Algorithmic Methods for Enumerative Combinatorics

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ALEA Days @ CIRM, Luminy

ÖAW RICAM

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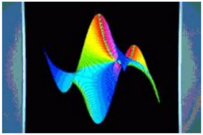
Computer algebra for Combinatorics

Alin Bostan & Bruno Salvy



Algorithms Project, INRIA

ALEA 2012

Overview

Today

1. Introduction
2. High Precision Approximations
   – Fast multiplication, binary splitting, Newton iteration
3. Tools for Conjectures
   – Hermite-Padé approximants, p-curvature

Tomorrow morning

4. Tools for Proofs
   – Symbolic method, resultants, D-finiteness, creative telescoping

Tomorrow night

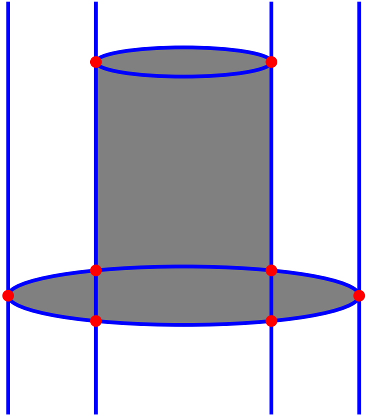
– Exercises with Maple

Plan of the Talk

1. Cylindrical Algebraic Decomposition (CAD)
   - unimodality of q-binomial coefficients
   - exact lower bounds for monochromatic Schur triples
   - proving inequalities among sequences
2. Lattice Reduction (LLL)
   - finding integer relations
   - guessing with little data
3. Creative Telescoping
   - D-finite functions and P-recursive sequences
   - proving special function identities
   - recurrences for balanced / pattern-avoiding matrices

Part 1

Cylindrical Algebraic Decomposition



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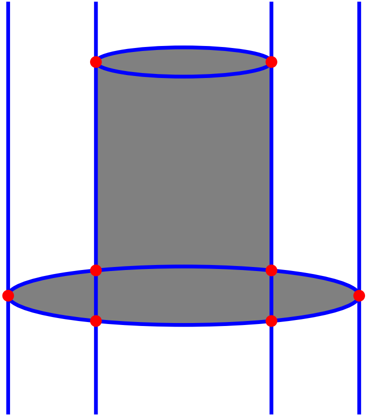

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Tarski–Seidenberg theorem. Every set in $\mathbb{R}^{n+1}$ defined by polynomial equations and inequalities can be projected down onto $\mathbb{R}^n$ such that the resulting set is still definable in terms of polynomial equations and inequalities.

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- Nowadays this is a fundamental algorithm for computer algebra and real algebraic geometry.
- It has much better complexity (but still doubly exponential).

Tarski Formulas

Definition. A Tarski formula is constructed from

- polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$
- relational symbols $<, \leqslant, >, \geqslant, \neq, =$
- logical connectives $\neg, \wedge, \vee, \implies, \iff$
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$$x^2 + y^2 + z^2 \leqslant 1 \wedge 2x - 3y + 5z > 0$$

$$\forall x \in \mathbb{R}\colon \exists y > 0\colon y^3 + xy^2 - 3xy \geqslant x^2(2y - 1)$$

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$$\forall x \in \mathbb{R} \colon x^4 + ax^2 + 2x + a \geqslant 1$$

Cylindrical Algebraic Decomposition

Algebraic decomposition: A set $p_1, \ldots, p_m \in \mathbb{Q}[x_1, \ldots, x_n]$ induces a partition of $\mathbb{R}^n$ into sign-invariant cells, i.e., connected sets in which the signs $(+, -, 0)$ of all p_i don't change.

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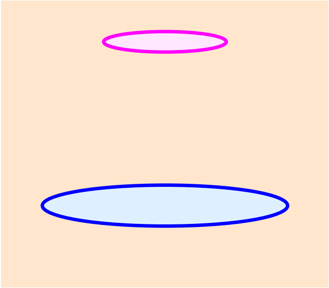
Example. $p_1(x, y) = x^2 + (6y)^2 - 9$, $p_2(x, y) = (2x)^2 + (12(y - 4))^2 - 9$. How many cells do we get?



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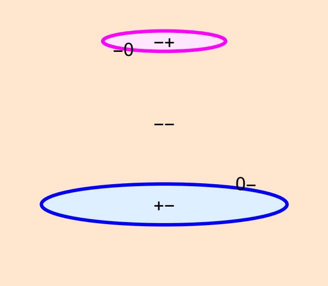

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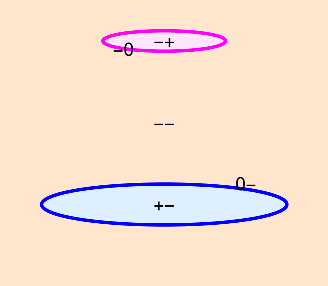

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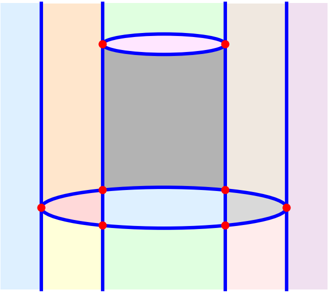

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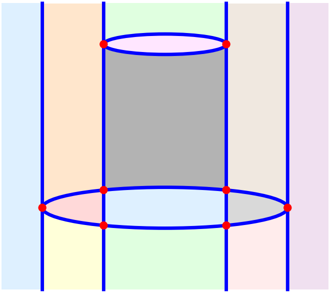

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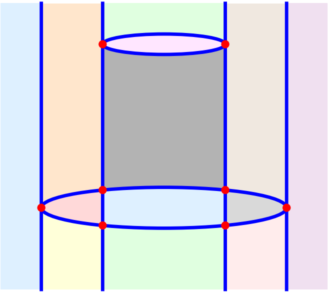

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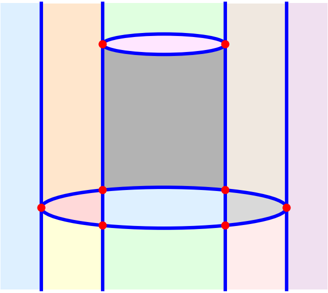

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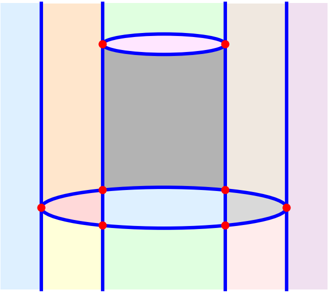

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How many cells do we get? 13 (2D) + 20 (1D) + 8 (0D) = 41.

Structure of CAD Formulas

A formula in a single variable x is in CAD format if it is of the form

$$\Phi_1 \vee \Phi_2 \vee \cdots \vee \Phi_m,$$

where each Φ_k is either $x < \alpha$ or $\alpha < x < \beta$ or $x > \beta$ or $x = \gamma$ for some real algebraic numbers α, β, γ with $\alpha < \beta$, such that any two Φ_k are inconsistent.

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A formula in n variables $x_1, \ldots, x_n$ is in CAD format if it is of the form

$$(\Phi_1 \wedge \Psi_1) \vee (\Phi_2 \wedge \Psi_2) \vee \cdots \vee (\Phi_m \wedge \Psi_m),$$

where the Φ_k are such that $\Phi_1 \vee \Phi_2 \vee \cdots \vee \Phi_m$ is in CAD format with respect to x_1 and the Ψ_k are satisfiable formulas which are in CAD format with respect to $x_2, \ldots, x_n$ whenever x_1 is replaced by a real algebraic number satisfying Φ_k.

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$$w > 0 \wedge 2w < a \leqslant w + \sqrt{3}w$$

Exercises

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Exercise 2. Is the bound given in Example 2 sharp? If not, determine such a sharp bound.

Unimodality of Gaussian Polynomials
(joint work with Ali Uncu and Elaine Wong)

Definition. A finite sequence of real numbers $a_1, \ldots, a_n$ is called d-strictly increasing (resp. decreasing) if $a_{k+1} - a_k \geqslant d$ (resp. $a_k - a_{k+1} \geqslant d$) holds for all $1 \leqslant k < n$.

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Definition. A sequence is called **unimodal** if for some $m \in \mathbb{N}$ we have non-decreasing (i.e., 0-strictly increasing) behavior up to m and subsequently non-increasing behavior:

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It is called **d-strictly unimodal** if the subsequence $a_1, \ldots, a_m$ is d-strictly increasing and $a_m, \ldots, a_n$ is d-strictly decreasing.

