Cellular automata and percolation

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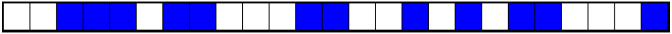
Content

Configurations

- **1-dimensional** configuration: **infinite** tape divided in regular **cells**, each one being in a given colour (finite number of possible colours).

. . .  . . .

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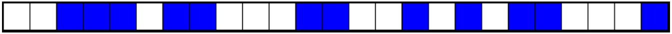
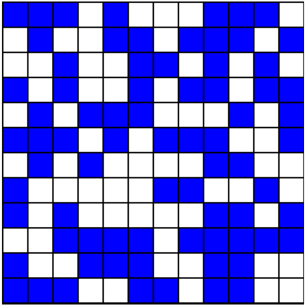
. . .  . . .

- **2-dimensional** configuration:



- We start from a **configuration**.
- All the cells update **simultaneously** their colour, and choose their new colour in function of the colours they observe in a **finite neighbourhood**.

If all cells apply simultaneously the same local rule, the update dynamics is called a **cellular automaton**.

Mathematical definition

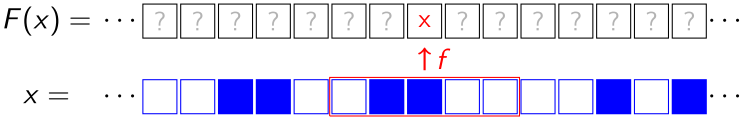
Let $\mathcal{A}$ be a finite set of symbols, called the **alphabet**.

We denote by $\mathcal{A}^{\mathbb{Z}^d}$ the set of **configurations**.
An element of $\mathcal{A}^{\mathbb{Z}^d}$ is a sequence $(x_k)_{k\in\mathbb{Z}^d}$, with $x_k \in \mathcal{A}$ for $k \in \mathbb{Z}^d$.

Definition

A map $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is a **cellular automaton** if there exists a **neighbourhood** $\mathcal{N} = (n_1, \ldots, n_\ell)$ and a **local function** $f : \mathcal{A}^\ell \to \mathcal{A}$ such that:

$$\forall k \in \mathbb{Z}^d, \quad F(x)_k = f(x_{k+n_1}, \ldots, x_{k+n_\ell}).$$



$\mathcal{A} = \{\square, \blacksquare\}$, $\mathcal{N} = (-2, -1, 0, 1, 2)$

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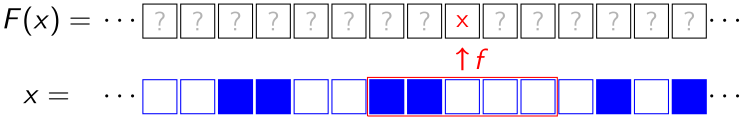

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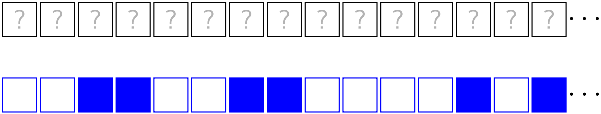
- One-dimensional majority CA of radius 1: each cell observes its own colour, and the colour of its left and right neighbours, and the new colour is the one that has a majority among the three.

$F(x) = \cdots$

$x = \cdots$



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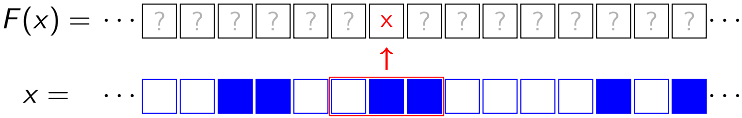

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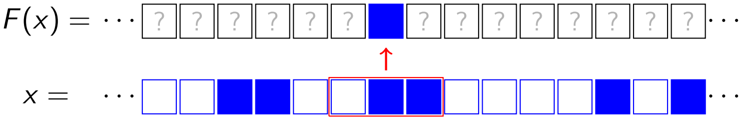

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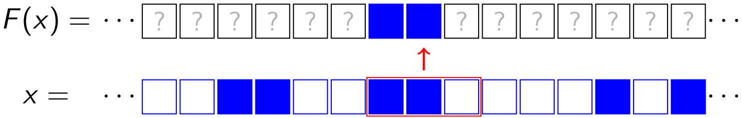

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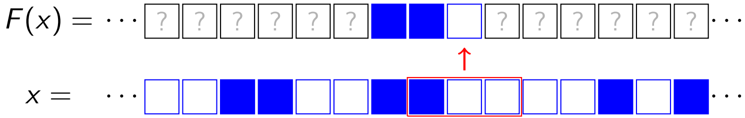

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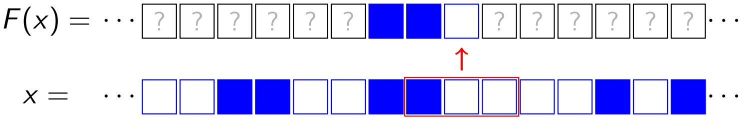
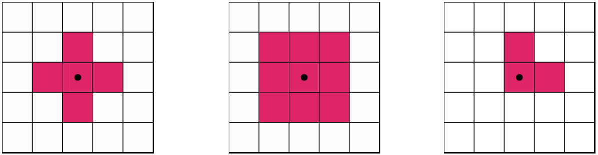


- Two-dimensional majority CA on various neighbourhoods...



Von Neumann   Moore   Toom

Parity cellular automaton (XOR)

A cell becomes blue if there is exactly one blue cell among its left and right neighbours.

$\mathcal{A} = \{0, 1\} \quad F(x)_k = x_{k-1} + x_{k+1} \mod 2$

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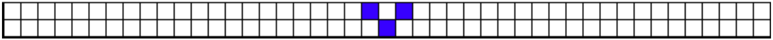

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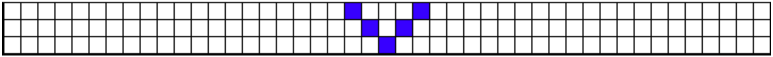

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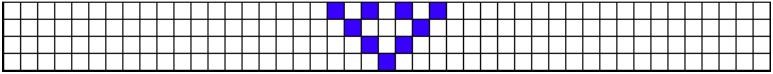

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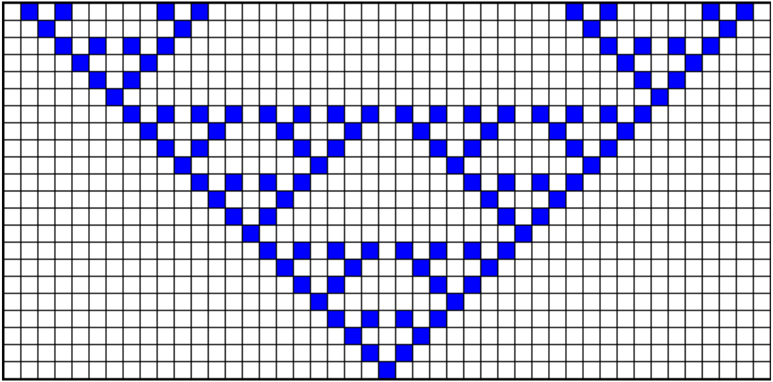

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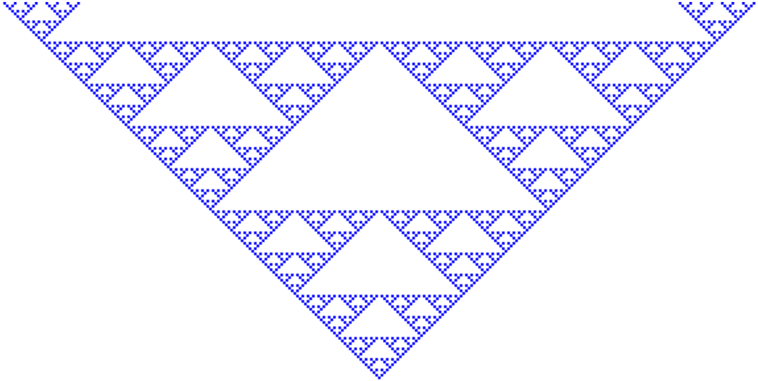
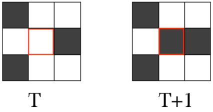
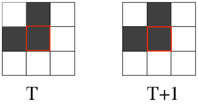


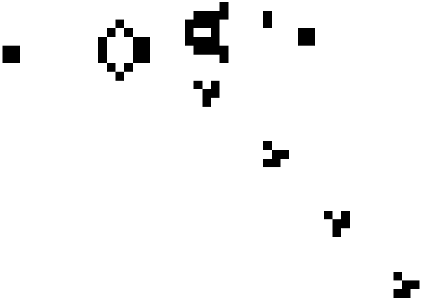
Conway's game of life

Birth
A dead cell with exactly three live neighbours becomes a live cell.



Survival
A live cell lives on to the next generation iff it has two or three live neighbours. Otherwise, it dies.





A Gosper's glider gun creating "gliders" Step 1 Images: Wikipedia

Conway's game of life

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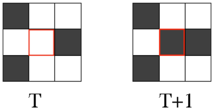

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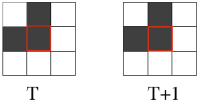

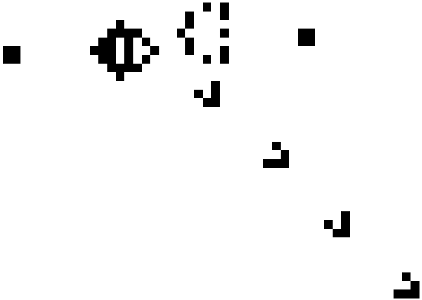

A Gosper's glider gun creating "gliders" Step 2 Images: Wikipedia

Conway's game of life

Birth

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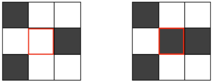

T T+1

Survival

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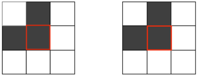

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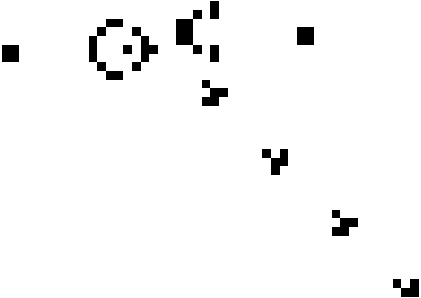

A Gosper's glider gun creating "gliders" Step 3 Images: Wikipedia

Conway's game of life

Birth

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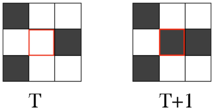

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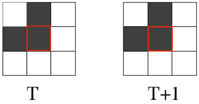

A Gosper's glider gun creating "gliders" Step 4 Images: Wikipedia

Conway's game of life

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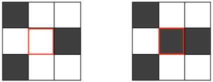

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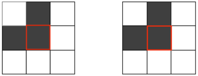

T T+1

A Gosper's glider gun creating "gliders" Step 5 Images: Wikipedia

Conway's game of life

Birth

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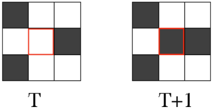

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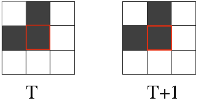
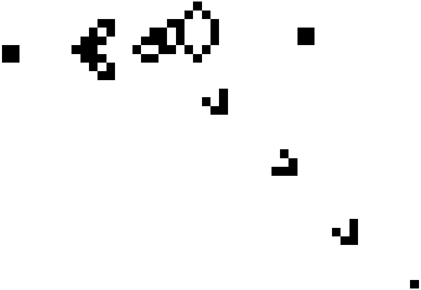


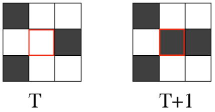


A Gosper's glider gun creating "gliders"   Step 6   Images: Wikipedia

Conway's game of life

Birth

A dead cell with exactly three live neighbours becomes a live cell.



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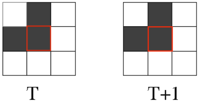
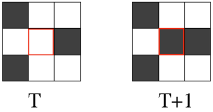


A Gosper's glider gun creating "gliders" Step 7 Images: Wikipedia

Conway's game of life

Birth

A dead cell with exactly three live neighbours becomes a live cell.



