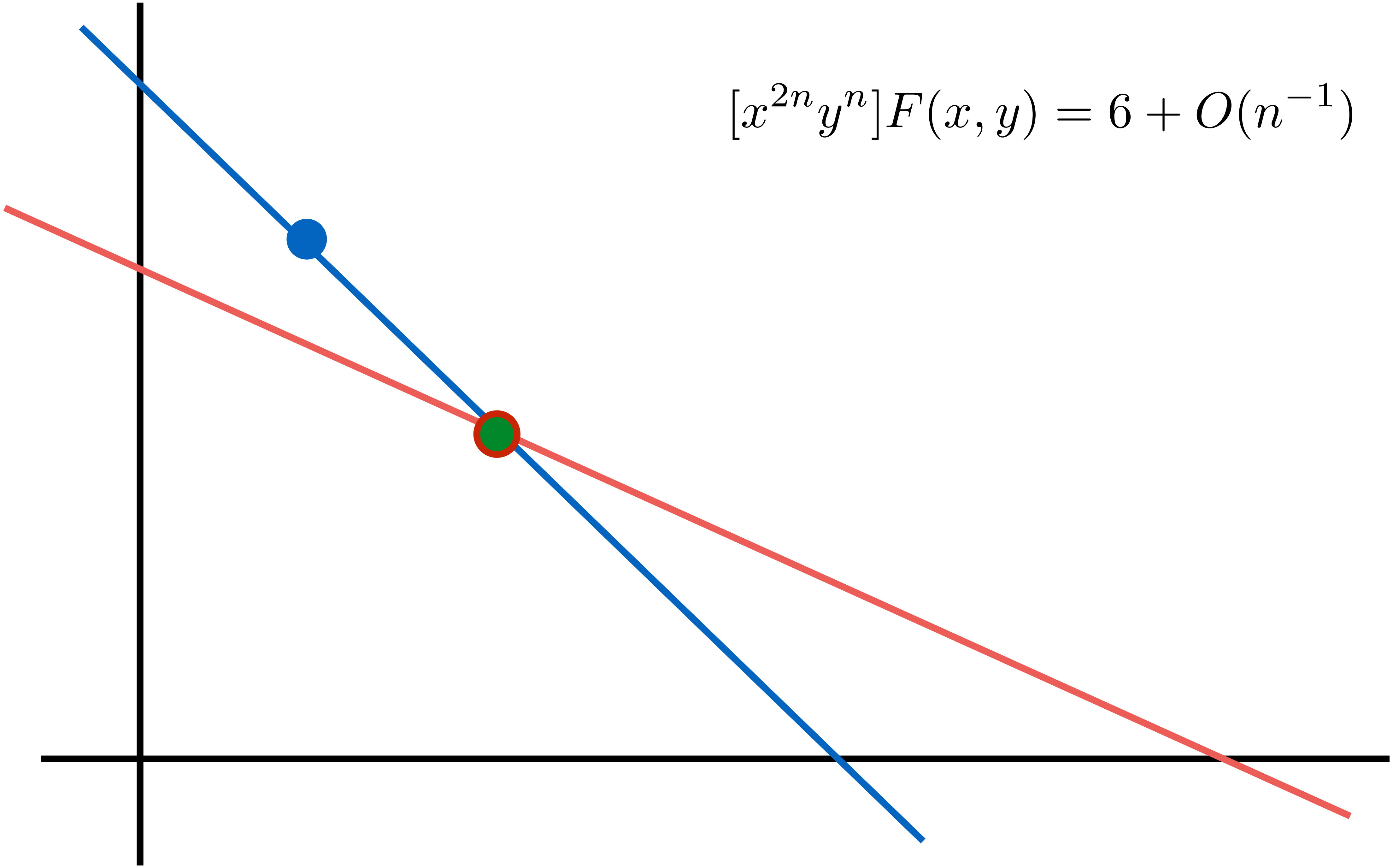
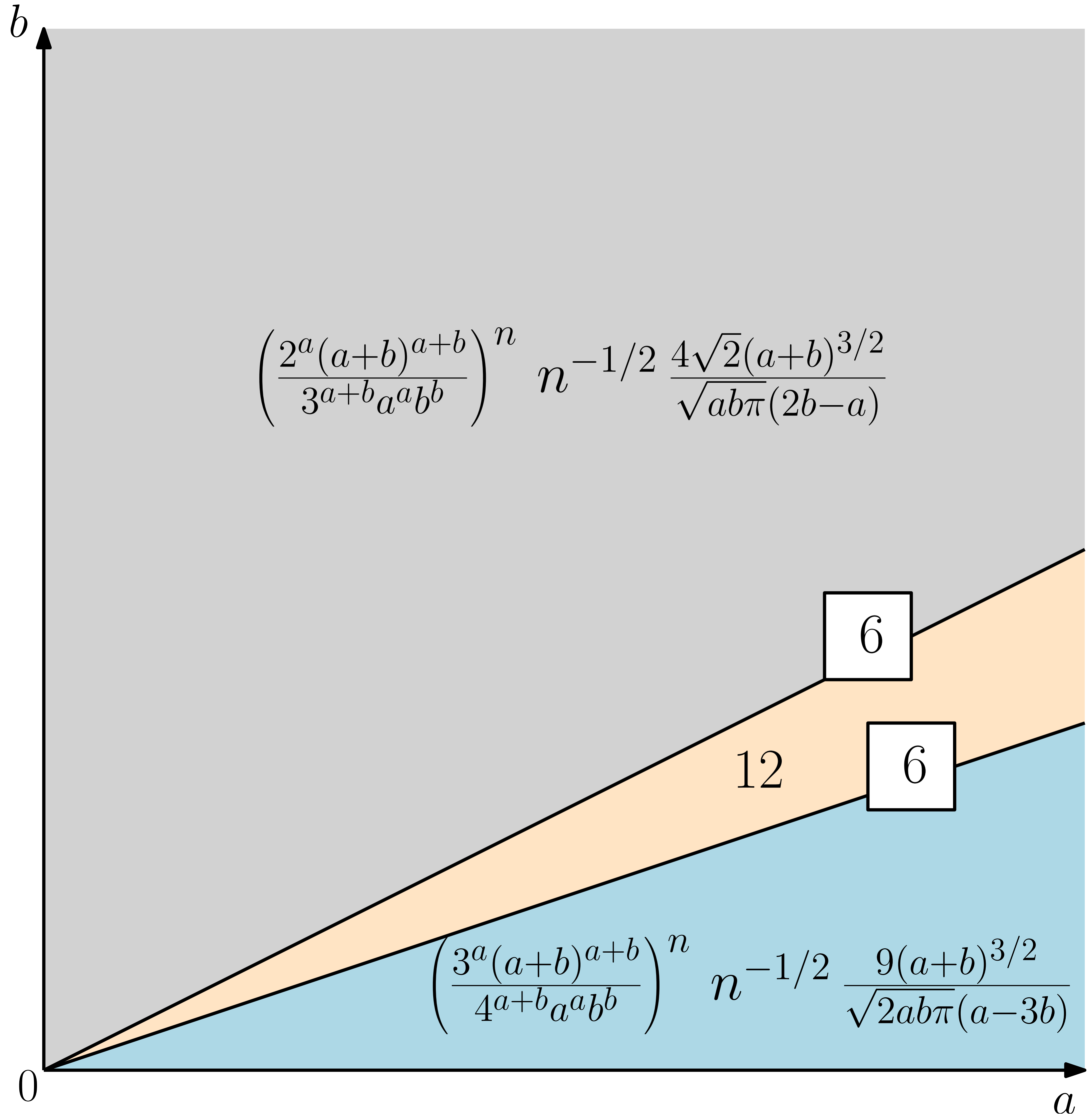