Gaussian Polynomials

Definition. For $\ell, m \in \mathbb{N}$, the **q-binomial coefficient**, defined by

$$\begin{bmatrix} \ell + m \\ m \end{bmatrix}_q := \prod_{i=1}^{m} \frac{1 - q^{\ell+i}}{1 - q^i} = \sum_{k=0}^{\ell m} p_k(\ell, m) \cdot q^k,$$

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Pak and Panova (2013) proved that the sequence $p_k(\ell, m)$, $1 \leqslant k \leqslant \ell m - 1$, is strictly unimodal for $\ell, m \geqslant 5$ with the following finite list of exceptional (ℓ, m) resp. (m, ℓ) pairs:

$$(5,6), (5,10), (5,14), (6,6), (6,7), (6,9), (6,11), (6,13), (7,10).$$

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$$(5,6),(5,10),(5,14),(6,6),(6,7),(6,9),(6,11),(6,13),(7,10).$$

Example. For $(\ell,m)=(6,5)$ we have that $\begin{bmatrix} 11 \\ 5 \end{bmatrix}_q = \begin{bmatrix} 11 \\ 6 \end{bmatrix}_q$ equals

$$q^{30}+q^{29}+2q^{28}+3q^{27}+5q^{26}+7q^{25}+10q^{24}+12q^{23}+16q^{22}+19q^{21}+23q^{20}$$
$$+25q^{19}+29q^{18}+30q^{17}+32q^{16}+32q^{15}+32q^{14}+30q^{13}+29q^{12}+25q^{11}$$
$$+23q^{10}+19q^{9}+16q^{8}+12q^{7}+10q^{6}+7q^{5}+5q^{4}+3q^{3}+2q^{2}+q+1.$$

Gaussian Polynomials

Definition. For $\ell, m \in \mathbb{N}$, the **q-binomial coefficient**, defined by

$$\begin{bmatrix} \ell+m \\ m \end{bmatrix}_q := \prod_{i=1}^{m} \frac{1-q^{\ell+i}}{1-q^i} = \sum_{k=0}^{\ell m} p_k(\ell,m)\cdot q^k,$$

is a polynomial in q.

Pak and Panova (2013) proved that the sequence $p_k(\ell,m)$, $1 \leqslant k \leqslant \ell m - 1$, is strictly unimodal for $\ell, m \geqslant 5$ with the following finite list of exceptional (ℓ, m) resp. (m, ℓ) pairs:

$$(5,6),(5,10),(5,14),(6,6),(6,7),(6,9),(6,11),(6,13),(7,10).$$

Example. For $(\ell, m) = (6,5)$ we have that $\begin{bmatrix} 11 \\ 5 \end{bmatrix}_q = \begin{bmatrix} 11 \\ 6 \end{bmatrix}_q$ equals

$$q^{30} + q^{29} + 2q^{28} + 3q^{27} + 5q^{26} + 7q^{25} + 10q^{24} + 12q^{23} + 16q^{22} + 19q^{21} + 23q^{20}$$
$$+ 25q^{19} + 29q^{18} + 30q^{17} + 32q^{16} + 32q^{15} + 32q^{14} + 30q^{13} + 29q^{12} + 25q^{11}$$
$$+ 23q^{10} + 19q^{9} + 16q^{8} + 12q^{7} + 10q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + q + 1.$$

Note. By symmetry, for ℓ, m odd, one has $p_{(\ell m-1)/2} = p_{(\ell m+1)/2}$.

General Setup

- Let $D \in \mathbb{Z}[q]$ be a univariate polynomial, all of whose zeros are roots of unity, i.e., $D(q) = \prod_{i=1}^{r} \big(1 - q^{e_i}\big)$, $e_1, \ldots, e_r \in \mathbb{N}$.

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- For $\ell_1, \ldots, \ell_n \in \mathbb{Z}$, let $c_k(\ell_1, \ldots, \ell_n)$ be the coefficient of q^k in the series expansion of the rational function

$$c_k := c_k(\ell_1, \ldots, \ell_n) := \langle q^k \rangle \frac{N\big(q, q^{\ell_1}, \ldots, q^{\ell_n}\big)}{D(q)}.$$

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Example. For concrete integer $m \in \mathbb{N}$ and $X = q^\ell$ one can define

$$N\bigl(q, q^\ell\bigr) = \bigl(1 - q^{\ell+1}\bigr)\bigl(1 - q^{\ell+2}\bigr) \cdots \bigl(1 - q^{\ell+m}\bigr)$$
$$D(q) = (1 - q)(1 - q^2) \cdots (1 - q^m)$$

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and obtain for c_k the partition numbers from before:

$$c_k = \langle q^k \rangle \frac{N\left(q, q^\ell\right)}{D(q)} = \langle q^k \rangle \begin{bmatrix} \ell + m \\ m \end{bmatrix}_q = p_k(\ell, m).$$

Strategy for Proving Unimodality

Goal. For a set $\Omega \subseteq \mathbb{Z}^n$ defined by polynomial inequalities, and for given $d \in \mathbb{Z}$, the goal is to prove that for all $(\ell_1, \ldots, \ell_n) \in \Omega$ the sequence (c_k) is d-strictly increasing in a certain range $a \leqslant k \leqslant b$, where the bounds a and b may depend on $\ell_1, \ldots, \ell_n$.

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2. Take the difference $c_{k+1} - c_k$ and perform an appropriate case distinction such that all complex roots of unity are eliminated. Each case is reduced to a polynomial in k and $\ell_1, \ldots, \ell_n$.
3. Apply CAD to each case to show that $c_{k+1} - c_k \geqslant d$ for all k in the corresponding range of interest.

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Coefficients d_k in the Taylor expansion of the denominator

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where $\omega_1, \ldots, \omega_s$ are the distinct roots of $D(q)$, and where $p_i \in \mathbb{Q}(\omega_1, \ldots, \omega_s)[k]$ of degree less than the multiplicity of ω_i.

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Let $L \in \mathbb{N}$ be the smallest integer such that $\omega_1^L = \cdots = \omega_s^L = 1$, and let $\omega = \exp(2\pi i/L)$. Then $\mathbb{Q}(\omega) \supseteq \mathbb{Q}(\omega_1, \ldots, \omega_s)$.

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We get the closed form $d_k = \dfrac{47}{72} + \dfrac{k}{2} + \dfrac{k^2}{12} + \dfrac{\omega^{3k}}{8} + \dfrac{\omega^{2k}}{9} + \dfrac{\omega^{4k}}{9}$.

Including the Numerator

By expanding, we find certain $a_{i,j}, b_i \in \mathbb{Z}$ such that

$$\frac{N\left(q, q^{\ell_1}, \ldots, q^{\ell_n}\right)}{D(q)} = \sum_{i=1}^{r} \gamma_i q^{a_{i,1}\ell_1 + \cdots + a_{i,n}\ell_n + b_i} \cdot \frac{1}{D(q)} = \sum_{k=0}^{\infty} c_k q^k.$$

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This translates into a representation of c_k in terms of d_k:

$$c_k = \sum_{i=1}^{r} \gamma_i d_{k - a_{i,1}\ell_1 - \cdots - a_{i,n}\ell_n - b_i}.$$

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Divide into finitely many regions such that in each region the expressions $k - a_{i,1}\ell_1 - \cdots - a_{i,n}\ell_n - b_i$, $1 \leqslant i \leqslant r$, are sign-invariant ($< 0$ or $\geqslant 0$).

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Example (cont'd). The expanded form of the numerator is

$$\begin{aligned} N\big(q, q^{\ell}\big) &= \big(1 - q^{\ell+1}\big)\big(1 - q^{\ell+2}\big)\big(1 - q^{\ell+3}\big) \\ &= 1 - q^{\ell+1} - q^{\ell+2} - q^{\ell+3} + q^{2\ell+3} + q^{2\ell+4} + q^{2\ell+5} - q^{3\ell+6}. \end{aligned}$$

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By the symmetry of the Gaussian polynomial, we focus on $k \leqslant \frac{3}{2}\ell$:

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Using the closed form for d_k, we get the piecewise expression

$$p_k(\ell, 3) = \begin{cases} \frac{47}{72} + \frac{1}{2}k + \frac{1}{12}k^2 + \frac{1}{8}\omega^{3k} + \frac{1}{9}\omega^{2k} + \frac{1}{9}\omega^{4k}, & 0 \leqslant k < \ell, \\ \frac{19}{36} + \frac{1}{2}\ell - \frac{1}{6}k^2 + \frac{1}{2}k\ell - \frac{1}{4}\ell^2 & \\ \quad + \frac{1}{8}\omega^{3k} + \frac{1}{8}\omega^{3k+3\ell} + \frac{1}{9}\omega^{2k} + \frac{1}{9}\omega^{4k}, & \ell \leqslant k < 2\ell. \end{cases}$$

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- Using $\omega^L = 1$, these powers can be eliminated by substituting

$$k \to Lk' + \kappa \quad \text{and} \quad \ell_i \to L\ell_i' + \lambda_i$$

where $k', \ell_1', \ldots, \ell_n'$ are new variables taking integral values, and $\kappa, \lambda_1, \ldots, \lambda_n \in \{0, \ldots, L-1\}$ are concrete integers.

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- All possible choices for κ and λ_i yield L^{n+1} case distinctions.
- Apply CAD to each of these $(n+1)$-variate polynomials, in order to show that it is $\geqslant d$ under the given assumptions.