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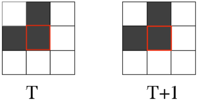

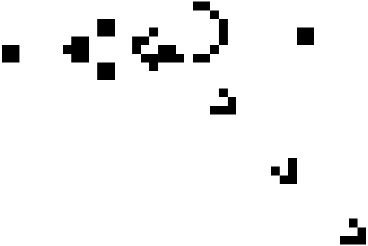

A Gosper's glider gun creating "gliders" Step 8 Images: Wikipedia

Conway's game of life

Birth

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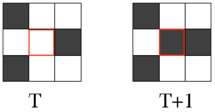

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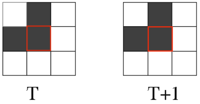

A Gosper's glider gun creating "gliders" Step 9 Images: Wikipedia

Birth

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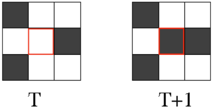

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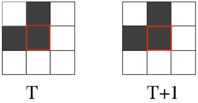

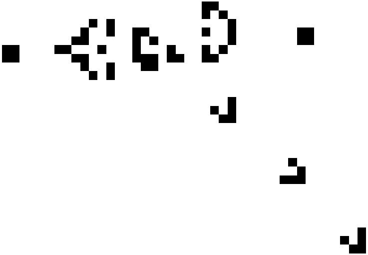

A Gosper's glider gun creating "gliders" Step 10 Images: Wikipedia

Conway's game of life

Birth

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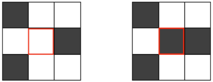

T T+1

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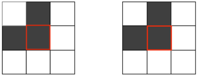

T T+1

A Gosper's glider gun creating "gliders" Step 11 Images: Wikipedia

Conway's game of life

Birth

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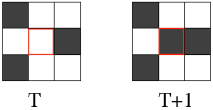

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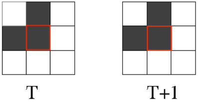

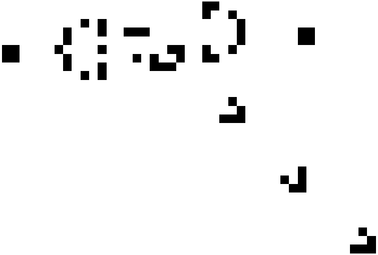

A Gosper's glider gun creating "gliders" Step 12 Images: Wikipedia

Conway's game of life

Birth	**Survival**
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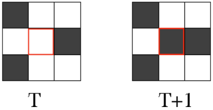
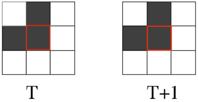
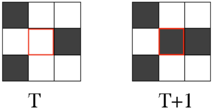




A Gosper's glider gun creating "gliders" Step 13 Images: Wikipedia

Conway's game of life

Birth
A dead cell with exactly three live neighbours becomes a live cell.



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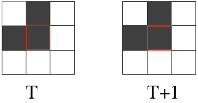
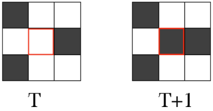


A Gosper's glider gun creating "gliders" Step 14 Images: Wikipedia

Conway's game of life

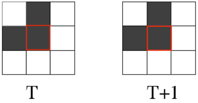
Birth

A dead cell with exactly three live neighbours becomes a live cell.



Survival

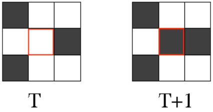
A live cell lives on to the next generation iff it has two or three live neighbours. Otherwise, it dies.

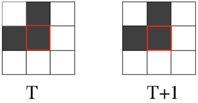


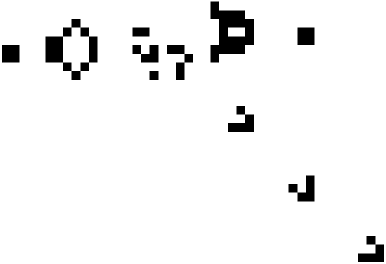
A Gosper's glider gun creating "gliders" Step 15 Images: Wikipedia

Conway's game of life

Birth	**Survival**
A dead cell with exactly three live neighbours becomes a live cell.	A live cell lives on to the next generation iff it has two or three live neighbours. Otherwise, it dies.





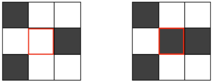


A Gosper's glider gun creating "gliders" Step 16 Images: Wikipedia

Conway's game of life

Birth

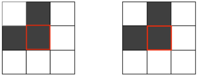
A dead cell with exactly three live neighbours becomes a live cell.



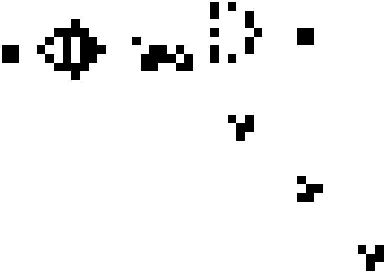
T T+1

Survival

A live cell lives on to the next generation iff it has two or three live neighbours. Otherwise, it dies.



T T+1

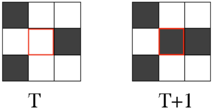


A Gosper's glider gun creating "gliders" Step 17 Images: Wikipedia

Conway's game of life

Birth

A dead cell with exactly three live neighbours becomes a live cell.



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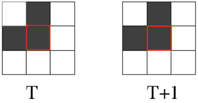
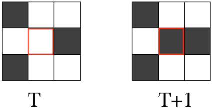


A Gosper's glider gun creating "gliders" Step 18 Images: Wikipedia

Conway's game of life

Birth

A dead cell with exactly three live neighbours becomes a live cell.



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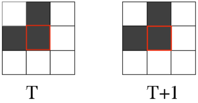
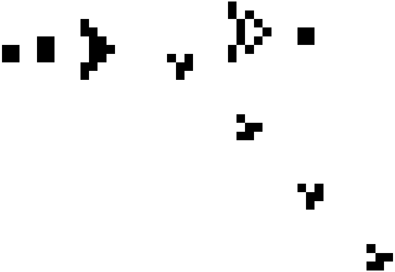


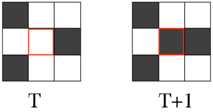


A Gosper's glider gun creating "gliders"   Step 19   Images: Wikipedia

Conway's game of life

Birth

A dead cell with exactly three live neighbours becomes a live cell.



Survival

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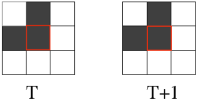
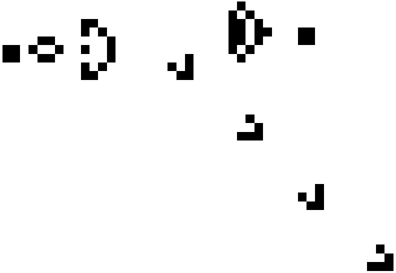


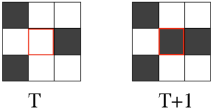


A Gosper's glider gun creating "gliders" Step 20 Images: Wikipedia

Conway's game of life

Birth

A dead cell with exactly three live neighbours becomes a live cell.



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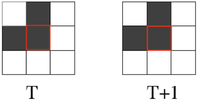
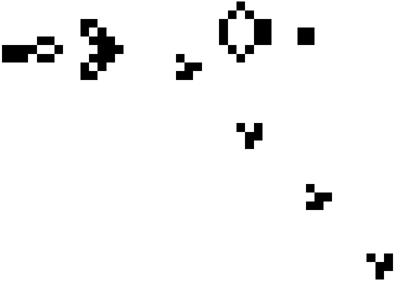




A Gosper's glider gun creating "gliders"

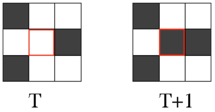
Step 21

Images: Wikipedia

Conway's game of life

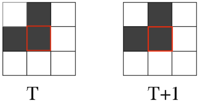
Birth

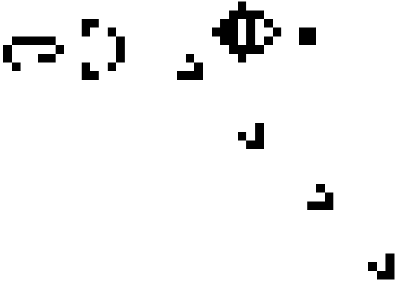
A dead cell with exactly three live neighbours becomes a live cell.



Survival

A live cell lives on to the next generation iff it has two or three live neighbours. Otherwise, it dies.



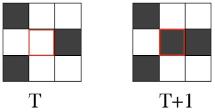


A Gosper's glider gun creating "gliders" Step 22 Images: Wikipedia

Conway's game of life

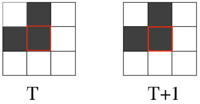
Birth

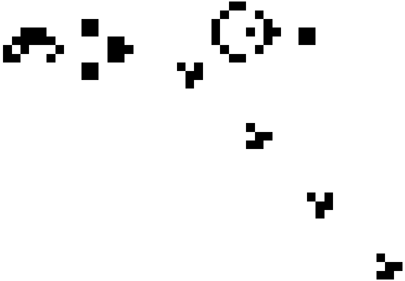
A dead cell with exactly three live neighbours becomes a live cell.



Survival

A live cell lives on to the next generation iff it has two or three live neighbours. Otherwise, it dies.

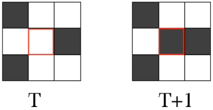


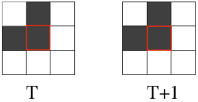


A Gosper's glider gun creating "gliders" Step 23 Images: Wikipedia

Conway's game of life

Birth	**Survival**
A dead cell with exactly three live neighbours becomes a live cell.	A live cell lives on to the next generation iff it has two or three live neighbours. Otherwise, it dies.

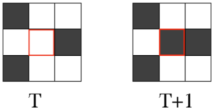




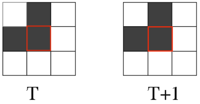
A Gosper's glider gun creating "gliders" Step 24 Images: Wikipedia

Conway's game of life

Birth
A dead cell with exactly three live neighbours becomes a live cell.



Survival
A live cell lives on to the next generation iff it has two or three live neighbours. Otherwise, it dies.

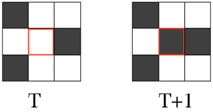


A Gosper's glider gun creating "gliders" Step 25 Images: Wikipedia

Conway's game of life

Birth

A dead cell with exactly three live neighbours becomes a live cell.



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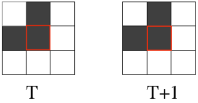
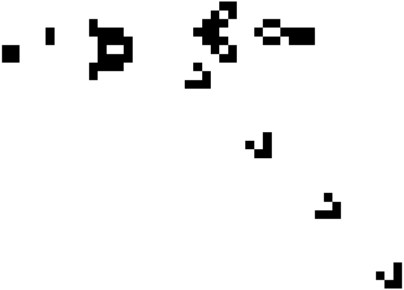


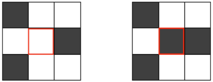


A Gosper's glider gun creating "gliders" Step 26 Images: Wikipedia

Conway's game of life

Birth

A dead cell with exactly three live neighbours becomes a live cell.



T T+1

Survival

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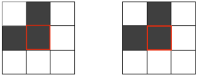
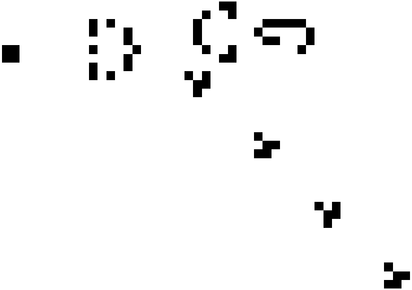


T T+1

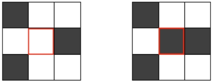


A Gosper's glider gun creating "gliders" Step 27 Images: Wikipedia

Conway's game of life

Birth

A dead cell with exactly three live neighbours becomes a live cell.



T T+1

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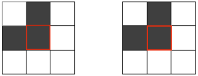
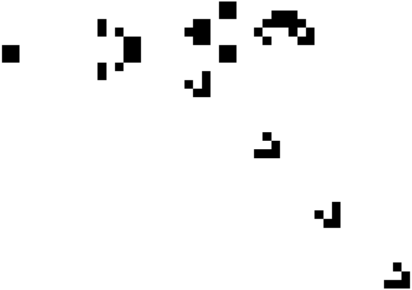


T T+1

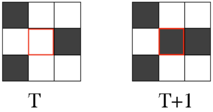


A Gosper's glider gun creating "gliders" Step 28 Images: Wikipedia

Conway's game of life

Birth

A dead cell with exactly three live neighbours becomes a live cell.



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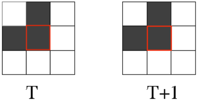

A Gosper's glider gun creating "gliders" Step 29 Images: Wikipedia

Conway's game of life

Birth

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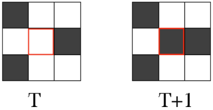

Survival

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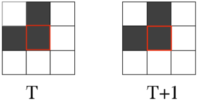

A Gosper's glider gun creating "gliders" Step 30 Images: Wikipedia

Modelling of **complex systems**, made of a large number of components, that evolve in time, and whose behaviour only depends on what they observe in some bounded neighbourhood.

Examples:

- Cellular tissues
- Computer networks
- Road traffic
- Swarms of birds
- Shell shapes

Complexity

Cellular automata: **simple definition** but **very complex evolutions**!

Cellular automata: **simple definition** but **very complex evolutions**!

Wolfram's classification (1981)

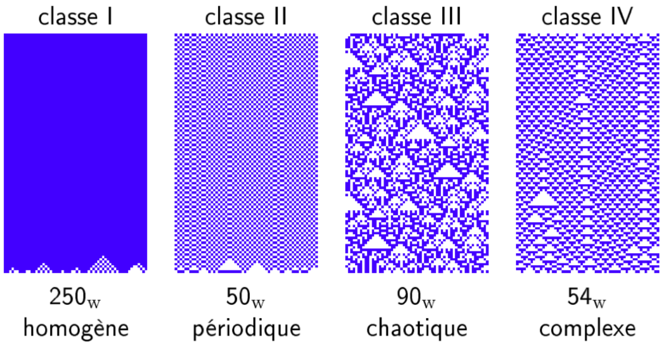

Image: N. Fatès

Curtis-Hedlund-Lyndon theorem

Theorem [Curtis-Hedlund-Lyndon 1969]

$F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a cellular automaton if and only if F is a **continous** function that comutes with the **shift map** σ.