$$[x^{2n}y^n]F(x,y) = 6 + O(n^{-1})$$



Asymptotics in direction $\mathbf{r} = (a,b)$

Beyond Smoothness

How do we generalize?

Need to properly **define critical points**

Their smooth definition is constructed to give saddle-point integrals

Alternative definitions

Assume $H(\mathbf{w}) = 0$ but $(\nabla H)(\mathbf{w}) \neq \mathbf{0}$. The following are equivalent

- $\mathbf{w}$ is critical in the direction $\mathbf{r}$
- $(\nabla \phi)(\mathbf{w}) = \lambda (\nabla H)(\mathbf{w})$ where $\phi(\mathbf{z}) = \mathbf{z}^{\mathbf{r}}$
- $\mathbf{r} = \lambda (\nabla_{\log} H)(\mathbf{w})$ where $(\nabla_{\log} H)(\mathbf{w}) = \Big(w_1 H_{z_1}(\mathbf{w}), \ldots, w_d H_{z_d}(\mathbf{w}) \Big)$
- the differential of ϕ restricted to the manifold $\mathcal{V}$ vanishes at $\mathbf{w}$

Multiple Points

In general, **partition** $\mathcal{V}$ into a finite collection of **smooth strata**
Need the strata to *fit together nicely*

Simplest Case

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H_1(\mathbf{z}) \cdots H_m(\mathbf{z})}$$

where

- $(\nabla H_k)(\mathbf{w}) \neq \mathbf{0}$ if $\mathbf{w}$ in $\mathcal{V}_k = \{\mathbf{z} : H_k(\mathbf{z}) = 0\}$
- $\nabla H_{k_1}(\mathbf{w}), \ldots, \nabla H_{k_s}(\mathbf{w})$ linearly independent if $\mathbf{w} \in \mathcal{V}_{k_1} \cap \cdots \cap \mathcal{V}_{k_s}$

This is an example of the **transverse multiple point** case

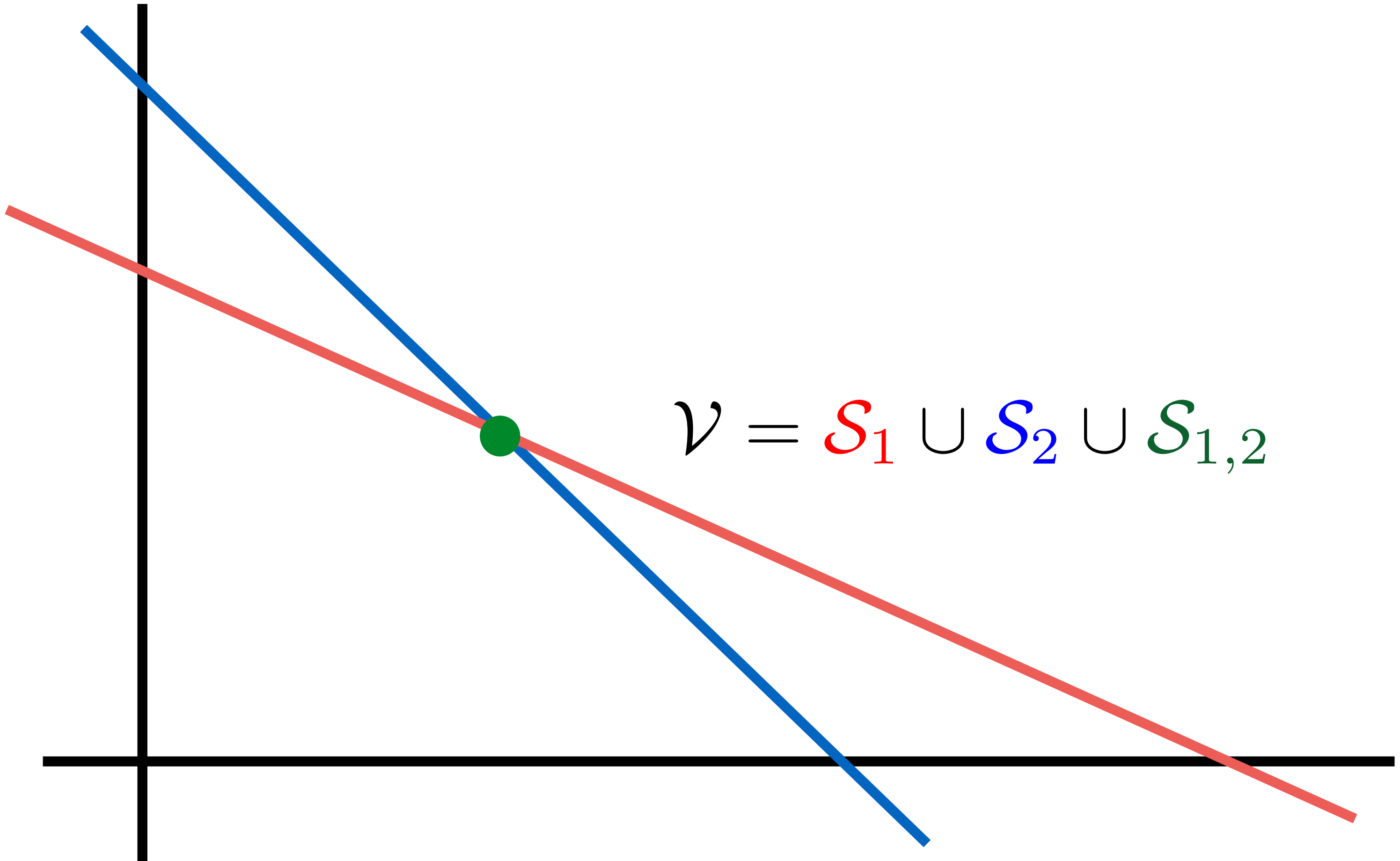
Strata

Under these assumptions, for any $S = \{k_1, \ldots, k_s\}$ we define the **flat**

$$\mathcal{V}_S = \mathcal{V}_{k_1,\ldots,k_s} = \mathcal{V}_{k_1} \cap \cdots \cap \mathcal{V}_{k_s}$$

and the **stratum**

$$\mathcal{S}_S = \mathcal{V}_S \setminus \bigcup_{\mathcal{V}_T \subsetneq \mathcal{V}_S} \mathcal{V}_T$$