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Example (cont'd). The difference $\Delta := p_{k+1}(\ell, 3) - p_k(\ell, 3)$ is

$$\Delta = \begin{cases} \frac{7}{12} + \frac{k}{6} - \frac{1}{4}\omega^{3k} + \frac{1}{9}(\omega - 2)\omega^{2k} - \frac{1}{9}(\omega + 1)\omega^{4k}, & 0 \leqslant k < \ell, \\ -\frac{1}{6} - \frac{1}{3}k + \frac{1}{2}l - \frac{1}{4}\omega^{3k} - \frac{1}{4}\omega^{3k+3l} & \\ \quad + \frac{1}{9}(\omega - 2)\omega^{2k} - \frac{1}{9}(\omega + 1)\omega^{4k}, & \ell \leqslant k < 2\ell. \end{cases}$$

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For instance, substituting $k \to 6k' + 4$ and $\ell \to 6\ell' + 2$ gives

$$\Delta_{4,2} = \begin{cases} k' + 1, & 0 \leqslant 6k' + 4 \leqslant 6\ell' + 1, \\ 3\ell' - 2k' - 1, & 6\ell' + 2 \leqslant 6k' + 4 \leqslant 12\ell' + 3. \end{cases}$$

Assume we want to prove strict monotonicity for $k \leqslant \frac{3}{2}\ell$.

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The second line of $\Delta_{4,2}$ translates into the formula:

$$k' \geqslant 0 \wedge \ell' \geqslant 0 \wedge 6\ell' \leqslant 6k' + 2 \leqslant 9\ell' \implies 3\ell' - 2k' - 1 \geqslant 1.$$

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- First and third clause: special cases $\ell' = 0$ and $\ell' = 1$
- Second clause: no integer ℓ', not relevant
- Last line says the formula is false if $\frac{3}{2}\ell' - 1 < k' \leqslant \frac{3}{2}\ell' - \frac{1}{3}$.
  There is no such k' if ℓ' is even, but there is for odd ℓ'.
  We get the infinite family $(k, \ell) = (18j + 10, 12j + 8)$, $j \geqslant 0$, of pairs where $p_k(\ell, 3)$ is not strictly increasing.

Unimodality Results for Gaussian Polynomials

Theorem. Let $d, \ell, m \in \mathbb{N}$ such that $1 \leqslant d \leqslant 5$ and $3 \leqslant m \leqslant 7$, and let $p_k(\ell, m)$ be as before. Then there exist positive integers $L(m, d)$ and $U(m, d)$ such that $p_{k+1}(\ell, m) - p_k(\ell, m) \geqslant d$ holds for all

$$L(m, d) \leqslant k \leqslant \lfloor \ell m/2 \rfloor - 1 - U(m, d)$$

and almost all $\ell \geqslant 1$, with only a finite number of exceptions.

d	m	$L(m,d)$	$U(m,d)$	Exceptions (ℓ)
	3	1	3	None
	4	1	2	4
1	5	1	0	$1, \ldots, 4, 6, 10, 14$
	6	1	0	$1, \ldots, 7, 9, 11, 13$
	7	1	0	$1, \ldots, 4, 6, 10$

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d	m	$L(m,d)$	$U(m,d)$	Exceptions (ℓ)
	3	7	6	None
	4	5	2	$5, \ldots, 8, 10$
2	5	3	0	$1, \ldots, 10, 14$
	6	3	0	$1, \ldots, 9, 11, 13, 15, 17$
	7	3	0	$1, \ldots, 5, 6, 10$

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	3	13	9	None
	4	7	2	$5, \ldots, 14, 16$
3	5	5	0	$1, \ldots, 12, 14, 18, 22, 26$
	6	5	0	$1, \ldots, 11, 13, 15, 17, 19$
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d	m	$L(m,d)$	$U(m,d)$	Exceptions (ℓ)
	3	19	12	None
	4	9	2	$6, \ldots, 20, 22$
4	5	7	0	$1, \ldots, 15, 18, 22, 26, 30$
	6	7	0	$1, \ldots, 11, 13, 15, 17, 19, 21$
	7	7	0	$1, \ldots, 8, 10$

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d	m	$L(m,d)$	$U(m,d)$	Exceptions (ℓ)
	3	25	15	None
	4	11	2	$7, \ldots, 26, 28$
5	5	7	0	$1, \ldots, 18, 22, 26, 30, 34$
	6	7	0	$1, \ldots, 13, 15, 17, 19, 21, 23$
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Monochromatic Schur triples

(joint work with Elaine Wong)

Schur triple:

$$(x, y, z) \in \mathbb{N}^3 \quad \text{with} \quad x + y = z$$

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Consider a 2-coloring χ of $[n] = \{1, \ldots, n\}$. E.g., for $n = 6$:

$$\chi(2) = \chi(4) = \text{red}, \qquad \chi(1) = \chi(3) = \chi(5) = \chi(6) = \text{blue}$$

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There are exactly 4 monochromatic Schur triples (MSTs):

$$(1, 5, 6), \quad (2, 2, 4), \quad (3, 3, 6), \quad (5, 1, 6).$$

We write $\mathcal{M}(6, \chi) = 4$.

Problem

Minimal number: Determine the minimal number $\mathcal{M}(n)$ of MSTs among all possible 2-colorings of $[n]$

$$\mathcal{M}(n) := \min_{\chi\colon [n]\to\{R,B\}} \mathcal{M}(n,\chi).$$

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Example. Consider again $[6] = \{1,2,3,4,5,6\}$.

- What is $\mathcal{M}(6)$?
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Answer: Choose the coloring $\chi = R^2B^3R = RRBBBR$:

$$\{1, 2, 3, 4, 5, 6\}$$

Then there exists only one single MST, namely $(1, 1, 2)$, hence $\mathcal{M}(6) = 1$.

Three blocks

It has been shown previously (RobertsonZeilberger 98, Schoen 99) that the number $\mathcal{M}(n,\chi)$ is minimized when χ is of the form

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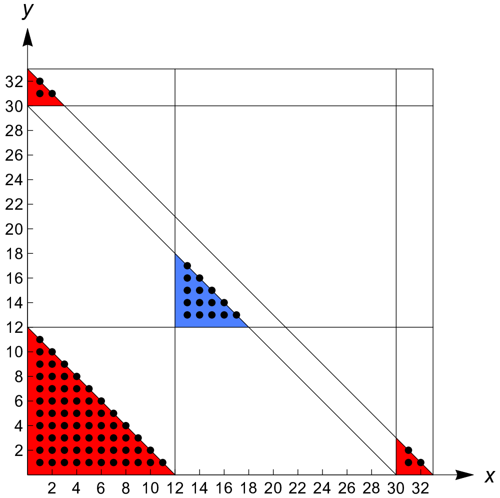
The optimal values for s and t are then easily derived using the techniques of multivariable calculus (assuming $n \to \infty$).

Example

$$\mathcal{M}(n,s,t) = \frac{s(s-1)}{2} + \frac{(t-2s)(t-2s-1)}{2} + (n-t)(n-t-1).$$



- $\chi = R^{12}B^{18}R^3$
- $s = 12,\ t = 30$
- $\mathcal{M}(33,12,30) = 66 + 15 + 6 = 87$
- Actually we have $\mathcal{M}(33) = 87$

Optimal values for discrete s and t

Lemma. For fixed $n \in \mathbb{N}$, the integers s_0 and t_0 that minimize the function $\mathcal{M}(n, s, t)$ are given by

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- Small adaptions to take into account that i, j are integers.

Proof (cont.)

Show that $\mathcal{M}(n, s_0 + i, t_0 + j)$ is minimal for $i = j = 0$:

$$\mathcal{M}(11k + 5, 4k + 2 + i, 10k + 4 + j) = \frac{1}{2}\big(2 + 5i + 5i^2 - 3j - 4ij + 3j^2 + 12k + 22k^2\big).$$

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$$p(i, j) = 5i + 5i^2 - 3j - 4ij + 3j^2$$

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- Such a task can, in principle, be routinely executed by cylindrical algebraic decomposition (CAD).
- In this method, the variables i and j are treated as real variables, which causes some problems here...