Curtis-Hedlund-Lyndon theorem

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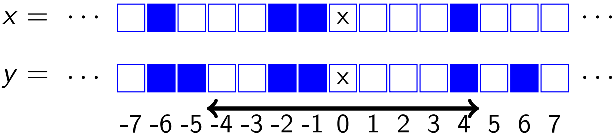
Example:
$d(x,y) = 2^{-5}$.



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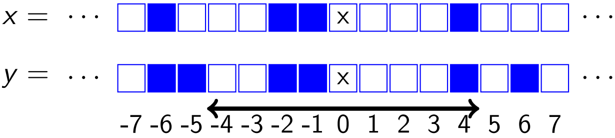
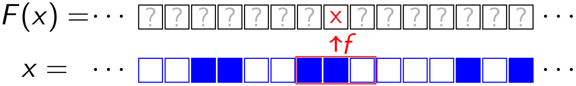


Continuity of CA: $d(x,y) < 2^{-n-r} \implies d(F(x),F(y)) < 2^{-n}$.

For **probabilistic** CA, the local function gives, for each element of $\mathcal{A}^\ell$, the probability of each symbol.

$$f : \mathcal{A}^\ell \to \mathcal{M}(\mathcal{A})$$



Probabilistic cellular automata

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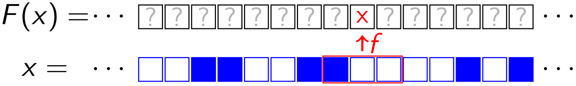

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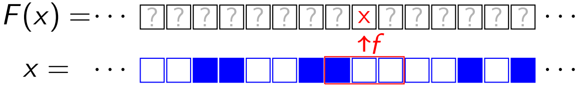

Probabilistic cellular automata

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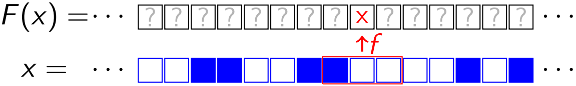
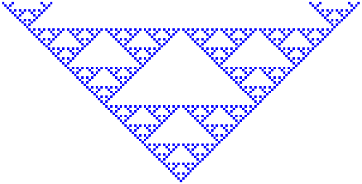


Example: parity CA with a probability ε of error.



$\varepsilon = 0$ $\qquad\qquad$ $\varepsilon = 0.01$

Probabilistic cellular automata

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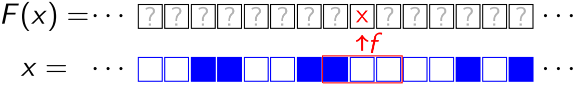

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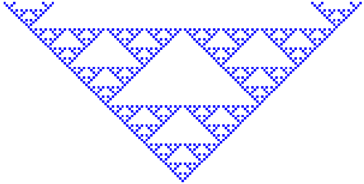
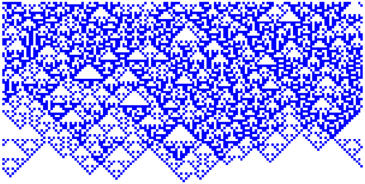


$\varepsilon = 0$



$\varepsilon = 0.01$

For a neighbourhood $\mathcal{N} = \{-r, \ldots, r\}$,

$$F\mu[y_m \ldots y_n] = \sum_{x_{m-r} \ldots x_{n+r} \in \mathcal{A}^{n-m+2r+1}} \mu[x_{m-r} \ldots x_{n+r}] \prod_{k=m}^{n} f(x_{k-r}, \ldots, x_{k+r})(y_k)$$

Probabilistic cellular automata

For a neighbourhood $\mathcal{N} = \{-r, \dots, r\}$,

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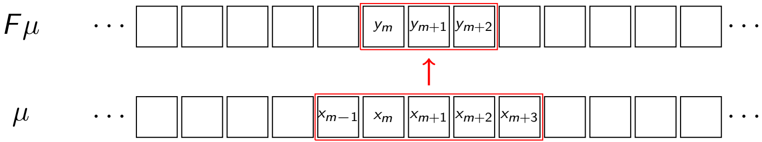
Illustration for $r = 1$ and $n = m + 2$:

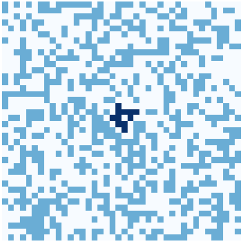


Content

Undirected site percolation

$p = 0.4$ $p = 0.5$ $p = 0.6$

Undirected site percolation



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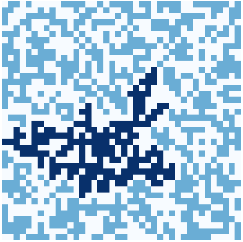

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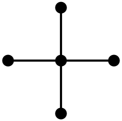
$p = 0.6$



Directed site percolation

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Directed site percolation

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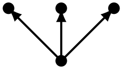
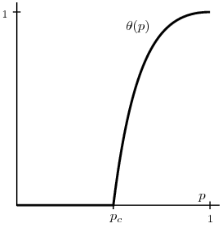


$\theta(p)$ = probability for the origin to belong to an infinite cluster

θ is a non-decreasing function

There exists a threshlod value p_c such that:

$$p < p_c \implies \theta(p) = 0$$
$$p > p_c \implies \theta(p) > 0$$



- Iteration of some 2D cellular automata from Bernoulli product configurations

- Space-time diagrams of 1D probabilistic cellular automata

Content

Bootstrap percolation

Definition

The bootstrap percolation CA is defined on $\{0,1\}^{\mathbb{Z}^2}$ as follows.

- A cell in state 1 always remains in state 1.
- A cell in state 0 having ≥ 2 neighbours in state 1 becomes in state 1.

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- A cell in state 1 always remains in state 1.
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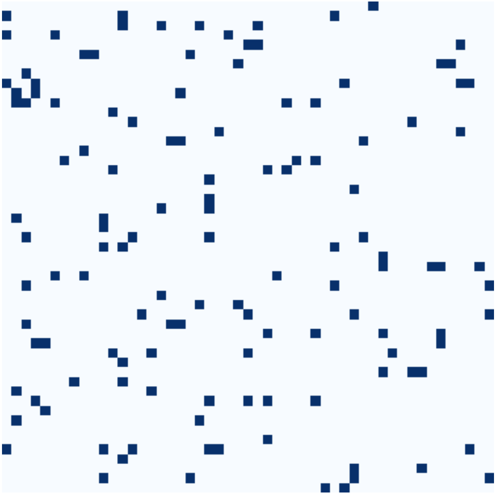
Let's choose the initial configuration according to the Bernoulli product measure of parameter p.

Experimentally, if p is not too small (say $p > 0.10$), the state 1 invades quickly the whole grid.
For p small, the picture is quite different...

Bootstrap percolation

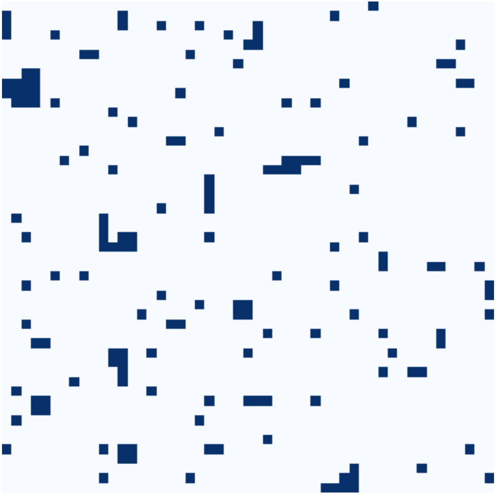
Simulation with $p = 0.05$.



Step 0

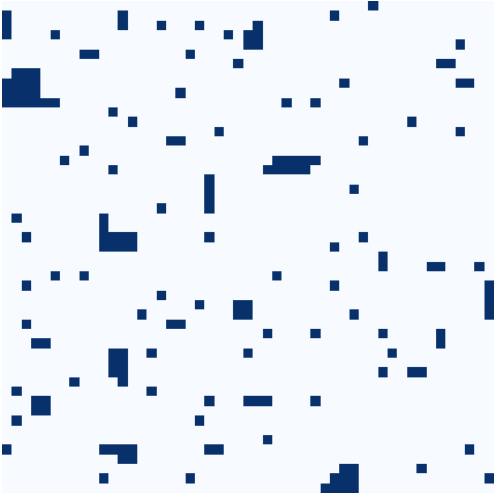
Bootstrap percolation

Simulation with $p = 0.05$.



Step 1

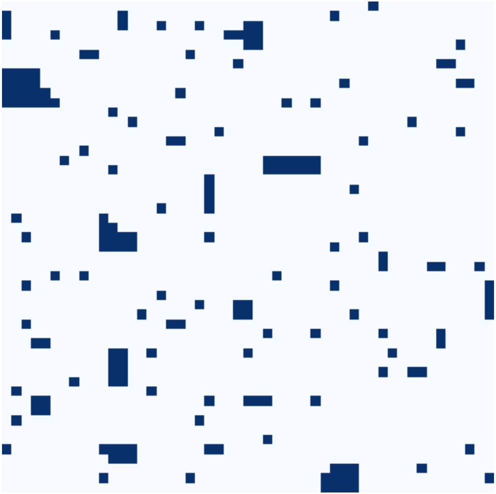
Simulation with $p = 0.05$.



Step 2

Bootstrap percolation

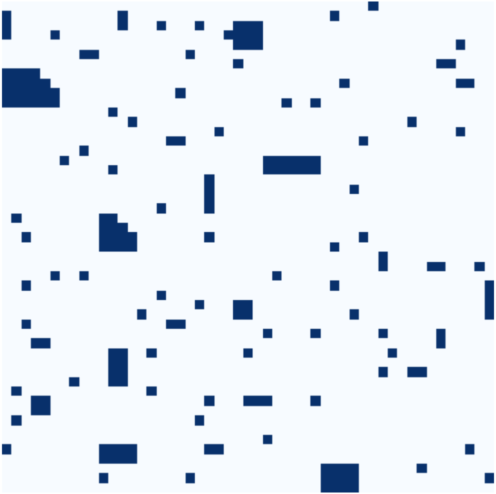
Simulation with $p = 0.05$.



Step 3

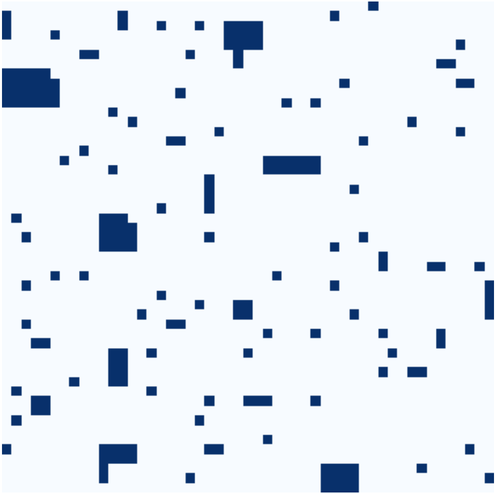
Bootstrap percolation

Simulation with $p = 0.05$.



Step 4

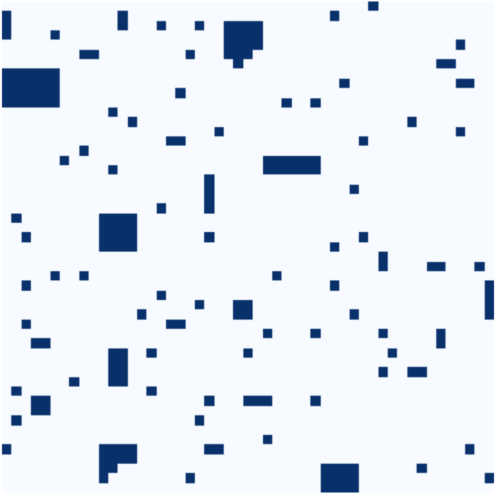
Simulation with $p = 0.05$.



Step 5

Bootstrap percolation

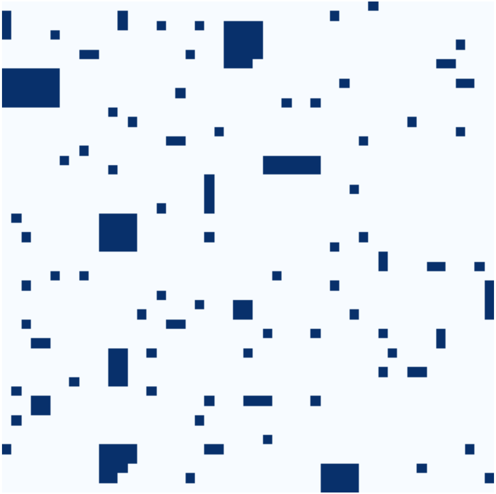
Simulation with $p = 0.05$.