Strata

Under these assumptions, for any $S = \{k_1, \ldots, k_s\}$ we define the **flat**

$$\mathcal{V}_S = \mathcal{V}_{k_1,\ldots,k_s} = \mathcal{V}_{k_1} \cap \cdots \cap \mathcal{V}_{k_s}$$

and the **stratum**

$$\mathcal{S}_S = \mathcal{V}_S \setminus \bigcup_{\mathcal{V}_T \subsetneq \mathcal{V}_S} \mathcal{V}_T$$

Definition

$\mathbf{w} \in \mathcal{S}_S$ is a **critical point** in the direction $\mathbf{r}$ if

$$\mathbf{r} = \lambda_1 (\nabla_{\log} H_1)(\mathbf{w}) + \cdots + \lambda_s (\nabla_{\log} H_s)(\mathbf{w})$$

Strata

Under these assumptions, for any $S = \{k_1, \ldots, k_s\}$ we define the **flat**

$$\mathcal{V}_S = \mathcal{V}_{k_1,\ldots,k_s} = \mathcal{V}_{k_1} \cap \cdots \cap \mathcal{V}_{k_s}$$

and the **stratum**

$$\mathcal{S}_S = \mathcal{V}_S \setminus \bigcup_{\mathcal{V}_T \subsetneq \mathcal{V}_S} \mathcal{V}_T$$

The critical points on $\mathcal{S}_S$ are defined by the vanishing of $H_{k_1}, \ldots, H_{k_s}$ and the $(s+1) \times (s+1)$ minors of

$$\begin{pmatrix} (\nabla_{\log} H_{k_1})(\mathbf{w}) \\ \vdots \\ (\nabla_{\log} H_{k_s})(\mathbf{w}) \\ \mathbf{r} \end{pmatrix}$$

Contributing Points

Suppose $\mathbf{w}$ is a minimal critical point on the stratum $\mathcal{S}_{1,\dots,s}$
If $(\partial H_j/\partial z_{k_j})(\mathbf{w}) \neq 0$ then define

$$\mathbf{v}_j = \frac{(\nabla_{\log} H_j)(\mathbf{w})}{w_{k_j}(\partial H_j/\partial z_{k_j})(\mathbf{w})} = \left(\frac{w_1(\partial H_j/\partial z_1)(\mathbf{w})}{w_{k_j}(\partial H_j/\partial z_{k_j})(\mathbf{w})}, \dots, \frac{w_d(\partial H_j/\partial z_d)(\mathbf{w})}{w_{k_j}(\partial H_j/\partial z_{k_j})(\mathbf{w})} \right)$$

It turns out $\mathbf{v}_j \in \mathbb{R}^d$

Definition
If $\mathbf{r} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_s \mathbf{v}_s$ where each $\lambda_k > 0$ then $\mathbf{w}$ is called a **contributing point**.

Note: Smooth minimal critical points are always contributing!

Multiple Point Asymptotics

Suppose $\mathbf{w}$ is a minimal contributing singularity of

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H_1(\mathbf{z}) \cdots H_m(\mathbf{z})}$$

with no other singularity with the same coordinate-wise modulus.

If w lies on a stratum of codimension s then there exist explicit matrices $\mathcal{M}_{\mathbf{w}}$ and $\Gamma_{\mathbf{w}}$ such that

$$f_{n\mathbf{r}} = \mathbf{w}^{-n\mathbf{r}}\, n^{(s-d)/2}\, (2\pi r_d)^{(s-d)/2} \det(\mathcal{M}_{\mathbf{w}})^{-1/2} \left(\frac{G(\mathbf{w})}{\det \Gamma_{\mathbf{w}}} + O\left(\frac{1}{n}\right)\right)$$

when $\det \mathcal{M}_{\mathbf{w}} \neq 0$

Proof Idea

Locally, $\mathcal{V}$ looks like a union of s hyperplanes near $\mathbf{w}$

1) Introduce asymptotically negligible integrals

2) Take residues in s dimensions

3) Approximate a $(d-s)-$dimensional saddle-point integral

Proof Idea

Locally, $\mathcal{V}$ looks like a union of s hyperplanes near $\mathbf{w}$

1) Introduce asymptotically negligible integrals

2) Take residues in s dimensions

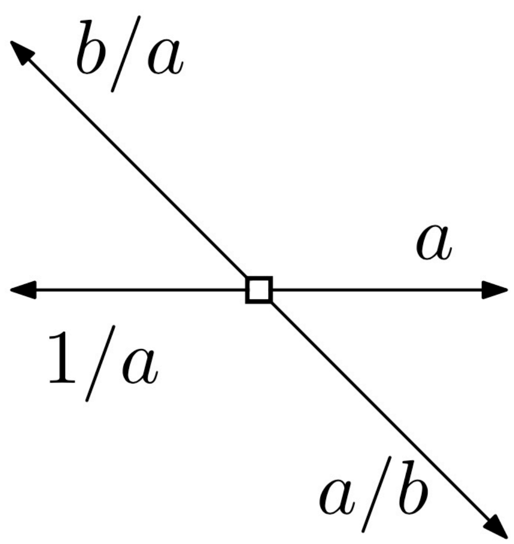
3) Approximate a $(d-s)-$dimensional saddle-point integral