CAD

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CylindricalDecomposition[
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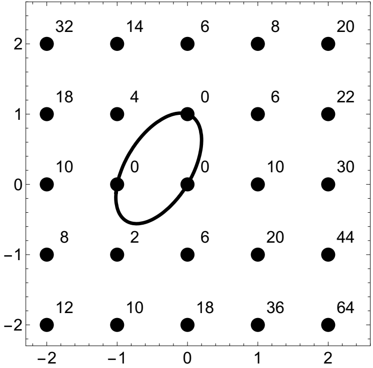

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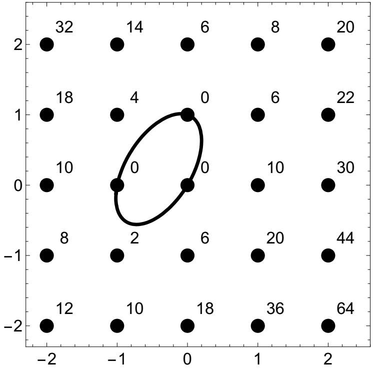

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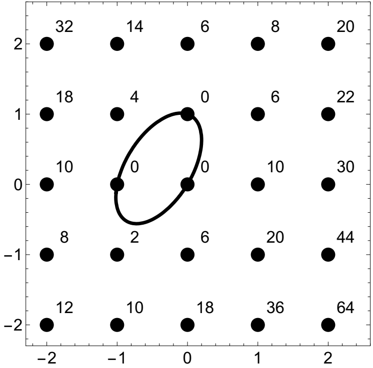

- Show that $p(i, j) \geqslant 0$ for all integer points that are close to $(0, 0)$, e.g., for all (i, j) with $-2 \leqslant i \leqslant 2$ and $-2 \leqslant j \leqslant 2$.
- Invoke cylindrical algebraic decomposition on the following formula

$$\forall i, j \in \mathbb{R}\colon (-2 \leqslant i \leqslant 2 \wedge -2 \leqslant j \leqslant 2) \vee p(i, j) \geqslant 0,$$

Exact lower bound

Theorem. The minimal number of monochromatic Schur triples that can be attained under any 2-coloring of $[n]$ is

$$\mathcal{M}(n) = \left\lfloor \frac{n^2 - 4n + 6}{11} \right\rfloor.$$

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Proof.

$$\begin{aligned}
\ell = 0\colon \mathcal{M}(11k, 4k, 10k) &= 11k^2 - 4k &&= \tfrac{1}{11}(n^2 - 4n) \\
\ell = 1\colon \mathcal{M}(11k+1, 4k, 10k) &= 11k^2 - 2k &&= \tfrac{1}{11}(n^2 - 4n + 3) \\
\ell = 2\colon \mathcal{M}(11k+2, 4k, 10k+1) &= 11k^2 &&= \tfrac{1}{11}(n^2 - 4n + 4) \\
\ell = 3\colon \mathcal{M}(11k+3, 4k+1, 10k+2) &= 11k^2 + 2k &&= \tfrac{1}{11}(n^2 - 4n + 3) \\
\ell = 4\colon \mathcal{M}(11k+4, 4k+1, 10k+3) &= 11k^2 + 4k &&= \tfrac{1}{11}(n^2 - 4n) \\
&\vdots &&\vdots \\
\ell = 9\colon \mathcal{M}(11k+9, 4k+3, 10k+8) &= 11k^2 + 14k + 4 &&= \tfrac{1}{11}(n^2 - 4n - 1) \\
\ell = 10\colon \mathcal{M}(11k+10, 4k+3, 10k+9) &= 11k^2 + 16k + 6 &&= \tfrac{1}{11}(n^2 - 4n + 6)
\end{aligned}$$

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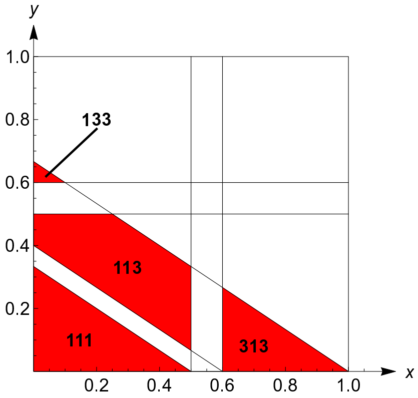

Example.
$s = \frac{1}{2}$, $t = \frac{3}{5}$, $a = \frac{3}{2}$

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Theorem. The minimal number of monochromatic generalized Schur triples of the form $(x, y, x+4y)$ that can be attained under any 2-coloring of $[n]$ of the form $R^s B^{t-s} R^{n-t}$ is

$$\mathcal{M}^{(4)}(n) = \left\lfloor \frac{n^2 - 28n + 245}{216} \right\rfloor - \begin{cases} 1, & \text{if } n = 108k + i \text{ for } i \in I, \\ 0, & \text{otherwise,} \end{cases}$$

where the set I is given by

$$\{0, 1, 27, 28, 43, 47, 48, 53, 58, 63, 67, 68, 69, 73, 78, 83, 88, 89, 93\}.$$

The Gerhold–Kauers Method

Aim. Prove inequalities both sides of which are polynomial expressions in sequences defined by recurrences.

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Example. For the Fibonacci polynomials $F_n(x)$, defined by $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ and $F_1(x) = 1, F_2(x) = x$, prove

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Idea. Set up an induction with respect to n, replace all non-polynomial expressions by new (real) variables, and try to prove the resulting formula by CAD (Gerhold, Kauers, 2005).

Algorithm Sketch

Input. Let $F(n) := F\big(n, \boldsymbol{x}, f_1(n, \boldsymbol{x}), \ldots, f_j(n, \boldsymbol{x})\big)$ and let $C(n, \boldsymbol{x})$ be (polynomial) constraints and $n_0 \in \mathbb{N}$ a bound.

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6. If $m = n_0$, then return FAIL, otherwise, increase m and loop.

Example: Bernoulli's inequality

We apply the Gerhold–Kauers method to the inequality

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This is fed into CAD, yielding `True` almost instantaneously.

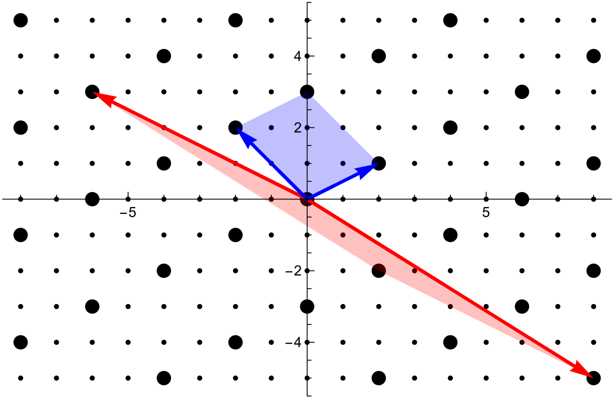
Exercise

Exercise 3. Prove the previously stated inequality for Fibonacci polynomials:

$$\big(F_n(x)\big)^2 \leqslant \big(x^2-1\big)^2\big(x^2+2\big)^{n-3} \qquad \text{(for } n \geqslant 3\text{).}$$

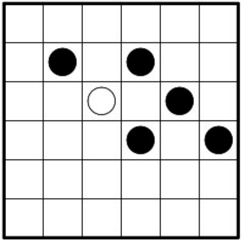
Part 2

Lattice Reduction and Guessing



The Not-Alone Puzzle

Published by Presanna Seshadri in the New York Times magazine



Rules:

Place a circle into each cell of the grid; some white, and some black. Each row and column must contain equally as many white circles as black circles. No individual circle may be sandwiched horizontally or vertically by circles of the opposite color.

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The puzzle can be turned into different enumeration problems:

- binary matrices without any restrictions (boring)

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Exercise 4. Compute as many terms as you can for $b_3(n)$ and $b_4(n)$ (without cheating, not using our recurrences).

A Guess

Example. It looks like $b_3(n)$ satisfies the following recurrence:

$$\begin{aligned}
&51200(2n+7)(2n+5)(2n+3)(2n+1)(n+2)(n+1) \\
&\qquad \times \left(33n^2+242n+445\right) b_3(n) \\
&-128(2n+7)(2n+5)(2n+3)(n+2)\big(7491n^4+84898n^3 \\
&\qquad +351364n^2+628997n+414370\big)\, b_3(n+1) \\
&+16(2n+5)(2n+7)\big(2772n^6+48048n^5+344379n^4 \\
&\qquad +1307394n^3+2775099n^2+3125336n+1460132\big)\, b_3(n+2) \\
&+2(2n+7)(n+3)\big(3201n^6+61886n^5+497179n^4+2124170n^3 \\
&\qquad +5089654n^2+6484024n+3431096\big)\, b_3(n+3) \\
&-(n+3)(n+4)^5\big(33n^2+176n+236\big)\, b_3(n+4)=0
\end{aligned}$$

Naive Guessing

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Ansatz: $x_0 a_n + x_1 a_{n+1} + \cdots + x_r a_{n+r} = 0$
leads to a linear system $M \cdot x = 0$ with

$$M = \begin{pmatrix} a_0 & a_1 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_{r+1} \\ a_2 & a_3 & \cdots & a_{r+2} \\ a_3 & a_4 & \cdots & a_{r+3} \\ a_4 & a_5 & \cdots & a_{r+4} \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

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- To trust the result, we need $N - r + 1 \geqslant (r+1)(d+1)$.

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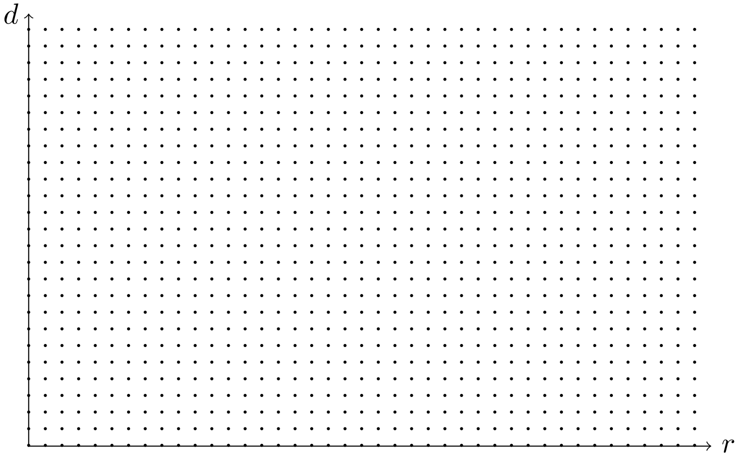
Exercise 5. Guess the recurrence for $b_3(n)$ (Hint: it is A172556 in the OEIS). How many terms are needed (a) with naive guessing, (b) with order-degree trading (see next slide)?