Step 6

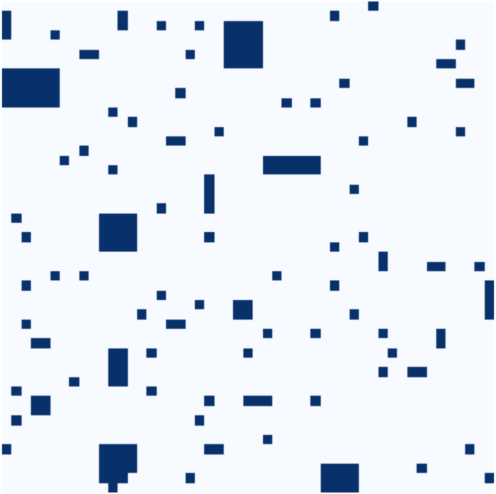
Bootstrap percolation

Simulation with $p = 0.05$.



Step 7

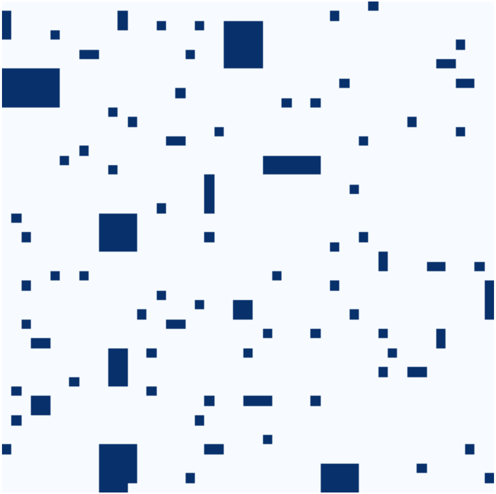
Simulation with $p = 0.05$.



Step 8

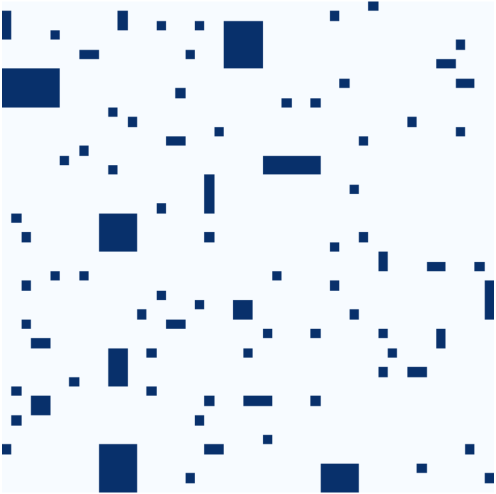
Bootstrap percolation

Simulation with $p = 0.05$.



Step 9

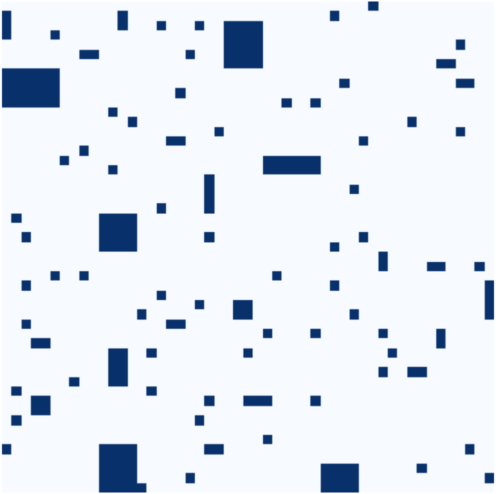
Simulation with $p = 0.05$.



Step 10

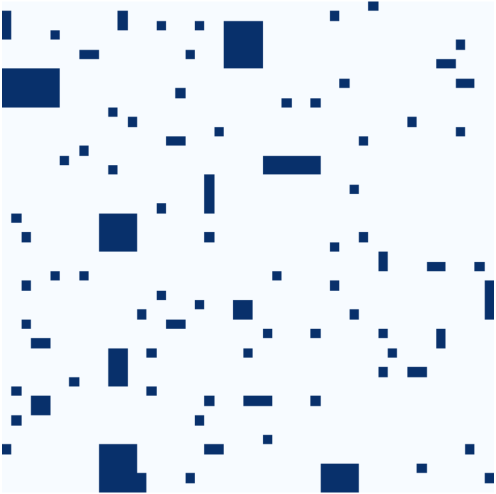
Bootstrap percolation

Simulation with $p = 0.05$.



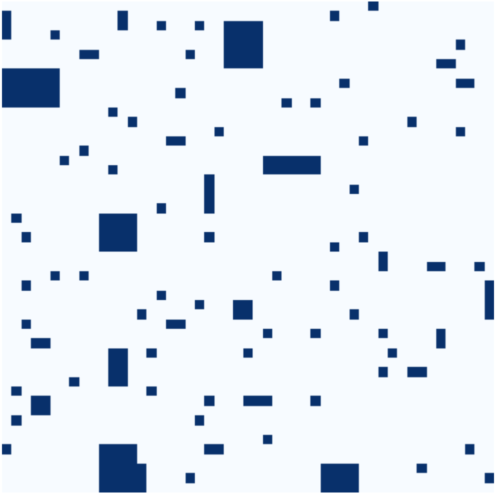
Step 11

Simulation with $p = 0.05$.



Step 12

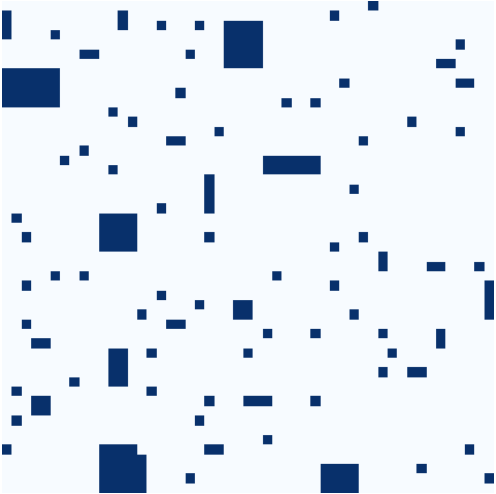
Simulation with $p = 0.05$.



Step 13

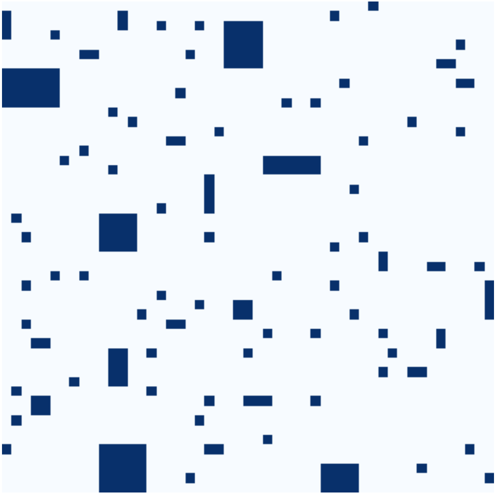
Bootstrap percolation

Simulation with $p = 0.05$.



Step 14

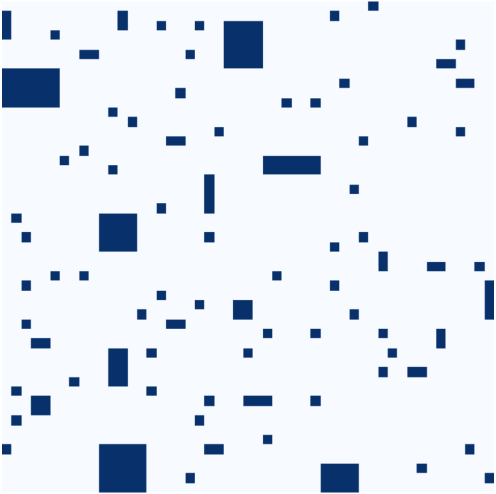
Simulation with $p = 0.05$.



Step 15

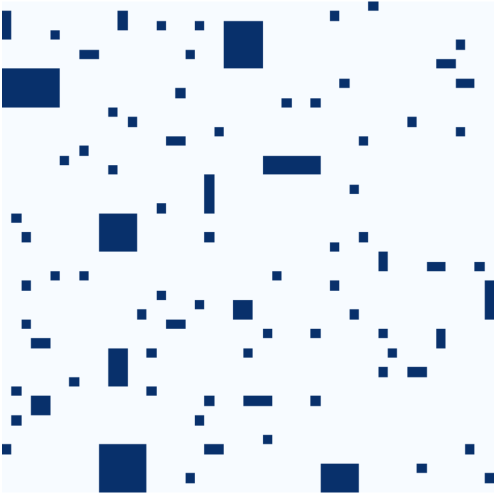
Bootstrap percolation

Simulation with $p = 0.05$.



Step 16

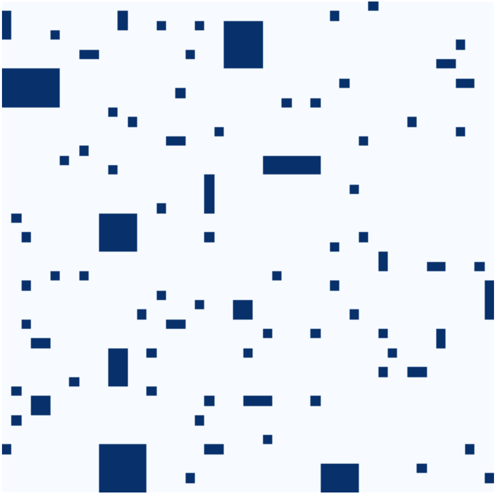
Simulation with $p = 0.05$.



Step 17

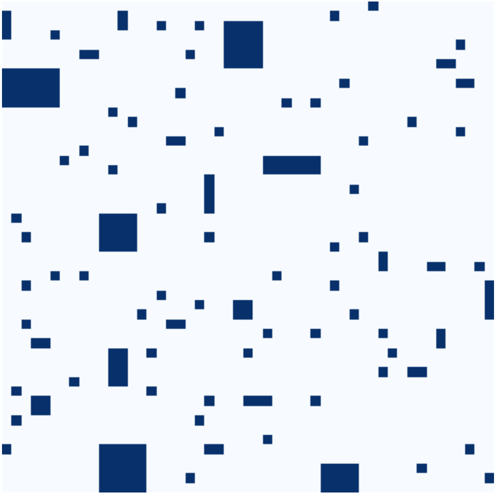
Bootstrap percolation

Simulation with $p = 0.05$.



Step 18

Simulation with $p = 0.05$.



Step 19

Bootstrap percolation

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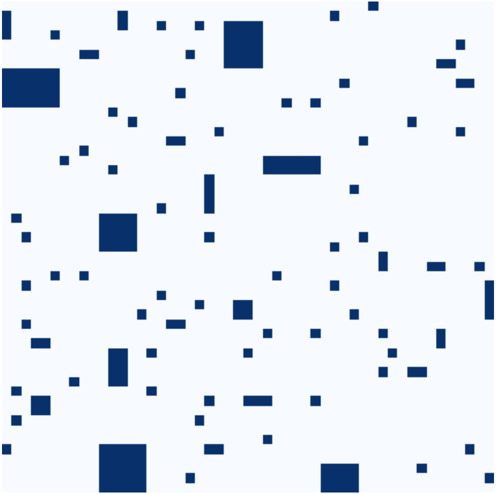

Step 20

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Theorem (van Enter 1987)

For any $p > 0$, the bootstrap CA on $\mathbb{Z}^2$ converges to the "all 1" configuration.

Bootstrap percolation

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For any $p > 0$, the bootstrap CA on $\mathbb{Z}^2$ converges to the "all 1" configuration.

Idea: prove that there is somewhere in the initial configuration an "all 1" square from which the whole configuration will be invaded.

This will happen iff this square is not surrounded by an "all 0" rectangle.

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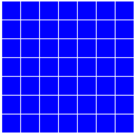

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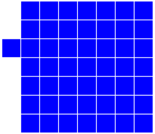

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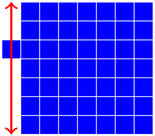

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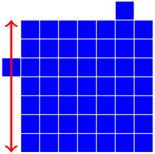

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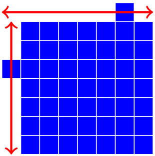

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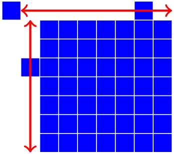

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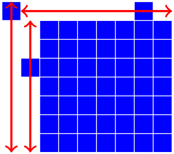

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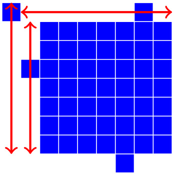

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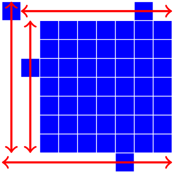

Take a fixed square C_N of $N \times N$ cells, and let $\varepsilon \in (0,1)$.

$$\mathbb{P}_p(C_N \text{ is surrounded by an empty rectangle}) < \sum_{k=4N}^{\infty} (1-p)^k \, \alpha_k,$$

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The number of shapes for a rectangle of length 2ℓ equals $\ell - 1$, and the number of rectangles of length 2ℓ and fixed shape surrounding the origin is $\leq (\ell-1)^2$.