If $s = d$ then

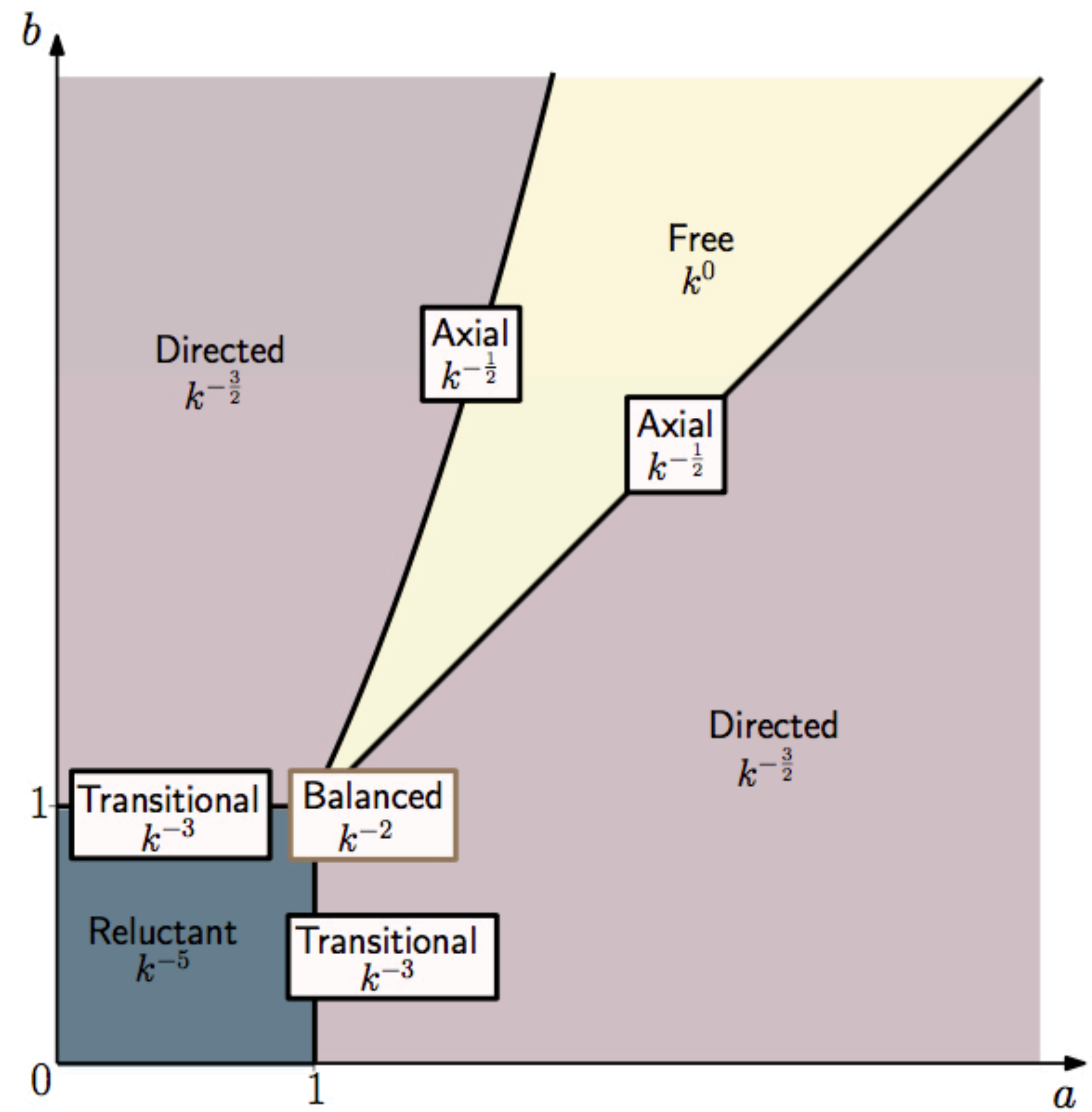
$$f_{n\mathbf{r}} = \mathbf{w}^{-n\mathbf{r}} \cdot \frac{G(\mathbf{w})}{|\det \Gamma_{\mathbf{w}}|} + O\left(\tau^n\right)$$

where $0 < \tau < \left|\mathbf{w}^{-\mathbf{r}}\right|$ and $\Gamma_{\mathbf{w}} = \begin{pmatrix} (\nabla_{\log} H_1)(\mathbf{w}) \\ \vdots \\ (\nabla_{\log} H_s)(\mathbf{w}) \end{pmatrix}$

Quadrant Walks on Weighed Stepsets



$$\frac{yt^2(y-b)(a-x)(a+x)(a^2y-bx^2)(ay-bx)(ay+bx)}{a^4b^3\left(1-txy\left(\frac{1}{ax}+ax+\frac{ax}{by}+\frac{by}{ax}\right)\right)(1-x)(1-y)}$$



Courtiel, M., Mishna, and Raschel (2017)

Topic 6
Geometric Approach to ACSV

melczer.ca/ALEA22

Whitney Stratifications

In general, we partition $\mathcal{V}$ into a finite collection of **strata** that are manifolds satisfying the *Whitney conditions*

The **critical points** of F in direction $\mathbf{r}$ are the critical points of $\phi(\mathbf{z}) = \mathbf{z}^{\mathbf{r}}$ restricted to each of these manifolds

Fact: The exist effectively computable algebraic sets

$$\mathcal{V} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_m = \varnothing$$

such that the connected components of $\mathcal{F}_k \setminus \mathcal{F}_{k+1}$ form a Whitney stratification.

Corollary: All critical points defined by algebraic (in)equations

Height Functions

For large n the modulus of

$$f_{n\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_{\mathcal{C}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{n\mathbf{r}+\mathbf{1}}}$$

is captured by

$$h(\mathbf{z}) = -r_1 \log |z_1| - \cdots - r_d \log |z_d|$$

Idea 1: We start with points of C with high height
Try to push them down as far as possible while avoiding $\mathcal{V}$

Idea 2: Use *Leray residues* to reduce to integral 'on' $\mathcal{V}$
Try to push down resulting *intersection cycles*

Pushing Down Cycle

Start with the expression

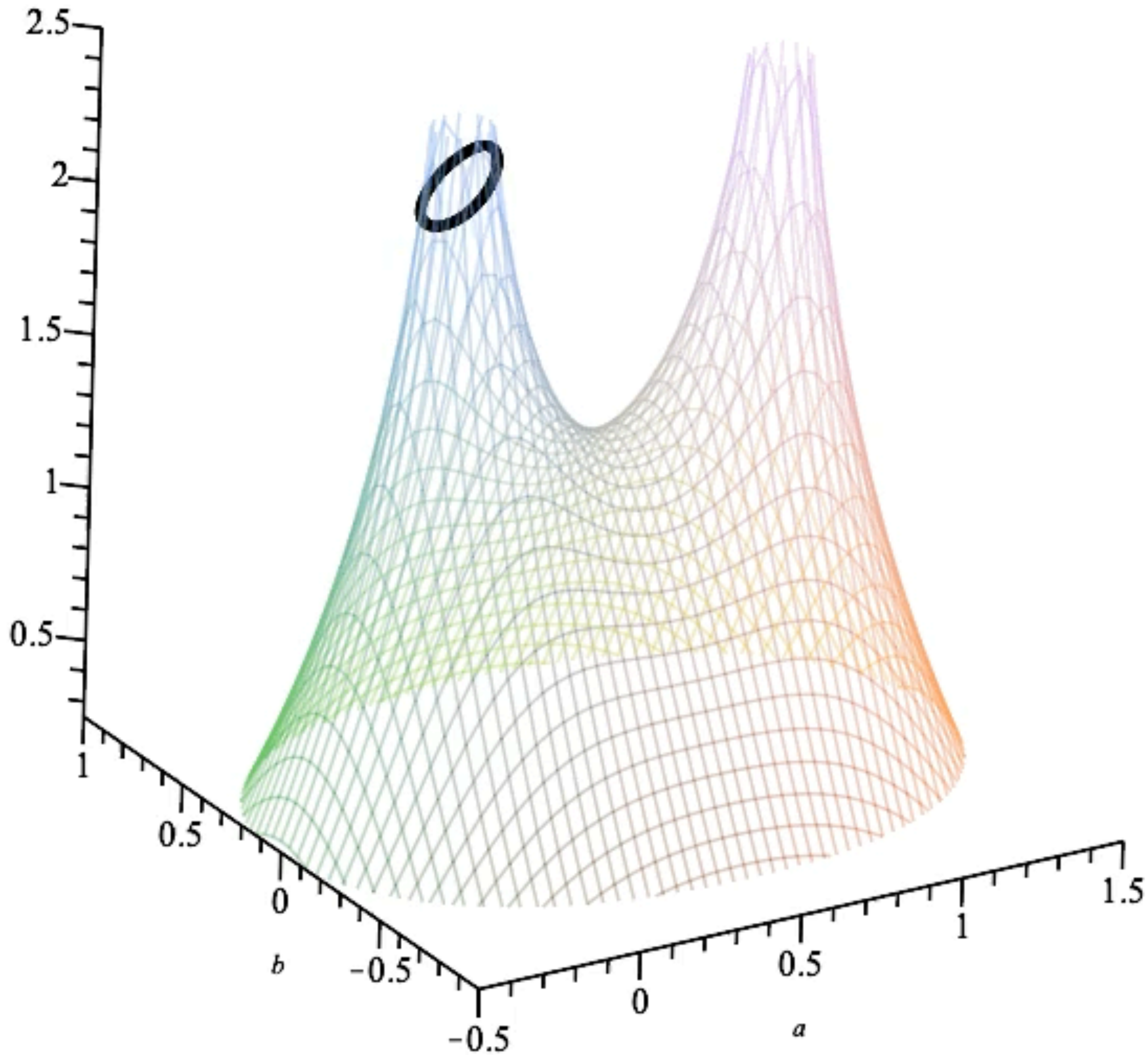
$$\binom{2n}{n} = \frac{1}{(2\pi i)^2} \int_{|x|=\varepsilon, |y|=\varepsilon} \frac{1}{1-x-y} \frac{dx dy}{x^{n+1} y^{n+1}}$$