Trading Order vs. Degree



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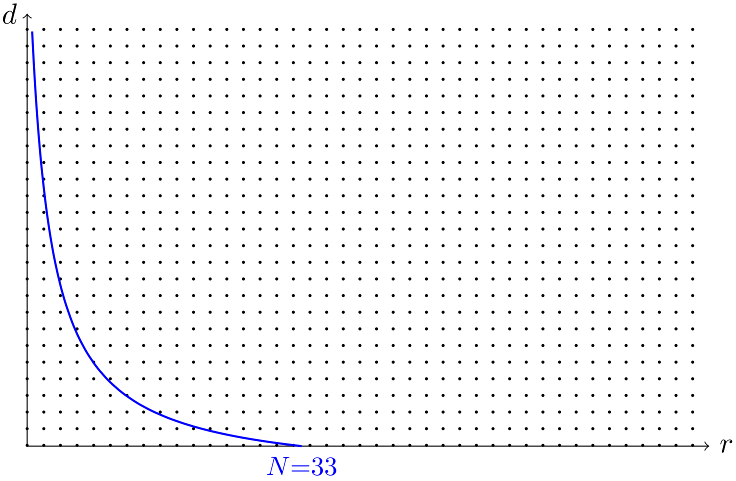

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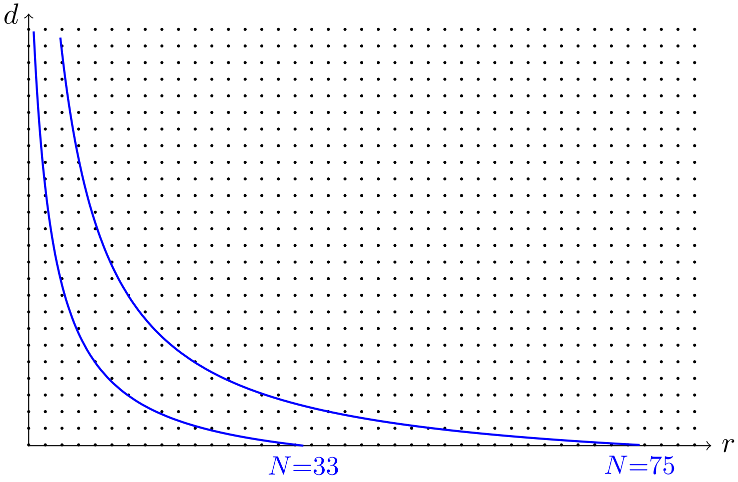

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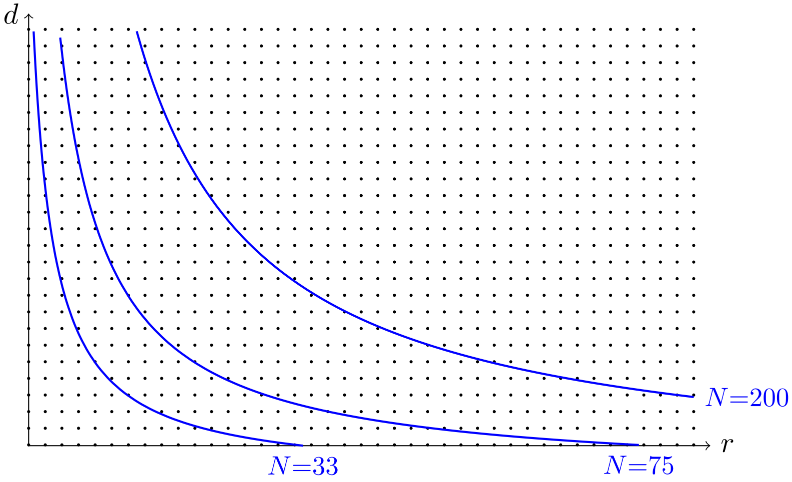

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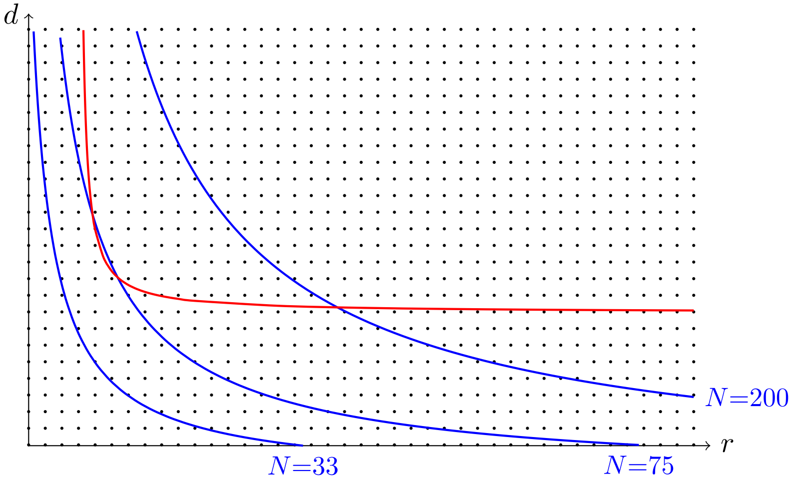

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- The coefficients of the recurrence involve "small" integers.

$\longrightarrow$ Employ a lattice reduction algorithm (LLL, BKZ, ...).

Lattice Basis with Short Vectors

Let $v_1, \ldots, v_\ell \in \mathbb{Z}^m$. They generate a lattice

$$\mathcal{L} = \{c_1 v_1 + \cdots + c_\ell v_\ell \mid c_1, \ldots, c_\ell \in \mathbb{Z}\}.$$

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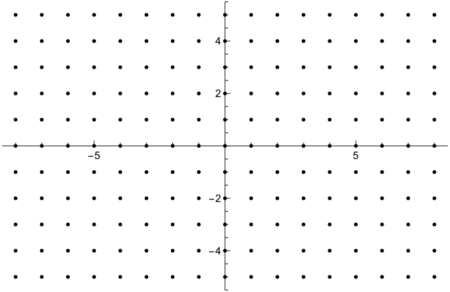

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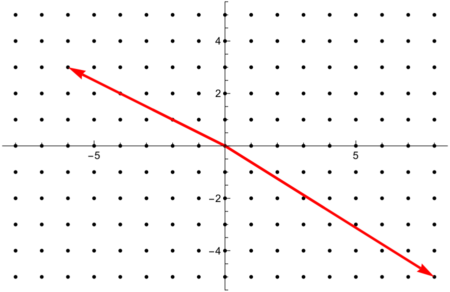

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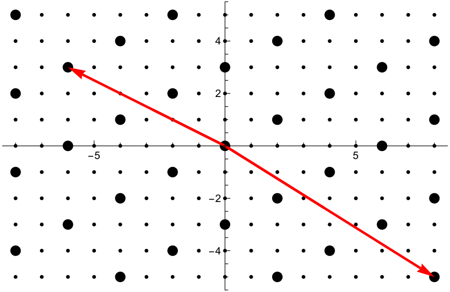

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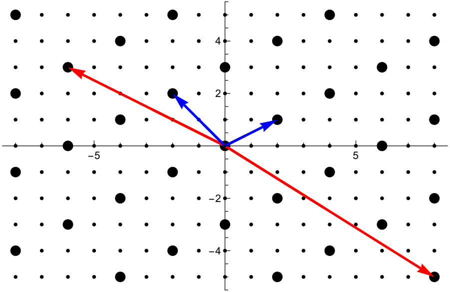

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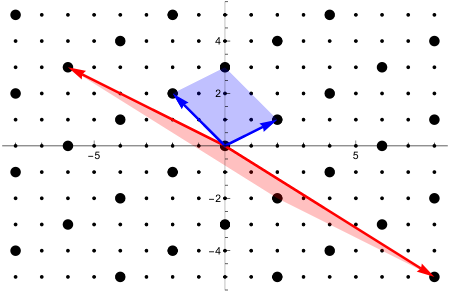

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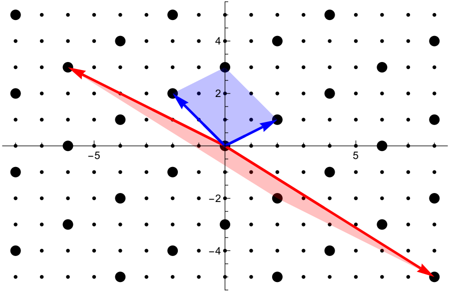

Idea. LLL works similar as the Gram–Schmidt orthogonalization, but over the integers.