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For any fixed p and ε, if N is large enough:

$$\mathbb{P}_p(C_N \text{ is surrounded by an empty rectangle }) < \varepsilon$$
$$\Longrightarrow \mathbb{P}_p(C_N \text{ is not surrounded by any empty rectangle}) > 1 - \varepsilon.$$

$$\mathbb{P}_p(C_N \text{ is all occupied and not surrounded by an empty rectangle})$$
$$> p^{N^2}(1-\varepsilon) > 0$$

By ergodicity of $\mathbb{P}_p$, the occurence of such a square C_N somewhere has probability 1.

Bootstrap percolation on finite grids

On a square grid of $N \times N$ cells, let:
$\alpha(N, p) =$ probability that the entire square is eventually occuppied

Let $(L_n)_{n \geq 0}, (p_n)_{n \geq 0}$ be such that $L_n \underset{n \to \infty}{\longrightarrow} \infty$ and $p_n \underset{n \to \infty}{\longrightarrow} 0$.

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Theorem (Holroyd 2003)

(i) If $\liminf_{n\to\infty} \; p_n \log L_n > \frac{\pi^2}{18}$, then $\lim_{n\to\infty} \alpha(L_n, p_n) = 1$.

(ii) If $\limsup_{n\to\infty} \; p_n \log L_n < \frac{\pi^2}{18}$, then $\lim_{n\to\infty} \alpha(L_n, p_n) = 0$.

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In other words, on a large $N \times N$ grid:
$p > \frac{\pi^2}{18 \log N} \implies$ convergence to total occupancy with high prob.
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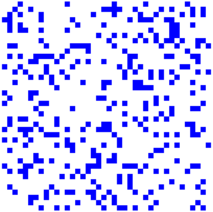
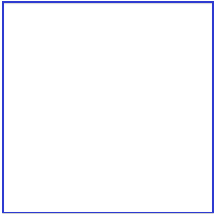
But the convergence is very slow: experimentally, the threshold value observed is not $\pi^2/18 \approx 0.548...$ but rather 0.245 ± 0.015 (!).

Content

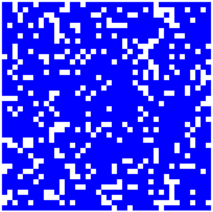
Density classification



evolution $\xrightarrow[t\to\infty]{}$



p=0.2



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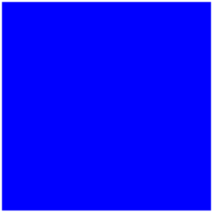
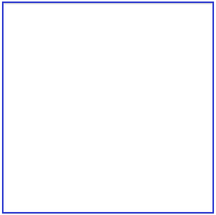


p=0.8

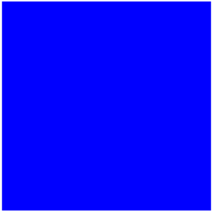
Density classification

$$\xrightarrow[t\to\infty]{\text{evolution}}$$



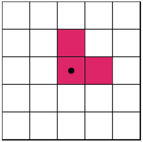
p=0.49

$$\xrightarrow[t\to\infty]{\text{evolution}}$$



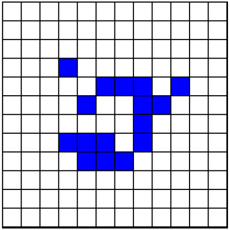
p=0.51

Majority CA



Proposition

The **majority CA on Toom's neighbourhood** is an eroder.



Rectangle of size $m \times n$: blue cells are erased in at most $m + n$ steps.

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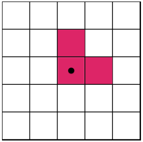

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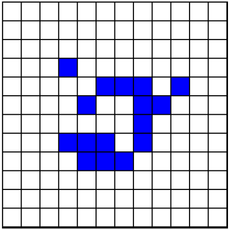

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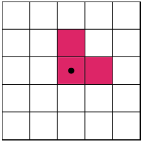

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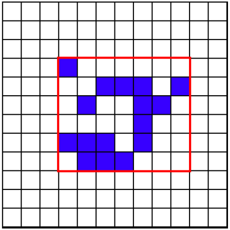

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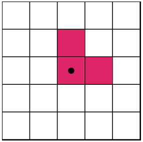

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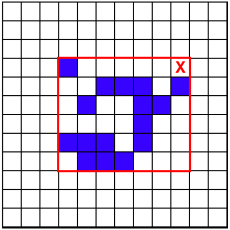

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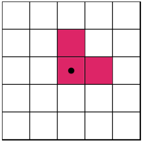

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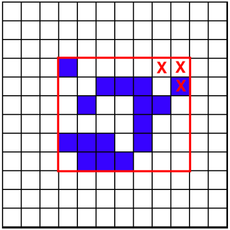

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Toom's CA classifies the density

Theorem (Bušić-Fatès-Mairesse-M. 2013)

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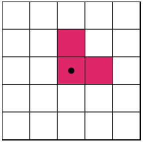

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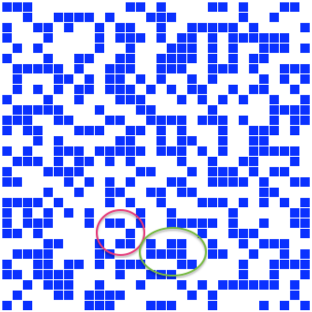

$t = 0$

Percolation theory

p = proportion of blue sites
$\theta(p)$ = probability for the origin to belong to an infinite cluster

$$p < p_c \implies \theta(p) = 0$$
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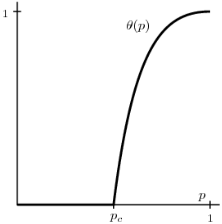

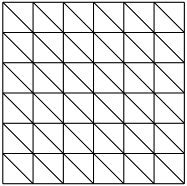

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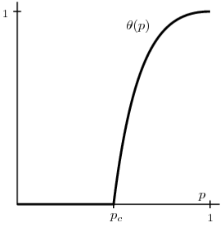

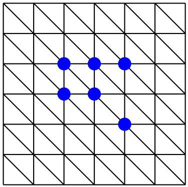

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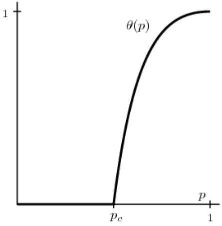

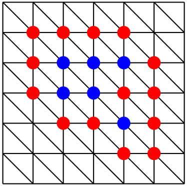

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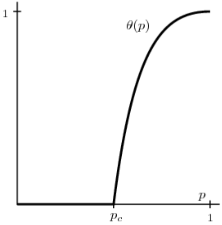

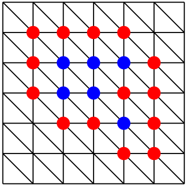

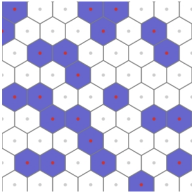

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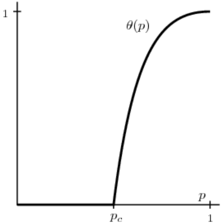

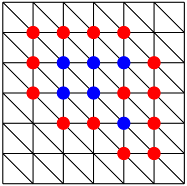

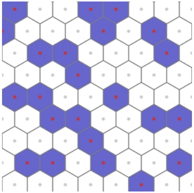

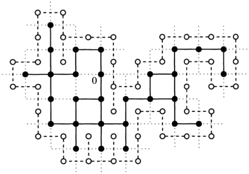

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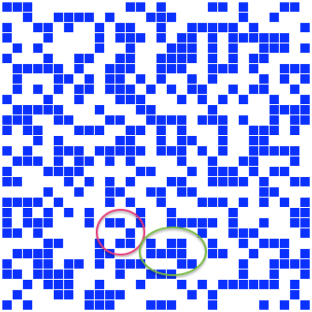

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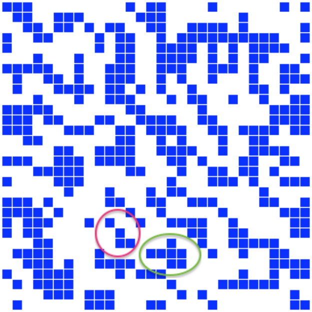

$t = 1$

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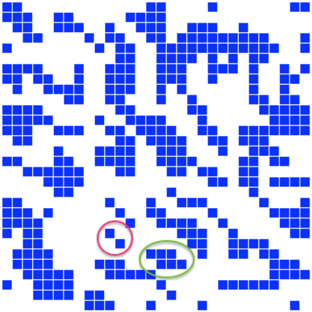

$t = 2$

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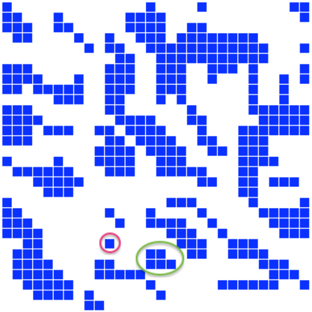

$t = 3$

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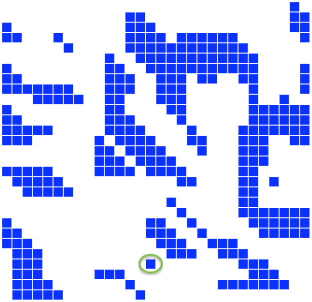

$t = 5$

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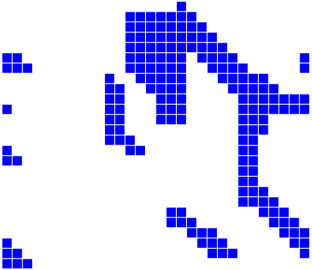

$t = 10$

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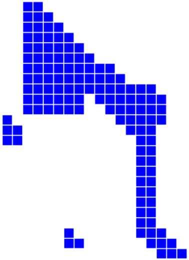

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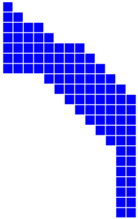

$t = 20$

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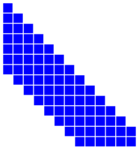

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And in dimension 1?

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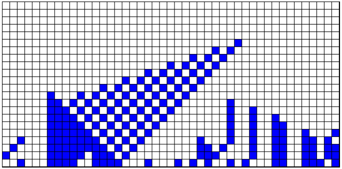

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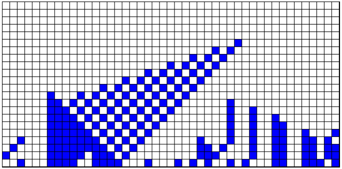

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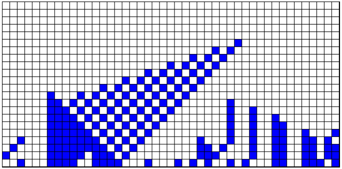

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In dim. 1, it is an open problem whether there exists a CA classifying the density!

Content

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Which PCA are ergodic?
How to compute their equilibrium distribution?

Positive rates conjecture

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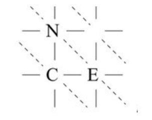
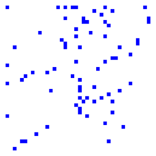
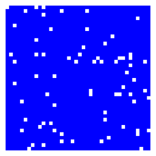
- Remark: **false** for $d \geq 2$

Counter-example for $d = 2$:
noisy version of Toom's majority CA







Positive rates conjecture

- Positive rates conjecture: for $d = 1$, if all the transition rates are > 0, then the PCA is ergodic.
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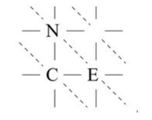

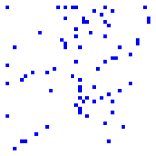

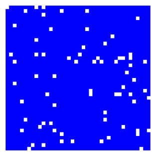

Counter-example for $d = 1$: very complicated! (Gács 2001)

Content

Perfect sampling for PCA

Let F be an ergodic PCA of invariant measure π.

Perfect sampling of π: probabilistic algorithm returning a sequence $a_1 \ldots a_n$ with *exactly* the probability it has to appear under the measure π (that is to say, with probability $\pi(\{x \in \mathcal{A}^{\mathbb{Z}} / x_1 \ldots x_n = a_1 \ldots a_n\}))$.

Aim: simulating the behaviour of the PCA after an infinity of iterations with a (hopefully) finite-time algorithm!