Expand $|y|$ until you hit $\mathcal{V}$ and take a residue

$$\binom{2n}{n} = \frac{-1}{2\pi i} \int_{|x|=\varepsilon} \frac{dx}{x^{n+1}(1-x)^{n+1}}$$

Then flow the cycle $|x| = \varepsilon$ to points of lower height

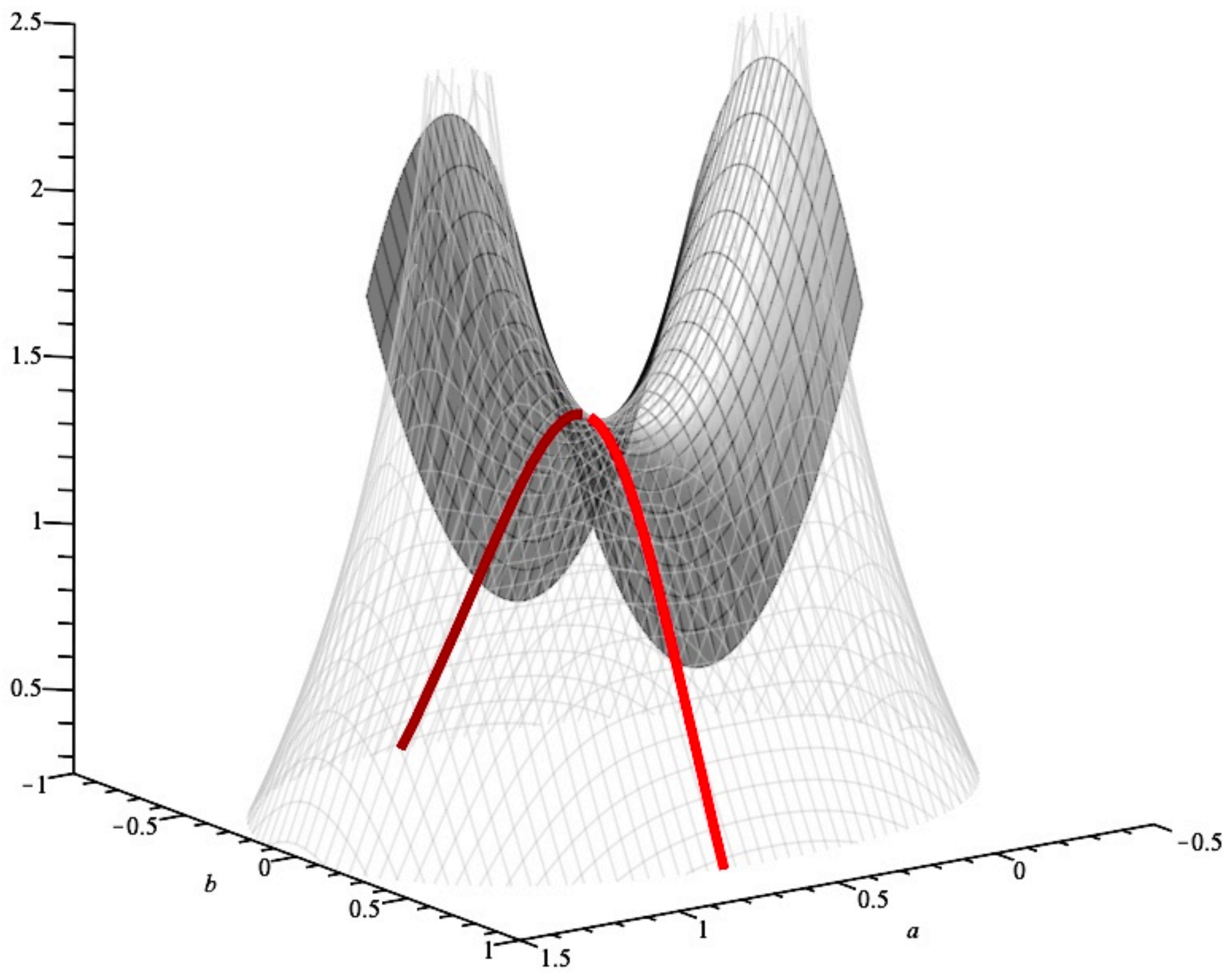
Maximum height = 2.408



Flow of $x = a + ib$ starting with $|x| = 1/10$

with $h(a,b) = -\log|x| - \log|1-x| = -\frac{1}{2}\log(a^2+b^2) - \frac{1}{2}\log((1-a)^2+b^2)$