Integer Relations via LLL

Definition. Real numbers $x_1, \ldots, x_n \in \mathbb{R}$ satisfy an integer relation if there exist $a_1, \ldots, a_n \in \mathbb{Z}$ such that

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Exercise 6. Use LLL to identify the number

0.60819681587412188135682003077628677069061840980889

as a linear combination of π, π^2, $\zeta(3)$, and $\log(2)$.

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Back to guessing with little data:

Goal. Find a vector with small (integer) entries in $\ker(M)$.

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- Clearing denominators is not enough.
- We need a basis of the $\mathbb{Z}$-module $\ker_{\mathbb{Z}}(M)$.
- It can be computed, e.g., using the Hermite normal form:

$$\left(\begin{array}{c|c} M^{\mathrm{T}} & I_m \end{array}\right) \xrightarrow{\text{HNF}} \left(\begin{array}{c|c} * & * \\ \hline 0 & K \end{array}\right)$$

Then the rows of K form a $\mathbb{Z}$-module basis of $\ker_{\mathbb{Z}}(M)$.

Algorithm: Guessing with Little Data

(joint work with Manuel Kauers)

Input: $a_0, \ldots, a_N \in \mathbb{Q}$, and $r, d \in \mathbb{N}$

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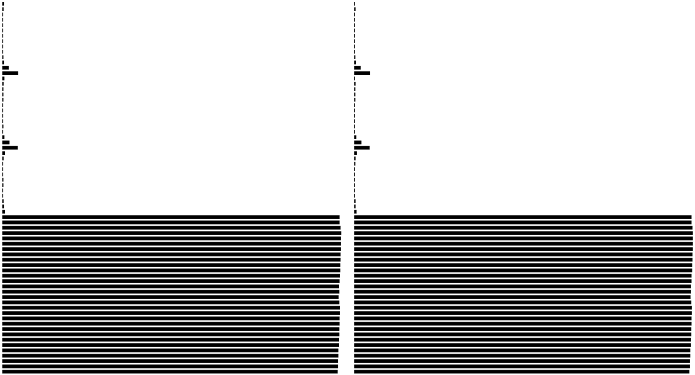
- Use different basis than standard monomials (binomials, ...)
- Incorporate homomorphic images and Chinese remaindering
- Recycle LLL-output when trying a range of degrees d

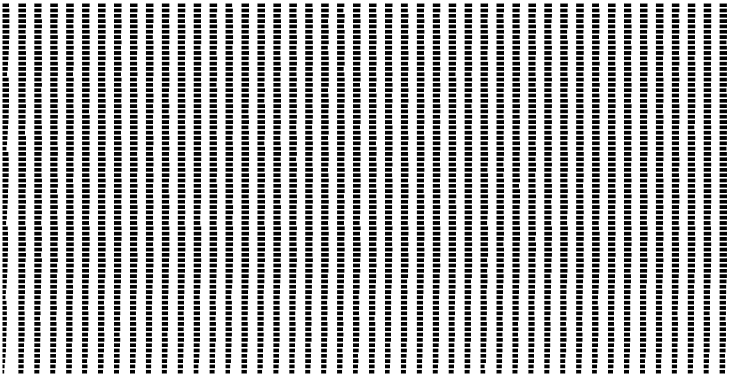
Behavior of the Algorithm

First two vectors of $\ker_{\mathbb{Z}} M$:



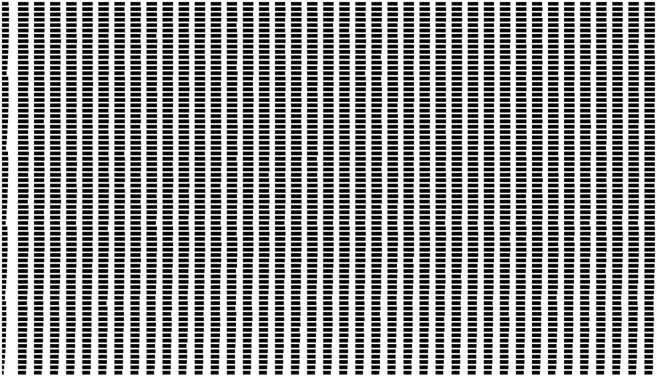
Behavior of the Algorithm

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 28$:



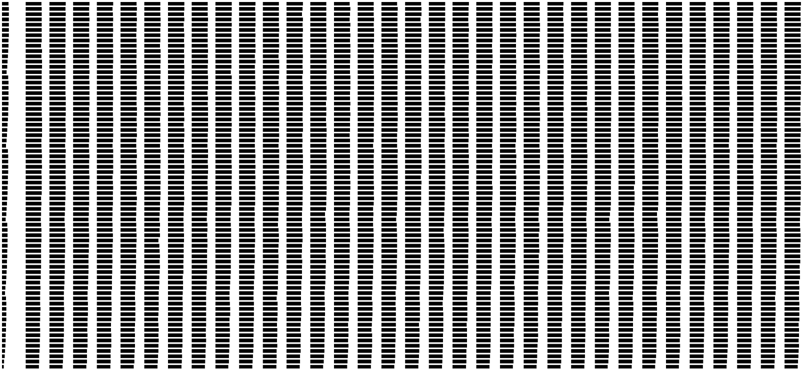
Behavior of the Algorithm

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 33$:



Behavior of the Algorithm

LLL-basis of $\ker_{\mathbb{Z}} M$, using $N = 40$:



Guessing for an OEIS Sequence

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000

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202410
747558000
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737005538936597762145600
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343350312732551834388000013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
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1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000

$$
\begin{aligned}
&(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10} \\
&+795176466036180480n^{9} + 4485660756765878340n^{8} + 18521224670025594405n^{7} \\
&+56639217843614362320n^{6} + 128197997261515989990n^{5} + 211964073373172447460n^{4} \\
&+248660072114197834440n^{3} + 195845152107619591920n^{2} \\
&+92743576895010081600n + 19927056990544704000)a_n \\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10} \\
&+19409301171931086n^{9} + 118570454113296582n^{8} + 533897028046714761n^{7} \\
&+1794118103056008945n^{6} + 4499490897537212457n^{5} + 8317813242144219813n^{4} \\
&+11017108466619178896n^{3} + 9901273828612752684n^{2} \\
&+5411908796200065936n + 1358800904704763520)a_{n+1} \\
&+(-7905964176n^{13} - 375533298360n^{12} - 8210014228350n^{11} - 109384917208164n^{10} \\
&-990927551678562n^{9} - 6445641158908164n^{8} - 30971993224981077n^{7} \\
&-111314492026841106n^{6} - 299240095376493090n^{5} - 594271149013691226n^{4} \\
&-847459848696773373n^{3} - 821800045816910820n^{2} \\
&-485718284438018172n - 132150596906568240)a_{n+2} \\
&+(-34192224n^{13} - 1709611200n^{12} - 39348646744n^{11} - 551960207552n^{10} \\
&-5264405804862n^{9} - 36048494147578n^{8} - 182315015737541n^{7} \\
&-689472630263907n^{6} - 1949560872565283n^{5} - 4070539427181535n^{4} \\
&-6099491170412670n^{3} - 6211013227585736n^{2} \\
&-3851899366258336n - 1098712786184832)a_{n+3} \\
&+(3784n^{13} + 198660n^{12} + 4794801n^{11} + 70437960n^{10} \\
&+702635490n^{9} + 5025358332n^{8} + 26510256652n^{7} \\
&+104430770292n^{6} + 307166340054n^{5} \\
&+666220125600n^{4} + 1035598237875n^{3} \\
&+1092435142500n^{2} + 700889050000n \\
&+206542200000)a_{n+4}
\end{aligned}
$$

Guessing for an OEIS Sequence

Trustworthy?