Idea: adapt the *coupling from the past* algorithm (Propp-Wilson 1996)

Update function of a PCA

A way to run a PCA (on $\mathcal{A} = \{0, 1\}$) from configuration $x \in \mathcal{A}^{\mathbb{Z}}$:

- generate for each cell k independently and uniformly a random number r_k in $[0, 1]$,
- choose the new state of the cell k to be

  $\mathbf{0}$ if $r_k < f((x_{k+v})_{v \in \mathcal{N}})(0)$, and $\mathbf{1}$ otherwise.

$$\ldots \quad x_{-3} \quad x_{-2} \quad x_{-1} \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad \ldots$$

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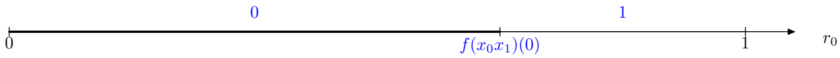

$$\mathbf{y_0}$$

$$\begin{array}{ccccccccc} \dots & x_{-3} & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & x_3 & \dots \\ \dots & r_{-3} & r_{-2} & r_{-1} & r_0 & r_1 & r_2 & r_3 & \dots \end{array}$$

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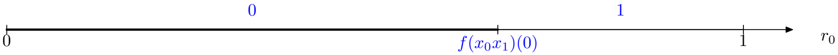


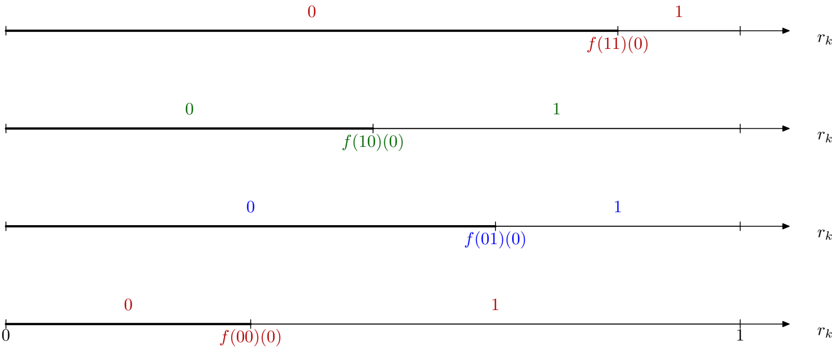
It defines an update function for F, given by:

$$\phi : \mathcal{A}^{\mathbb{Z}} \times [0,1]^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$$

$$\phi(x,r)_k = \begin{cases} \mathbf{0} \text{ if } r_k < f((x_i)_{i\in k+\mathcal{N}})(0) \\ \mathbf{1} \text{ otherwise.} \end{cases}$$

Update function of a PCA

Example: $\mathcal{A} = \{0, 1\}$, neighbourhood $\mathcal{N} = \{0, 1\}$



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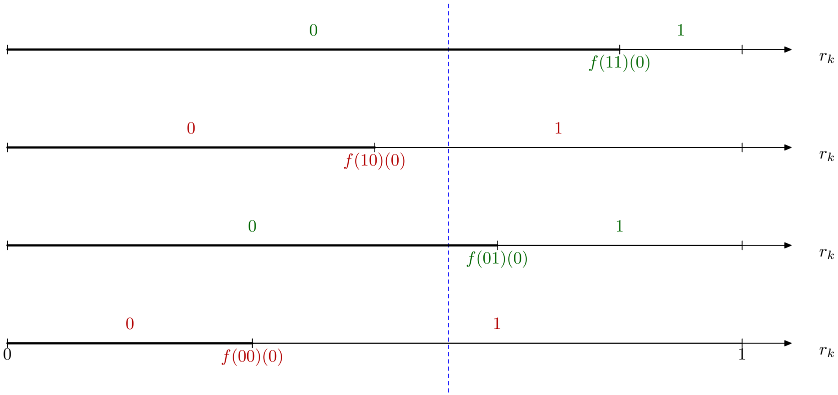

Update function of a PCA

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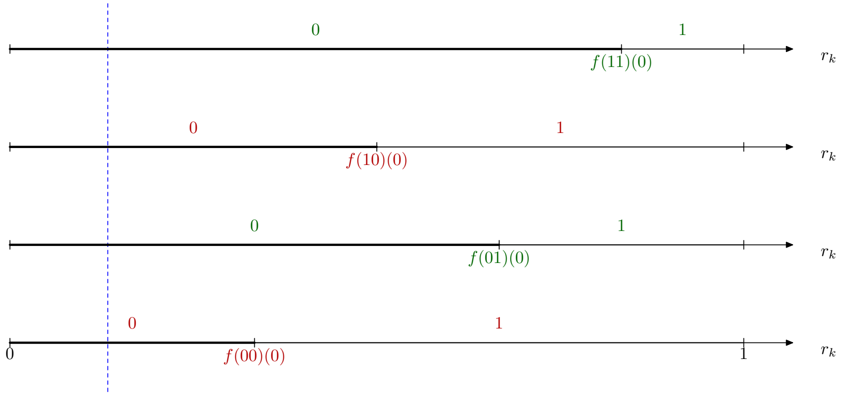

Update function of a PCA

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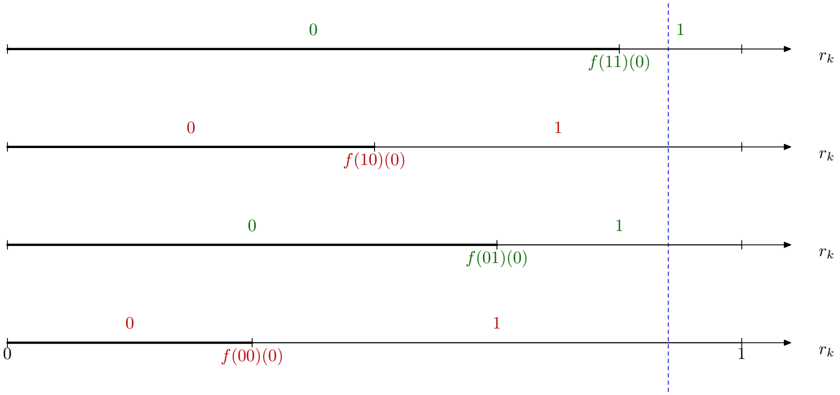
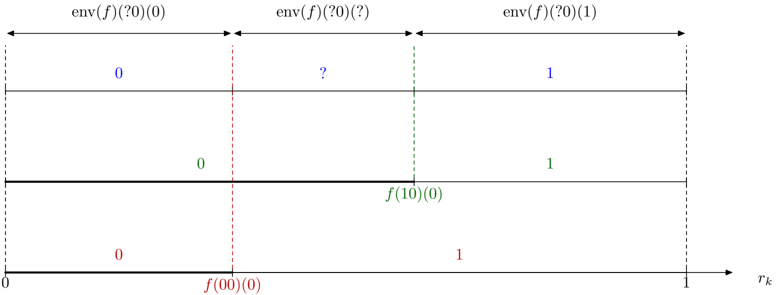


Envelope PCA

Introduction of an envelope PCA defined on the alphabet

$$\mathcal{B} = \{0 = \{0\}, 1 = \{1\}, ? = \{0, 1\}\},$$

to handle configurations partially known.



The update function $\tilde{\phi}$ of env(P) satisfies for $x \in \mathcal{A}^{\mathbb{Z}}$ and $y \in \mathcal{B}^{\mathbb{Z}}$,

$$x \in y \Rightarrow \forall r \in [0, 1]^{\mathbb{Z}}, \phi(x, r) \in \tilde{\phi}(y, r).$$

The envelope PCA

Definition of the envelope PCA

The PCA env(F) of alphabet $\mathcal{B} = \{0, 1, ?\}$, neighborhood $\mathcal{N}$, and local function env(f) is defined for $y \in \mathcal{B}^{\mathcal{N}}$ by

$$\text{env}(f)(y)(0) = \min_{x \in \mathcal{A}^{\mathcal{N}},\, x \in y} f(x)(0)$$

$$\text{env}(f)(y)(1) = \min_{x \in \mathcal{A}^{\mathcal{N}},\, x \in y} f(x)(1)$$

$$\text{env}(f)(y)(?) = 1 - \min_{x \in \mathcal{A}^{\mathcal{N}},\, x \in y} f(x)(0) - \min_{x \in \mathcal{A}^{\mathcal{N}}, x \in y} f(x)(1)$$

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In particular,

$$\mathrm{env}(f)(?^{\mathcal{N}})(?) = 1 - \min_{x \in \mathcal{A}^{\mathcal{N}}} f(x)(0) - \min_{x \in \mathcal{A}^{\mathcal{N}}} f(x)(1)$$

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In particular,

$$\begin{aligned} \text{env}(f)(?^{\mathcal{N}})(?) &= 1 - \min_{x \in \mathcal{A}^{\mathcal{N}}} f(x)(0) - \min_{x \in \mathcal{A}^{\mathcal{N}}} f(x)(1) \\ &= \max_{x \in \mathcal{A}^{\mathcal{N}}} f(x)(1) - \min_{x \in \mathcal{A}^{\mathcal{N}}} f(x)(1) \end{aligned}$$

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Let F be a PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

$a \quad b$

Coupling from the past algorithm

Let F be a PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0,1\}$, with $\mathcal{N} = \{0,1\}$.

? ?

Coupling from the past algorithm

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?   ?

?   ?   ?

$(r_i^1)_{0 \leq i \leq 2}$

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Let F be a PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

$$\begin{array}{ccc} ? & \mathbf{1} & \\ ? & ? & ? \end{array} \qquad (r_i^1)_{0 \le i \le 2}$$

Coupling from the past algorithm

Let F be a PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

?	1			
?	?	?		$(r_i^1)_{0 \leq i \leq 2}$
?	?	?	?	$(r_i^2)_{0 \leq i \leq 3}$

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?	1				
?	0	1			$(r_i^1)_{0 \leq i \leq 2}$
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?	?	?	?	?	$(r_i^3)_{0 \leq i \leq 4}$

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?	1				
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1	?	?	0		$(r_i^2)_{0 \leq i \leq 3}$
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?	1				
?	0	1			$(r_i^1)_{0 \leq i \leq 2}$
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?	?	?	?	?	$(r_i^3)_{0 \leq i \leq 4}$

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?	1					
?	0	1				$(r_i^1)_{0 \le i \le 2}$
1	?	?	0			$(r_i^2)_{0 \le i \le 3}$
?	?	?	?	?		$(r_i^3)_{0 \le i \le 4}$
?	?	?	?	?	?	$(r_i^4)_{0 \le i \le 5}$

Coupling from the past algorithm

Let F be a PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0,1\}$, with $\mathcal{N} = \{0,1\}$.

?	1					
?	0	1				$(r_i^1)_{0 \leq i \leq 2}$
1	?	?	0			$(r_i^2)_{0 \leq i \leq 3}$
?	0	?	1	?		$(r_i^3)_{0 \leq i \leq 4}$
?	?	?	?	?	?	$(r_i^4)_{0 \leq i \leq 5}$

Coupling from the past algorithm

Let F be a PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0,1\}$, with $\mathcal{N} = \{0,1\}$.

?	1					
?	0	1				$(r_i^1)_{0 \leq i \leq 2}$
1	1	?	0			$(r_i^2)_{0 \leq i \leq 3}$
?	0	?	1	?		$(r_i^3)_{0 \leq i \leq 4}$
?	?	?	?	?	?	$(r_i^4)_{0 \leq i \leq 5}$

Coupling from the past algorithm

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?	1					
1	0	1				$(r_i^1)_{0\leq i\leq 2}$
1	1	?	0			$(r_i^2)_{0\leq i\leq 3}$
?	0	?	1	?		$(r_i^3)_{0\leq i\leq 4}$
?	?	?	?	?	?	$(r_i^4)_{0\leq i\leq 5}$

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0	**1**					
1	0	1				$(r_i^1)_{0 \leq i \leq 2}$
1	1	?	0			$(r_i^2)_{0 \leq i \leq 3}$
?	0	?	1	?		$(r_i^3)_{0 \leq i \leq 4}$
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0	**1**					
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1	1	?	0			$(r_i^2)_{0 \leq i \leq 3}$
?	0	?	1	?		$(r_i^3)_{0 \leq i \leq 4}$
?	?	?	?	?	?	$(r_i^4)_{0 \leq i \leq 5}$

Proposition

If this algorithm stops a.s. then the PCA is ergodic, and the algorithm samples perfectly its unique invariant distribution.

Ergodicity criterion

Let F be a PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

?

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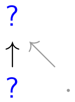

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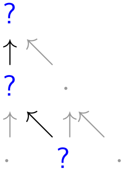

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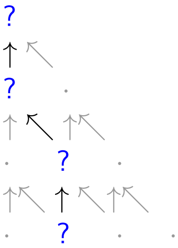

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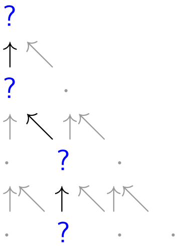

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Recall that: $p_? = 1-\min_{x\in\mathcal{A}^\mathcal{N}} f(x)(0) - \min_{x\in\mathcal{A}^\mathcal{N}} f(x)(1)$.