We get stuck at saddle-point



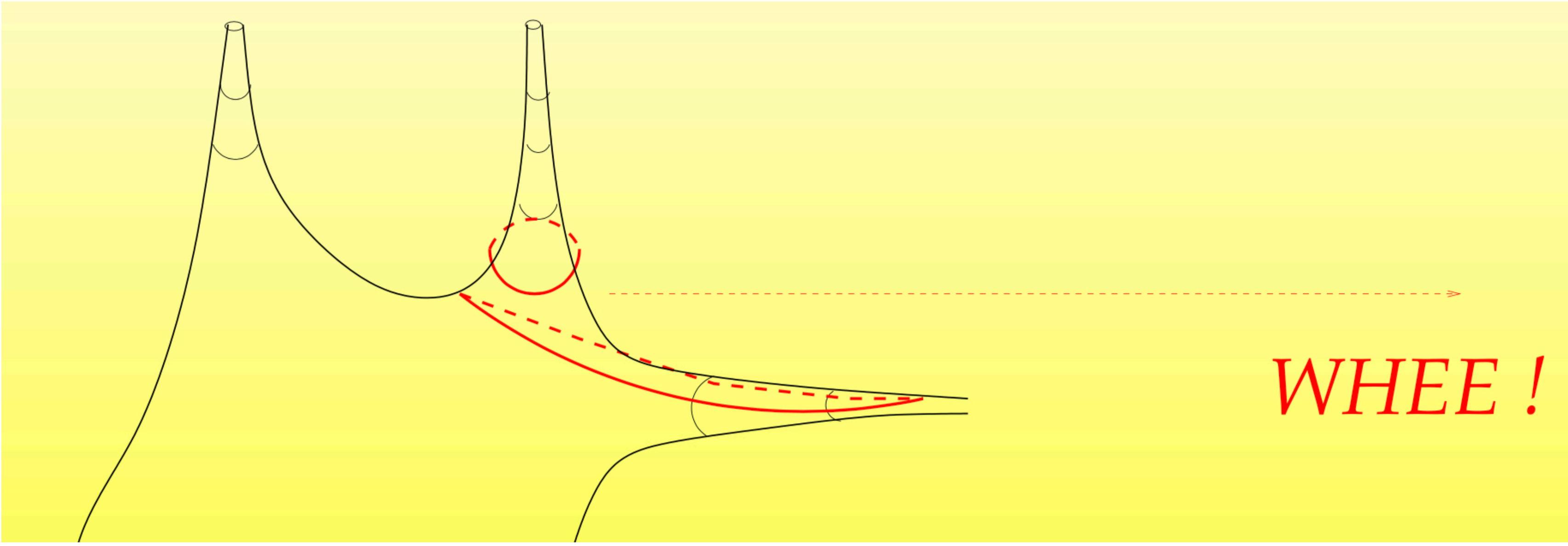
$$\binom{2n}{n} = \frac{-1}{2\pi i}\int_{\mathcal{C}'} \frac{dx}{x^{n+1}(1-x)^{n+1}} = \frac{4^n}{\sqrt{\pi n}}\left(1 + O\left(n^{-1}\right)\right)$$

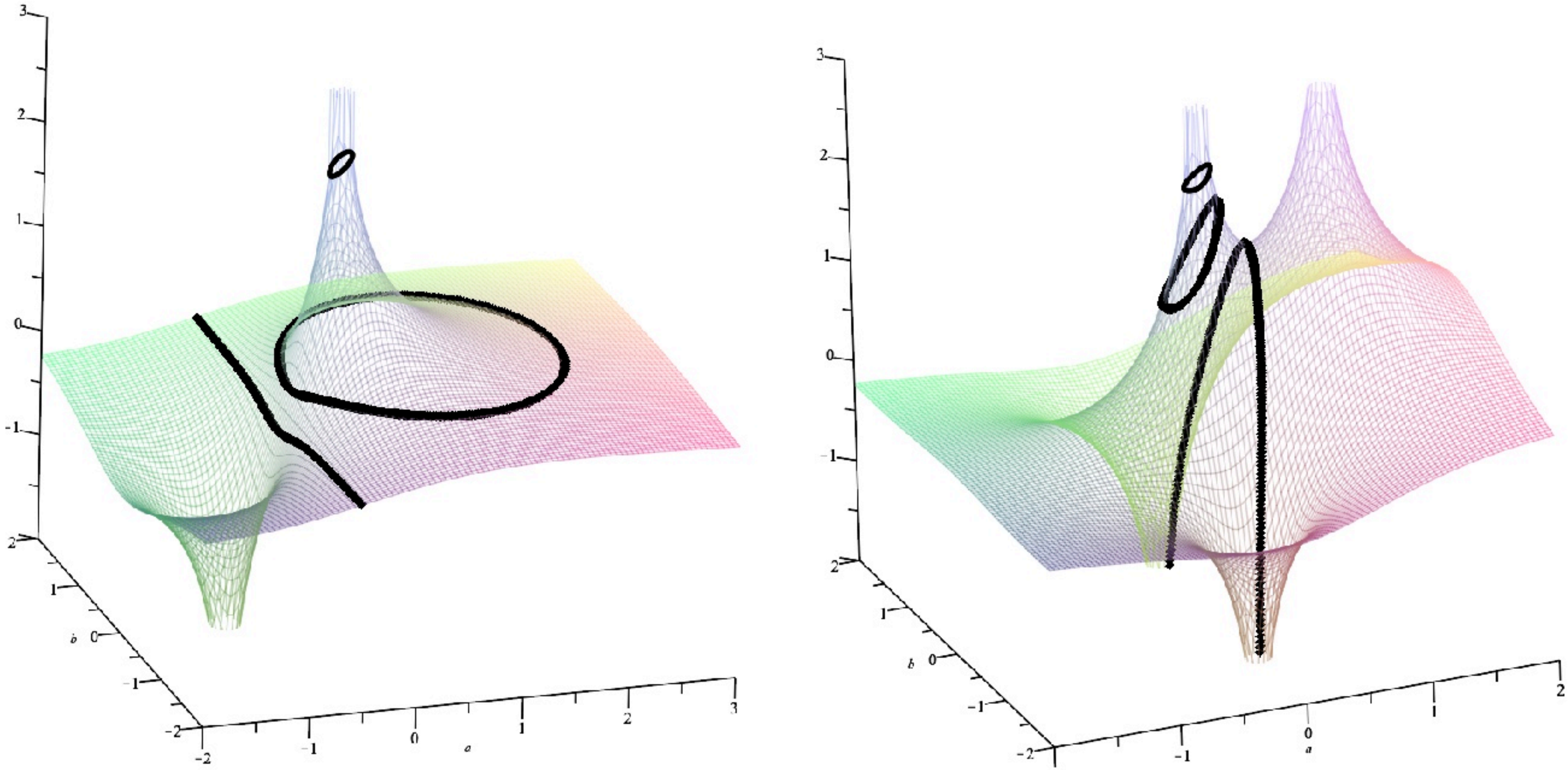
Morse theory

Actually performing these flows is inefficient!
But we can **predict** where they get stuck — at **critical points**!
And we will get **saddle-point integrals**!