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
34335031273255183438800013252500
245885257930209910994050195049583660
1803606070619313418263028665207782889600
13495472374334172242190334756526625738793200
102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000

$$
\begin{aligned}
&(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10} \\
&+795176466036180480n^{9} + 4485660756765878340n^{8} + 18521224670025594405n^{7} \\
&+56639217843614362320n^{6} + 128197997261515989990n^{5} + 211964073373172447460n^{4} \\
&+248660072114197834440n^{3} + 195845152107619591920n^{2} \\
&+92743576895010081600n + 19927056990544704000)a_{n} \\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10} \\
&+19409301171931086n^{9} + 118570454113296582n^{8} + 533897028046714761n^{7} \\
&+1794118103056008945n^{6} + 4499490897537212457n^{5} + 8317813242144219813n^{4} \\
&+11017108466619178896n^{3} + 9901273828612752684n^{2} \\
&+5411908796200065936n + 1358800904704763520)a_{n+1} \\
&+(-7905964176n^{13} - 375533298360n^{12} - 8210014228350n^{11} - 109384917208164n^{10} \\
&-990927551678562n^{9} - 6445641158908164n^{8} - 30971993224981077n^{7} \\
&-111314492026841106n^{6} - 299240095376493090n^{5} - 594271149013691226n^{4} \\
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&-485718284438018172n - 132150596906568240)a_{n+2} \\
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Guessing for an OEIS Sequence

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$$+248660072114197834440n^{3} + 195845152107619591920n^{2}$$
$$+92743576895010081600n + 199270569905447040000)a_n$$

$\ldots 092056n^{10}$ $\ldots 6714761n^{7}$ $\ldots 4219813n^{4}$ $\ldots 2752684n^{2}$ $\ldots 3520)a_{n+1}$ $\ldots 208164n^{10}$ $\ldots 4981077n^{7}$ $\ldots 3691226n^{4}$ $\ldots 6910820n^{2}$ $\ldots 8240)a_{n+2}$ $\ldots 207552n^{10}$ $\ldots 5737541n^{7}$ $\ldots 7181535n^{4}$ $\ldots 7585736n^{2}$ $\ldots 4832)a_{n+3}$ $\ldots 437960n^{10}$ $\ldots 0256652n^{7}$ $\ldots 6340054n^{5}$ $\ldots 8237875n^{3}$ $\ldots 89050000n$ $\ldots 0000)a_{n+4}$

90
202410
747558000
35369780063…
19292117692…
11542818594…
73700553893…
49379284270…
34335031273…
24588525793…
18036060700…
13495472374…
10268660945…
79260693690…
61940610460…
48930886220…
39022917847…
31384808443…
25432614293…
20749255870…
17032182384…
14058848882…
11663493370…

Neil Sloane (05.03.2022, about A189281): In the text of the paper you say the coefficients are small! Au contraire. In fact the amount of data in the g.f. is comparable with the data in the original 35-term b-file for the sequence.

If you print the g.f. and then print the data, the number of digits in the two printouts look about the same. When this happens, surely you should be worried. I am very worried, and I think the g.f. needs more justification.

In fact the g.f. looks wrong. I use gfun all the time, and when the g.f. looks like this, like something you would find in the dumpster behind a restaurant, then I would not even consider it :D

972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000

Guessing for an OEIS Sequence

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792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000

Guessing for an OEIS Sequence

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$$
\begin{aligned}
&(10346454767880n^{13} + 439724327634900n^{12} + 8541142111645605n^{11} + 100346408873891460n^{10}\\
&+795176466036180480n^9 + 4485660756765878340n^8 + 18521224670025594405n^7\\
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&+248660072114197834440n^3 + 195845152107619591920n^2\\
&+92743576895010081600n + 19927056990544704000)a_n\\
&+(194741607456n^{13} + 8763372335520n^{12} + 181116778854528n^{11} + 2276272139092056n^{10}\\
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&+1794118103056008945n^6 + 4499490897537212457n^5 + 8317813242144219813n^4\\
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&-847459848696773373n^3 - 821800045816910820n^2\\
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202410
747558000
3536978063850
19292117692187340
115428185943399529200
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4937928427617947420104982250
34335031273255183438800013252500
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3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
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8137021686675518646293987429238146291939698206862646669804019655299405410612322521011793199840
68378027287127596101538933052599954448793862727300484972893130374083314936140639370265791902301600
576696135477018756656097310539308206595297137655178128559217331447163987622287690653154248117571110400
4880259952199292008921826526312609348825249147788374851144194565482906902131493688167790266738802504840000
41428792196488801486282127539417868379239611007329360384215118568533632449531545568541320235439941375576624000

Guessing for an OEIS Sequence

Trustworthy !

$$
\begin{aligned}
&416745(n+2)(3n+4)(3n+5)(3n+7)(3n+8)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\quad\times(3784n^4+62436n^3+384549n^2+1047914n+1066254)a_n\\
&+9(3n+7)(3n+8)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\quad\times(29681696n^7+712360704n^6+7253307424n^5\\
&\quad+40621828312n^4+135172900470n^3+267337368752n^2\\
&\quad+291083104767n+134667010044)a_{n+1}\\
&-9(n+3)(3n+10)(3n+11)(3n+13)(3n+14)\\
&\quad\times(10844944n^8+309080904n^7+3833838118n^6\\
&\quad+27035659722n^5+118560795930n^4+331121212914n^3\\
&\quad+575194973415n^2+568260550317n+244478848756)a_{n+2}\\
&-(n+3)(n+4)^3(3n+13)(3n+14)\\
&\quad\times(3799136n^7+98777536n^6+1092573240n^5\\
&\quad+6662600832n^4+24184813590n^3+52244190090n^2\\
&\quad+62174897623n+31442101253)a_{n+3}\\
&+(n+3)(n+4)^3(n+5)^5\\
&\quad\times(3784n^4+47300n^3+219945n^2\\
&\quad+450988n+344237)a_{n+4}
\end{aligned}
$$

90
202410
747558000
3536978063850
19292117692187340
115428185943399529200
737005538936597762145600
4937928427617947420104982250
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102686609451774712441837258821702706690958244000
792606936905424716827805609592848631050897983368000
6194061046984488807137976612543476252072240088843168000
48930886220271330542271419741692768122929164062703692950250
390229178478432343758493287708395462786699986146463590205462500
3138480844349933121860864061245246387668619696538799391771830312500
25432614295681739433196618354669628742557464857190982677010381944500000
207492558790308966981127400374613926115883943143470298306753431997561245100
1703218238481833503830053446085753316816923905337688679320940617430053026793000
14058848882589179758130070400729131813439016621575276111626854605226450646014928000
116634933760657037542233232023342488551082357129978746187082171269726955508399331520000
972123687656328288735978572104329068283230362616209131997797645253144907352505487518710000
813702168667551864629398742923814629193969820686264666980401965529940541061232252101179319984O
6837802728712759610153893305259995444879386272730048497289313037408331493614063937026579190230160O
576696135477018756656097310539308206595297137655178128559217331447163987622287690653154248117571110400
48802599521992920089218265263126093488252491477883748511441945654829069021314936881677902667388025048400O0
414287921964888014862821275394178683792396110073293603842151185685336324495315455685413202354399413755766240O0

Guessing with Little Data for $b_4(n)$

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- With LLL-based guessing, the recurrence can be constructed from only 110 terms.
- The bit size of the guessed recurrence (after applying an “offset shift” and counting only its integer coefficients) is $46{,}599$, which compares with the bit size $70{,}955$ of the first 110 terms.

Experiments with OEIS Sequences

- There are more than 350,000 sequences in the OEIS.

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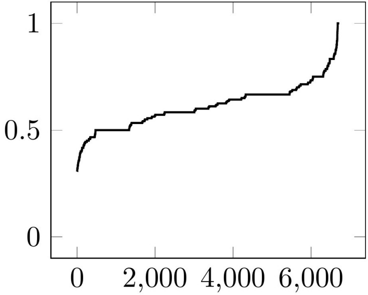

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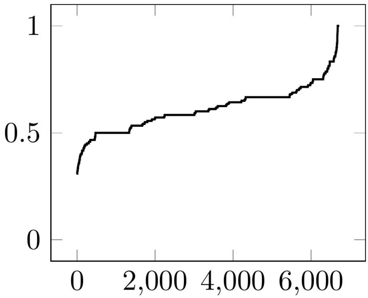

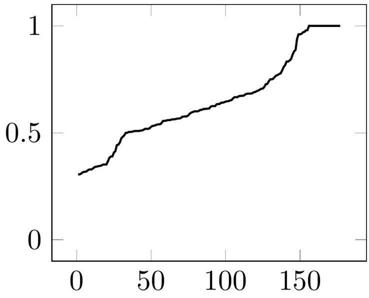

Experiments with OEIS Sequences

- There are more than 350,000 sequences in the OEIS.
- Select those for which at least 50 terms are given,
- which seem to be D-finite (LA finds recurrence),
- which are not too simple (LA needs more than 10 terms),
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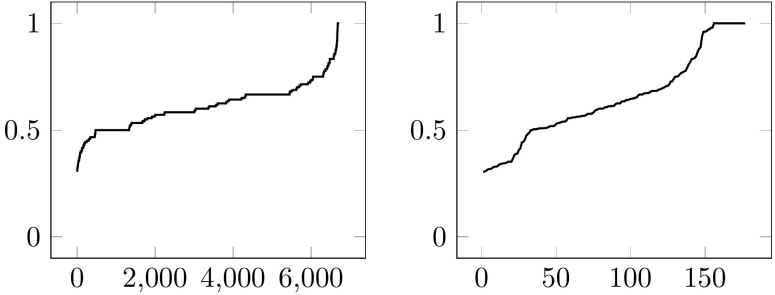

Exercise 7. Use the LLL approach to guess a recurrence for $b_3(n)$. What is the minimal number of terms needed?

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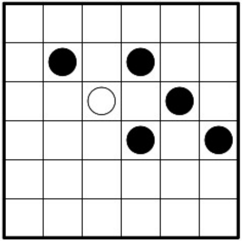
- We were not able to find a recurrence for the notorious Av(1324) sequence…

Back to the Not-Alone Puzzle



Rules:

Place a circle into each cell of the grid; some white, and some black. Each row and column must contain equally as many white circles as black circles. No individual circle may be sandwiched horizontally or vertically by circles of the opposite color.