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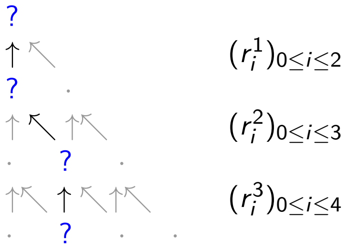

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For each cell, $\mathbb{P}(?) < p_?$ (domination by indep. Bernoulli)

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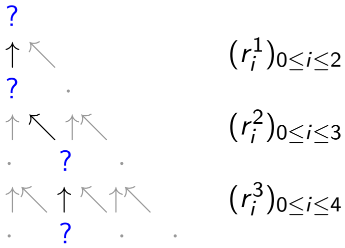

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For each cell, $\mathbb{P}(?) < p_?$ (domination by indep. Bernoulli)

If $p_? <$ directed percolation threshold, then the PCA is ergodic.

First ergodicity criterion

Proposition

Let $p_c(\mathcal{N})$ be the critical value of the two-dimensional directed site percolation of neighbourhood $\mathcal{N}$.

If $p_{?} < p_c(\mathcal{N})$, then the PCA is ergodic, and we can sample exactly its unique invariant measure using the CFTP algorithm.

Limits of the first ergodicity criterion

The algorithm stops a.s. iff the EPCA is ergodic. But there exist ergodic PCA for which the envelope PCA is not ergodic!

Limits of the first ergodicity criterion

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Example: parity CA with a probability ε of error.

$$f(x,y) = (1-\varepsilon)\,\delta_{x+y \bmod 2} + \varepsilon\,\delta_{x+y+1 \bmod 2}$$

For this PCA, we have $p_{?} = 1 - 2\varepsilon$.

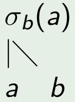
- This PCA is ergodic for all $\varepsilon \in (0,1)$ (convergence to the uniform measure).
- There exits $\varepsilon^* \in (0,1)$ such that the EPCA is ergodic if $\varepsilon > \varepsilon^*$, and non-ergodic if $\varepsilon < \varepsilon^*$.

Content

Permutive CA

Definition

A CA F of neighbourhood $\{0,1\}$ is **permutive** if:
$\forall b \in \mathcal{A},\ \exists \sigma_b \in \mathfrak{S}(\mathcal{A}),\ \forall a \in \mathcal{A},\ f(a,b) = \sigma_b(a).$



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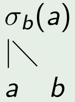

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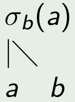

Example: $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$ and $f(a,b) = a + b$.

We consider the PCA F_ε that consists in applying the local rule of F with probability $1-\varepsilon$, and choosing a symbol uniformly at random with probability ε.

Proposition

For any $\varepsilon \in (0,1)$, the PCA F_ε is ergodic.

Permutive CA

For F, each $b \in \mathcal{A}$ induces a permutation $\sigma_b \in \mathfrak{S}(\mathcal{A}^n)$:

$$\begin{array}{llllll} t=1 & y_1 & y_2 & \dots & y_n & \\ t=0 & x_1 & x_2 & \dots & x_n & b \end{array}$$

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For F_ε, each $b \in \mathcal{A}$ induces a transition kernel P_b on $\mathcal{A}^n$: $\tilde{y}_1 \dots \tilde{y}_n \sim P_b\,(x_1 \dots x_n,\ \bullet\ )$.

$$\begin{array}{lccccc} t=1 & \tilde{y}_1 & \tilde{y}_2 & \cdots & \tilde{y}_n & \\ t=0 & x_1 & x_2 & \dots & x_n & b \end{array}$$

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P_b is aperiodic and irreducible, its invariant measure is the **uniform measure** on $\mathcal{A}^n$.

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$$\begin{array}{lccccc} t=2 & \tilde{z}_1 & \tilde{z}_2 & \cdots & \tilde{z}_n & \\ t=1 & \tilde{y}_1 & \tilde{y}_2 & \cdots & \tilde{y}_n & b \\ t=0 & x_1 & x_2 & \dots & x_n & b \end{array}$$

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$$\begin{array}{lccccc} t=3 & \cdot & \cdot & \cdots & \cdot & \\ t=2 & \tilde{z}_1 & \tilde{z}_2 & \cdots & \tilde{z}_n & b \\ t=1 & \tilde{y}_1 & \tilde{y}_2 & \cdots & \tilde{y}_n & b \\ t=0 & x_1 & x_2 & \dots & x_n & b \end{array}$$

P_b is aperiodic and irreducible, its invariant measure is the **uniform measure** on $\mathcal{A}^n$.

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For each $b \in \mathcal{A}$, there exists $\theta_b < 1$ such that:

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In particular, for $\nu = \lambda_n$ (uniform measure on $\mathcal{A}^n$), we obtain that for any distribution μ on $\mathcal{A}^n$ and any $b_1, \ldots, b_t \in \mathcal{A}$,

$$||P_{b_t} \ldots P_{b_2}P_{b_1}\mu - \lambda_n||_1 \leq \theta^t||\mu - \lambda_n||_1 \leq 2\theta^t.$$

Content

Nilpotent CA

F is **nilpotent** if there exists $N \geq 1$ such that F^N is constant.
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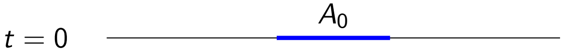

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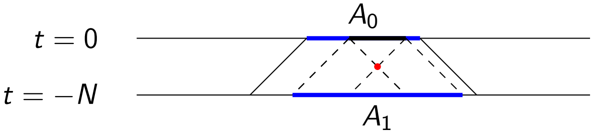

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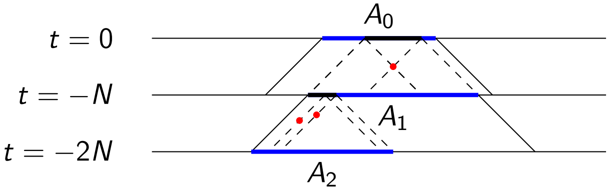

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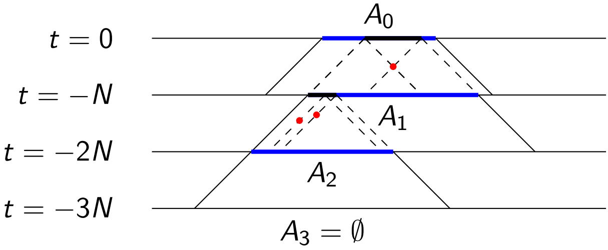
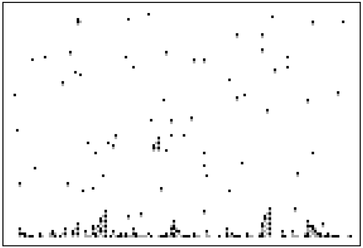


Nilpotent CA

$\varepsilon = 0$



$\varepsilon = 0.01$

$$F^{12}(x) = 0^{\mathbb{Z}} \text{ for all } x \in \{0,1,2\}^{\mathbb{Z}}$$

CA with a spreading state

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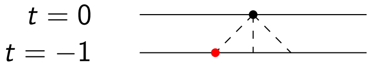

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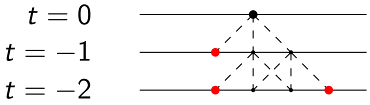

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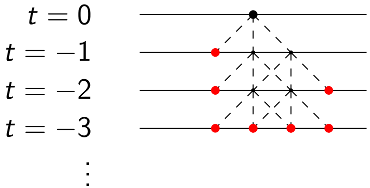

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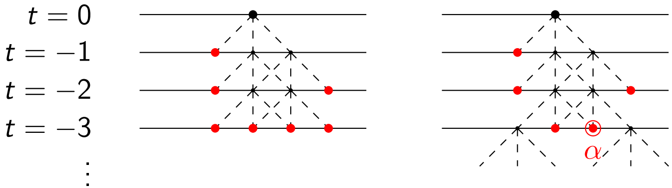

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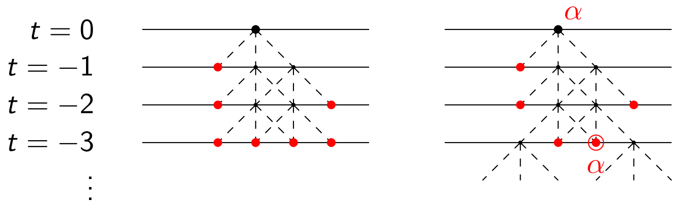

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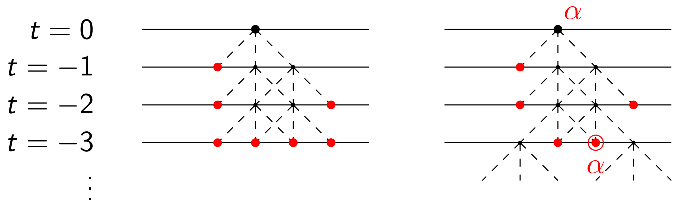
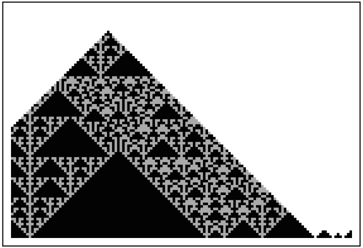


For ε small enough, also true for a general noise (with a different proof)...

CA with a spreading state



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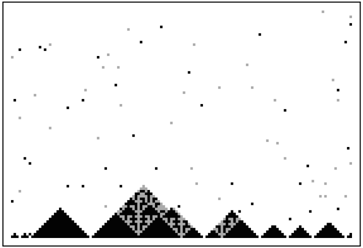
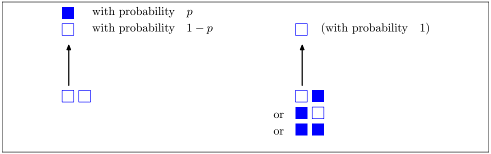


$\varepsilon = 0.01$

$$F(x)_i = x_{i-1} x_i x_{i+1} \mod 3$$

Content

The hardcore PCA



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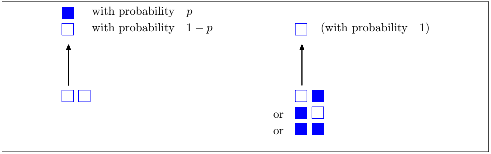

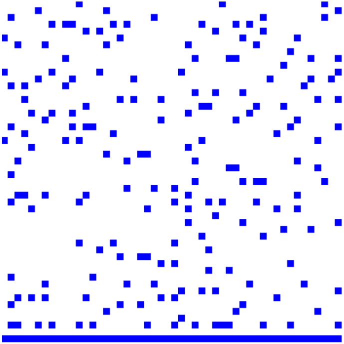

$p = 0.1$

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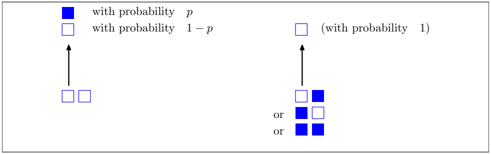

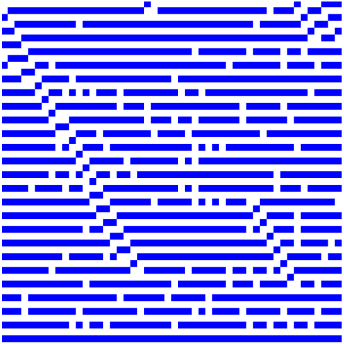

$p = 0.9$

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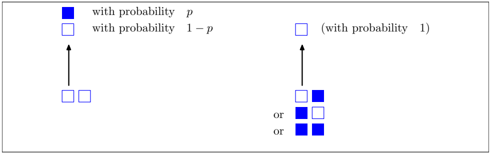
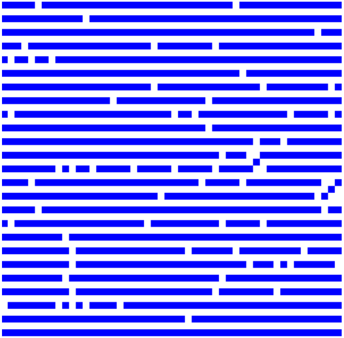
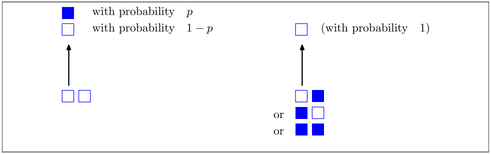




$p = 0.95$

The hardcore PCA



Here, $p_? = p$. The first ergodicity criterion proves the ergodicity only for $p < 0.7$ or so.

For which values of the parameter p is the PCA ergodic?

How can we describe its invariant measure(s)?

Motivations

- A model very easy to define!

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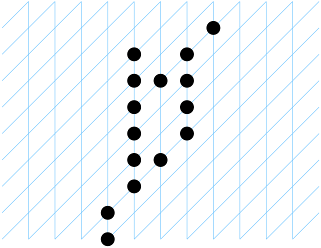
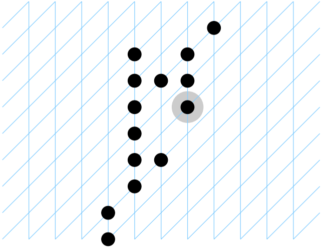
Definition of directed animals

Definition

A directed animal of **base** C is a finite subset of vertices of $\mathbb{Z} \times \mathbb{N}$, connected from $C \times \{0\}$ by links $\uparrow$ or $\nearrow$



A directed animal
(whose base has only one element)



Not a directed animal

Enumeration of directed animals

Counting series of directed animals of base C:

$$S_C(x) = \sum_{E:\text{DA of base } C} x^{|E|} = \sum_{n \geq 0} a_n(C) x^n,$$

where $a_n(C)$ = number of directed animals of base C and size n.