Morse theory: Topological study of manifolds via height functions
Stratified Morse theory: Study of varieties (and more)

Problem: Our height functions are not *proper*
Solution: Ignore parts not contributing to dominant asymptotics





Flows of $|x| = 1/10$ on $\mathcal{V}(1 - x - xy)$ and $\mathcal{V}(1 - x - y - x^2y)$

Morse Theory For Asymptotics

(Baryshnikov, M. and Pemantle 2021)

There exist checkable algebraic conditions, under which the results of Morse theory hold and we can write

$$f_{n\mathbf{r}} = \sum_{\mathbf{w} \in \text{crit}} \kappa_{\mathbf{w}} \Psi_{\mathbf{w}}(n) + \text{asymptotically negligible terms}$$

where $\kappa_{\mathbf{w}} \in \mathbb{Z}$

$\Psi_{\mathbf{w}}(n)$ is a local integral

Cannot always determine $\kappa_{\mathbf{w}}$ directly, but knowing they are integers we can sometimes use **rigorous numerical analytic continuation**

Two Complementary Approaches: Approach #1

Consider $F(w,x,y,z) = \dfrac{1}{1-(w+x+y+z)+27wxyz}$

There are **three critical points**, with asymptotic contributions

$$\Phi_1(n) = 81^n \cdot \Theta(n) \qquad (w=x=y=z=1/3)$$

$$\Phi_2(n) = \left(\frac{11621 - i\sqrt{30803599}}{20420}\right)\frac{\left(-7-4i\sqrt{2}\right)^n}{n^{3/2}\pi^{3/2}} \qquad (w=x=y=z=-1/3+i\sqrt{2}/3)$$

$$\Phi_3(n) = \left(\frac{11621 + i\sqrt{30803599}}{20420}\right)\frac{\left(-7+4i\sqrt{2}\right)^n}{n^{3/2}\pi^{3/2}} \qquad (w=x=y=z=-1/3-i\sqrt{2}/3)$$

Our Morse result immediately implies

$$\boxed{f_{n,n,n,n} = \kappa_1\Phi_1(n) + \kappa_2\Phi_2(n) + \kappa_3\Phi_3(n)}$$

Two Complementary Approaches: Approach #2

Consider $F(w,x,y,z) = \dfrac{1}{1-(w+x+y+z)+27wxyz}$

The diagonal coefficients satisfy a **linear recurrence**

$$(n^3+6n^2+12n+8)c_{n+2}+(14n^3+63n^2+97n+51)c_{n+1}+(81n^3+243n^2+243n+81)c_n=0$$

whose solutions form a **complex vector-space** with basis

$$\Psi_1(n)=\frac{(94i\sqrt{2}-7)^n}{n^{3/2}}\left(1+O\left(\frac{1}{n}\right)\right) \qquad \Psi_2(n)=\frac{(-94i\sqrt{2}-7)^n}{n^{3/2}}\left(1+O\left(\frac{1}{n}\right)\right)$$

so

$$\boxed{f_{n,n,n,n}=\sigma_1\Psi_1(n)+\sigma_2\Psi_2(n)}$$

Now the $\sigma_j \in \mathbb{C}$ but **we can rigorously approximate them to any desired accuracy using *numeric analytic continuation***

Two Complementary Approaches: Combined

The sequence in question is

Unknown integers

$$f_{n,n,n,n} = \kappa_1 \Phi_1(n) + \kappa_2 \Phi_2(n) + \kappa_3 \Phi_3(n)$$

and

$$f_{n,n,n,n} = \sigma_1 \Psi_1(n) + \sigma_2 \Psi_2(n)$$

Approximated Complex Numbers

By **combining methods** we can **exactly determine asymptotics**.

Arbitrary approximation cannot prove equalities without bounds!

Two Complementary Approaches: Combined

The sequence in question is

Unknown integers

$$f_{n,n,n,n} = \kappa_1 \Phi_1(n) + \kappa_2 \Phi_2(n) + \kappa_3 \Phi_3(n)$$

and

$$f_{n,n,n,n} = \sigma_1 \Psi_1(n) + \sigma_2 \Psi_2(n)$$

Approximated Complex Numbers

This process shows

$$\kappa_1 = 0.000\ldots$$

$$\kappa_2 = 2.999\ldots$$

$$\kappa_3 = 2.999\ldots$$

to hundreds of decimal places in seconds (on my laptop)

Two Complementary Approaches: Combined

The sequence in question is

Unknown integers

$$f_{n,n,n,n} = \kappa_1 \Phi_1(n) + \kappa_2 \Phi_2(n) + \kappa_3 \Phi_3(n)$$

and

$$f_{n,n,n,n} = \sigma_1 \Psi_1(n) + \sigma_2 \Psi_2(n)$$

Approximated Complex Numbers

Theorem 2.5. *The diagonal coefficients $a_{n,n,n,n}$ of the function in Example 2.4 have an asymptotic expansion in decreasing powers of n, beginning as follows.*

$$a_{n,n,n,n} = 3 \cdot \left(\frac{\left(4i\sqrt{2}-7\right)^n}{n^{3/2}} \frac{\left(5i-\sqrt{2}\right)\sqrt{-2i\sqrt{2}-8}}{24\pi^{3/2}} + \frac{\left(-4i\sqrt{2}-7\right)^n}{n^{3/2}} \frac{\left(-5i-\sqrt{2}\right)\sqrt{2i\sqrt{2}-8}}{24\pi^{3/2}} \right) + O\left(9^n\, n^{-5/2}\right) \quad (2.5)$$

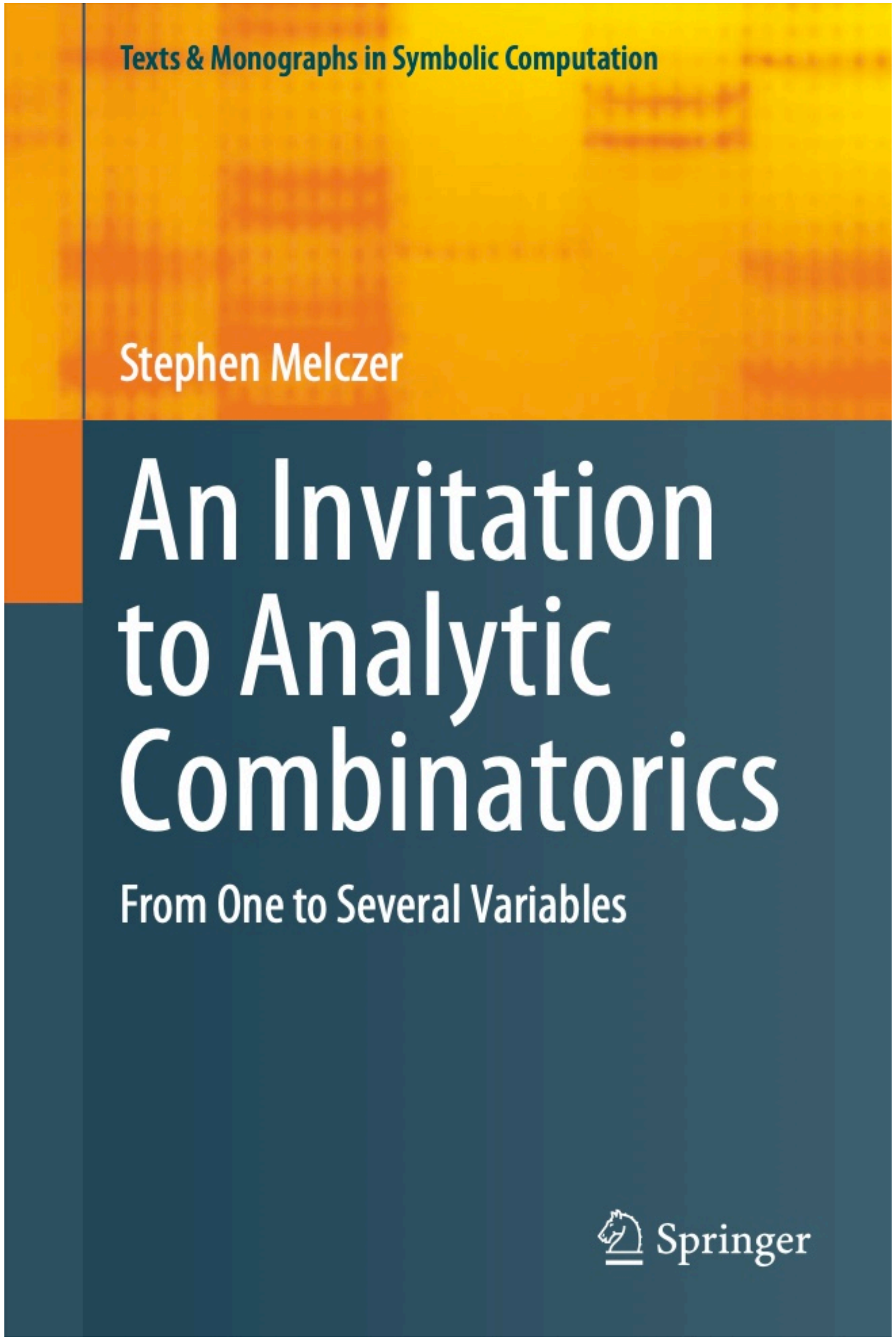
(Baryshnikov, M. and Pemantle 2021)

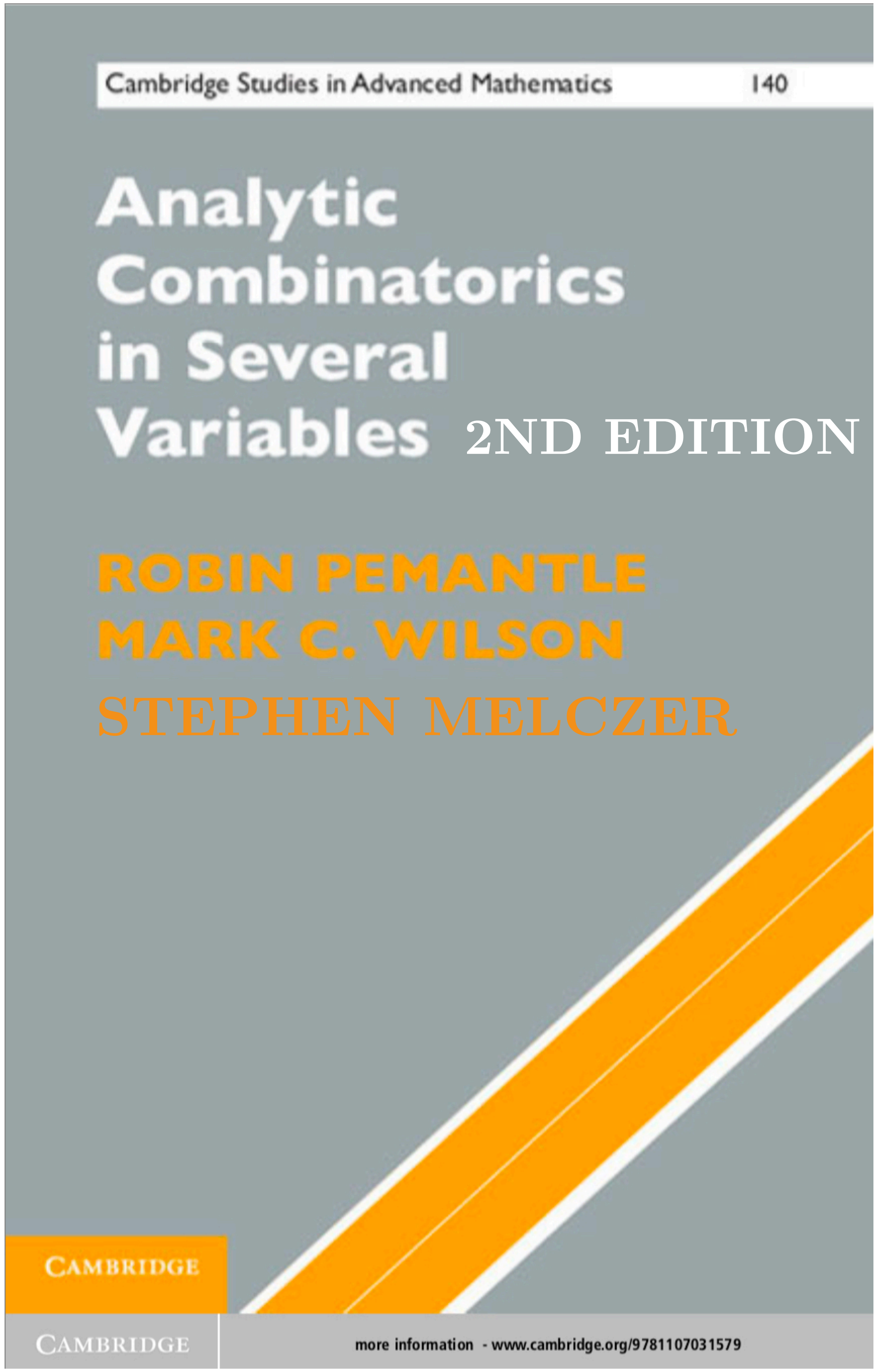
General Picture

- Compute critical points and sort by height
- Verify *critical points at infinity* don't interfere
- Determine the integer coefficients $\kappa_{\mathbf{w}}$ of highest crit pts
- Keep going until you get non-zero coefficients
- Try to approximate local integrals $\Psi_{\mathbf{w}}(n)$

Hardest part: Finding integer coefficients (and checking non-zero)

Currently we can only find the coefficients for **minimal critical points**, or in **dimension two**, or when H has only **linear factors**





For new researchers — focus more on explicit results and computation

Most general theory, covering topological approaches

THANK YOU!

An Invitation to Analytic Combinatorics
melczer.ca/textbook

melczer.ca/ALEA22