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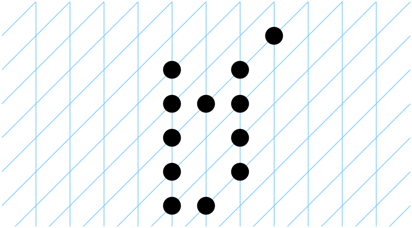
where $a_n(C)$ = number of directed animals of base C and size n.
Recurrence relation: $S_C(x) = x^{|C|} \left(\sum_{D \subset C + \{0,1\}} S_D(x) \right)$



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Let μ be an invariant measure of the PCA of parameter p, and let $X, Y \sim \mu$.

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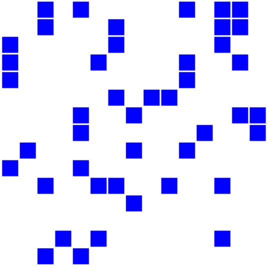
References: D. Dhar, M. Bousquet-Mélou, J.-F. Marckert, Y. Le Borgne, M. Albenque...

Content

Definition of the percolation game

Grid $\mathbb{N} \times \mathbb{N}$, with each site colored in blue independently with probability p (here, $p = 0.2$).



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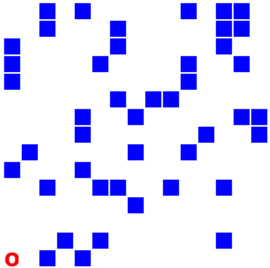

One token, that **two** players move alternatively, from position x to a white position among $x + (0, 1)$ or $x + (1, 0)$.

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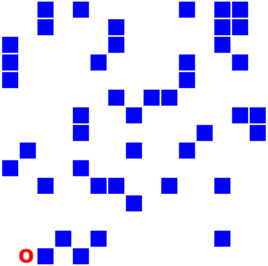

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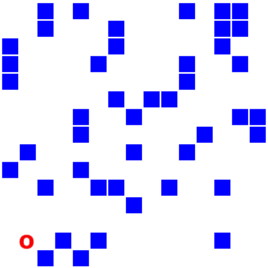

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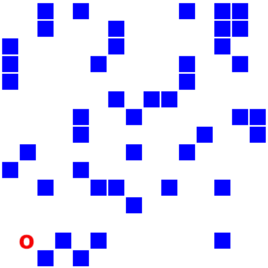
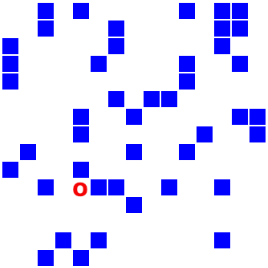


One token, that **two** players move alternatively, from position x to a white position among $x + (0, 1)$ or $x + (1, 0)$.
If both sites $x + (0, 1)$ and $x + (1, 0)$ are blue, then the player whose turn it is to play loses the game.

Definition of the percolation game

Grid $\mathbb{N} \times \mathbb{N}$, with each site colored in blue independently with probability p (here, $p = 0.2$).

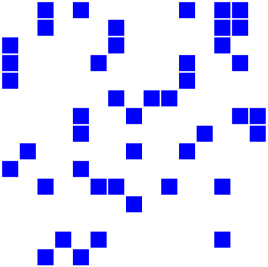


One token, that **two** players move alternatively, from position x to a white position among $x + (0,1)$ or $x + (1,0)$.
If both sites $x + (0,1)$ and $x + (1,0)$ are blue, then the player whose turn it is to play loses the game.

Winning positions, loss, and draws

A position is:

- a win (**W**) if from this position, the player whose turn it is to play has a winning strategy,
- a loss (**L**) if from this position, the other player has a winning strategy,
- a draw (**D**) if neither player has a winning strategy, so that with "best play", the game will continue for ever.



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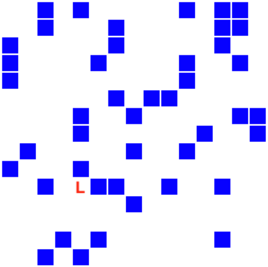

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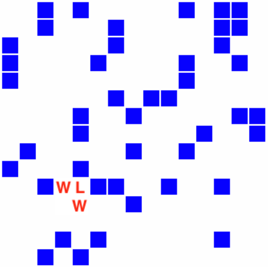

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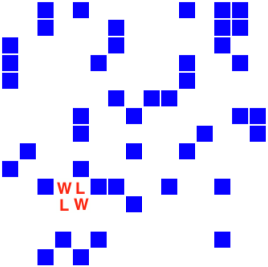

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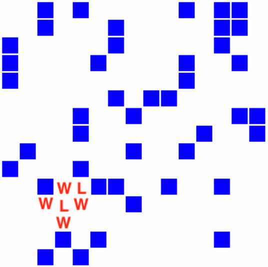

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Are there values of p for which there are **D** with a positive probability?
What is the probability for the origin to be **W**, **L**, or **D**?

The cellular automaton

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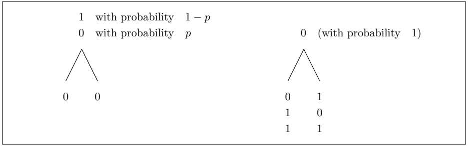
The **D** play the role of symbols "?".

Recoding

With the recoding ($\mathbf{L} = 1, \mathbf{W} = 0$), if we rotate the picture, we obtain the following PCA.



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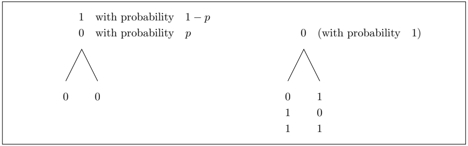

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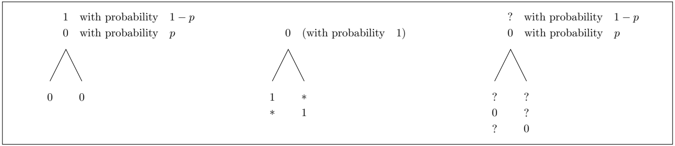

ACP F_p

Link with the ergodicity

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F_p ergodic $\Longleftrightarrow$ A_p ergodic

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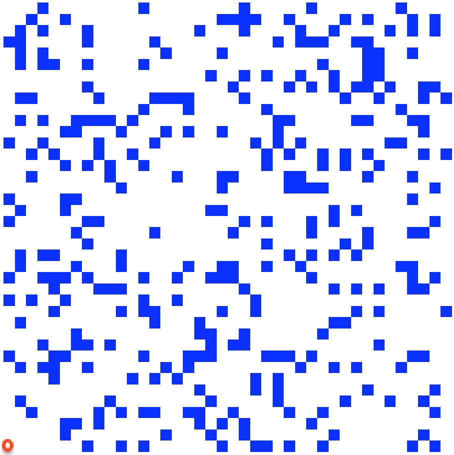

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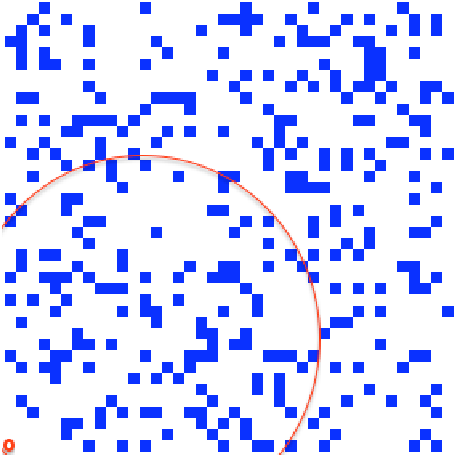

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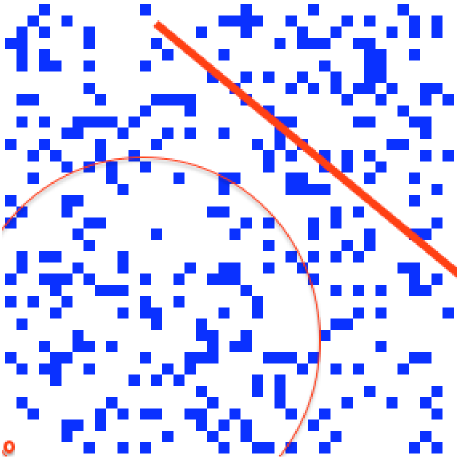

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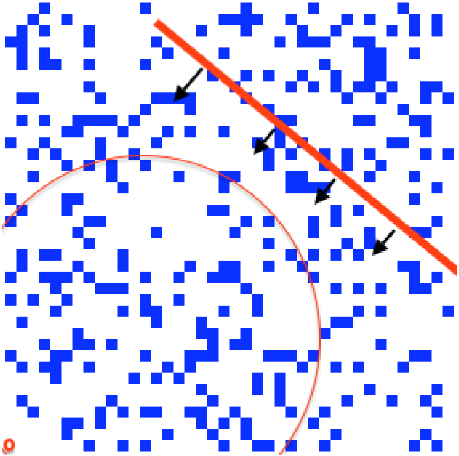

Content

Markovian invariant measure

One can show that for any value of p, the PCA has a Markovian invariant measure μ_p, given by the following transition matrix.

$$P = \begin{pmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{pmatrix} = \begin{pmatrix} \frac{2-p-\sqrt{p(4-3p)}}{2(1-p)^2} & \frac{2p^2-3p+\sqrt{p(4-3p)}}{2(1-p)^2} \\ \frac{-p+\sqrt{p(4-3p)}}{2(1-p)} & \frac{2-p-\sqrt{p(4-3p)}}{2(1-p)} \end{pmatrix}$$

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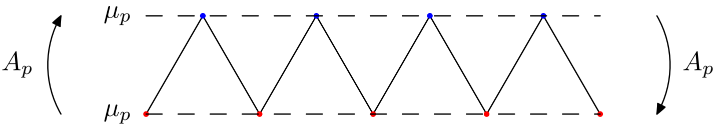

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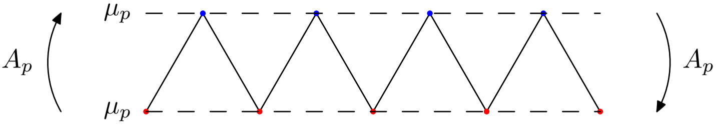
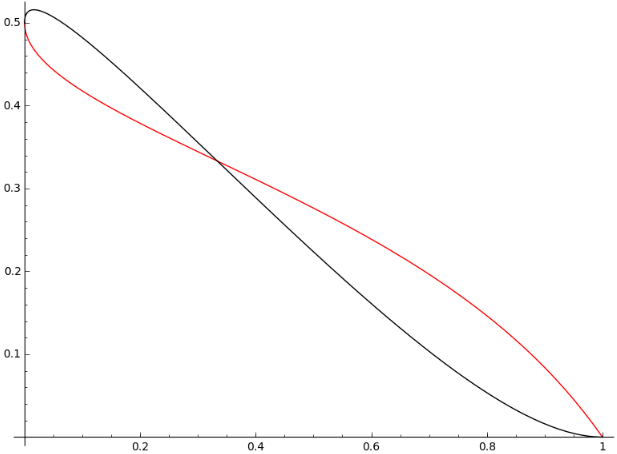
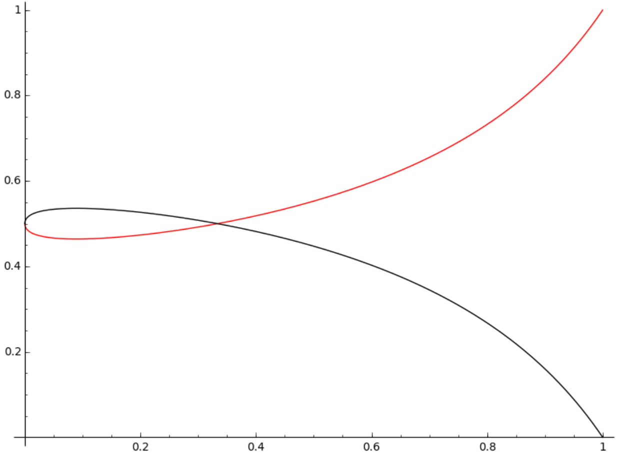


Theorem (Holroyd-M.-Martin 2018)

For any $p \in (0,1)$, the PCA A_p is ergodic.
Consequently, the probability of draws is 0 for the percolation game on $\mathbb{N}^2$.



Red: winning probability ; black: loss probability .



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Open questions

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- How can we describe the invariant measure(s) of a PCA?

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- In dimension 1, for **elementary PCA** (neighbourhood of size 2, binary states), is it true that if all the probability transitions are in $(0,1)$, then the PCA is ergodic?