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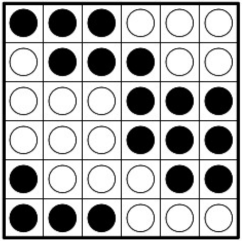

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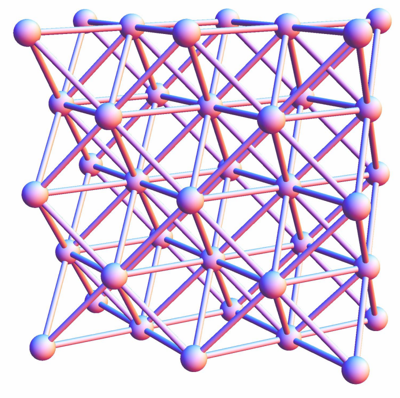
Part 3

D-Finite Functions and Creative Telescoping



Motivation: Proving Identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

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The Holonomic Systems Approach

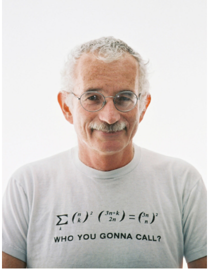
Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Received 14 November 1989

Abstract: We observe that many special functions are solutions of so-called holonomic systems. Bernstein's deep theory of holonomic systems is then invoked to show that any identity involving sums and integrals of products of these special functions can be verified in a finite number of steps. This is partially substantiated by an algorithm that proves terminating hypergeometric series identities, and that is given both in English and in MAPLE.



- seminal paper by Doron Zeilberger in 1990

Univariate D-finite Functions

Definition. A function $f(x)$ is called D-finite ("differentiably finite") if it satisfies a (nontrivial) linear ordinary differential equation with polynomial coefficients:

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Many Functions are D-Finite

ArcCsc, KelvinBei, HypergeometricPFQ, ExpIntegralE, ArcTanh, HankelH2, AngerJ, JacobiP, ChebyshevT, AiryBi, AiryAi, Sinc, CosIntegral, ArcSech, SphericalBesselY, Sin, WhittakerW, SphericalHankelH2, HermiteH, ExpIntegralEi, Beta, AiryBiPrime, SphericalBesselJ, ParabolicCylinderD, Erfc, EllipticK, Cos, Hypergeometric2F1, Erf, KelvinKer, BetaRegularized, HypergeometricPFQRegularized, Log, BesselY, Cosh, ArcSinh, CoshIntegral, ArcTan, ArcCoth, LegendreP, LaguerreL, EllipticE, SinhIntegral, Sinh, SphericalHankelH1, ArcSin, AiryAiPrime, EllipticThetaPrime, Root, AppellF1, FresnelC, LegendreQ, ChebyshevU, GammaRegularized, Erfi, BesselI, HypergeometricU, KelvinKei, Exp, ArcCot, Hypergeometric2F1Regularized, ArcSec, Hypergeometric0F1, EllipticPi, GegenbauerC, ArcCos, WeberE, FresnelS, EllipticF, ArcCosh, HankelH1, Sqrt, BesselK, BesselJ, Hypergeometric1F1Regularized, StruveL, KelvinBer, StruveH, WhittakerM, ArcCsch, Hypergeometric1F1, SinIntegral, . . .

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(vi) In particular, every algebraic function $h(x)$ is D-finite.

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&= \sum_{i=0}^{d_1-1} r_i(c_0, \ldots, c_d, x) f^{(i)}(x) + \sum_{i=0}^{d_2-1} s_i(c_0, \ldots, c_d, x) g^{(i)}(x)
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All coefficients r_i, s_i must vanish: this yields $d_1 + d_2$ equations for the unknowns $c_0, \ldots, c_d$. The choice $d := d_1 + d_2$ ensures a solution.

Quiz: Which functions are D-Finite?

- $\operatorname{erf}\left(\frac{1}{x^2+1}\right) \cdot \exp\left(\frac{1}{x^2+1}\right)$
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Exercise 8. For each of the functions that is D-finite, derive a linear differential equation that it satisfies.

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Let $L_1, L_2 \in K(x)\langle D_x\rangle$ both annihilate f. Then

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Definition. A sequence $(a_n)_{n\in\mathbb{N}}$ is called P-recursive if it satisfies a (nontrivial) linear ordinary recurrence equation with polynomial coefficients:

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Many Sequences are P-Recursive

Multinomial, KelvinBei, HypergeometricPFQ, HarmonicNumber, HankelH2, CatalanNumber, AngerJ, JacobiP, ChebyshevT, SphericalBesselY, WhittakerW, Gamma, Subfactorial, BesselJ, Pochhammer, SphericalHankelH2, Fibonacci, HermiteH, Beta, SphericalBesselJ, Tribonacci, StruveL, ParabolicCylinderD, Hypergeometric2F1, BesselK, BetaRegularized, KelvinKer, PolyGamma, HypergeometricPFQRegularized, SchröderNumber, SphericalHankelH1, LegendreP, LaguerreL, DelannoyNumber, BetaRegularized, AppellF1, LegendreQ, Binomial, ChebyshevU, GammaRegularized, BesselI, HypergeometricU, KelvinKei, Factorial, Hypergeometric2F1Regularized, GegenbauerC, KelvinBer, WeberE, HankelH1, Hypergeometric1F1Regularized, StruveH, WhittakerM, Hypergeometric0F1, Factorial2, Hypergeometric1F1, LucasL, MotzkinNumber, BesselY, . . .

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(iv) a_{cn+d}, where $c, d \in \mathbb{Z}$.

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Example. The three-term recurrence for Legendre polynomials

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

Operator Notation

Let S_n denote the forward shift operator w.r.t. n, i.e.,

$$S_n^0(a_n) = a_n, \quad S_n(a_n) = a_{n+1}, \quad S_n^2(a_n) = a_{n+2}, \quad \text{etc.}$$

Let $K(n)\langle S_n \rangle$ denote the polynomial ring in S_n with coeffs in $\mathbb{K}(n)$. It is not commutative:

$$S_n \cdot n = (n+1) \cdot S_n.$$

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translates to the operator

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Hence, P-recursiveness can be stated as follows:

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Theorem. A sequence $(a_n)_{n\in\mathbb{N}}$ is P-recursive iff its generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is D-finite.

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- q-special functions: q-Bessel functions, q-Legendre polynomials, q-Gegenbauer polynomials, etc.

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Examples: Bessel functions, orthogonal polynomials such as the Legendre polynomials $P_n(x)$, etc.

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Definition. A function $f_{n_1,\ldots,n_r}(x_1,\ldots,x_s)$ in the continuous variables $x_1,\ldots,x_s$ and in the discrete variables $n_1,\ldots,n_r$ is called D-finite if there is a **finite set** of basis functions of the form

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$$\mathbb{O} = \mathbb{K}(x,n,\ldots)\langle D_x, S_n, \ldots \rangle,$$

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Definition. We define the **annihilator** of a function f to be the set

$$\operatorname{Ann}_{\mathbb{O}} f := \left\{P\in\mathbb{O}\;\middle|\;P\cdot f=0\right\}$$

(it is a **left ideal** in the ring $\mathbb{O}$).

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Example. The Legendre polynomials $P_n(x)$ satisfy

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Here is a Gröbner basis:

$$(n+1)S_n + (1-x^2)D_x - (n+1)x, \quad (x^2-1)D_x^2 + 2xD_x - n(n+1).$$

Note. Gröbner bases (Buchberger 1965) are very useful!

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Definition. A function $f(x, y, \dots)$ is **D-finite** w.r.t. $\mathbb{O}$ if "all its shifts and derivatives"

$$\mathbb{O} \cdot f = \{P \cdot f \mid P \in \mathbb{O}\}$$

form a **finite-dimensional** $\mathbb{K}(x, y, \dots)$-vector space:

$$\dim_{\mathbb{K}(x,y,\dots)} \big(\mathbb{O}/\operatorname{Ann}_{\mathbb{O}}(f)\big) < \infty.$$

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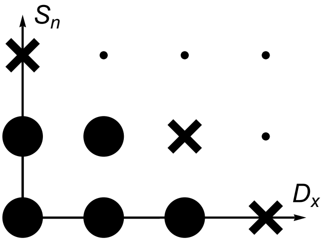

"monomials under the staircase" $(\dim = 5)$ = "holonomic rank"

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Indeed, we have $F(x) = K_\nu(x)$.

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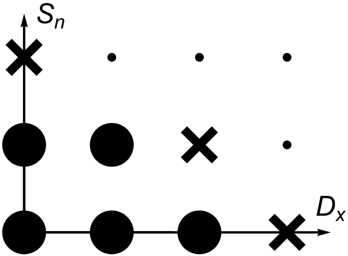

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Since $Q \in \mathbb{O}/\operatorname{Ann}_{\mathbb{O}}(f)$, we can set

$$Q = \sum_{u \in \mathfrak{U}} q_u(x, y)\, u \qquad \text{with unknowns } q_u \in \mathbb{K}(x, y).$$

Chyzak's Algorithm

Putting things together:

$$P - D_y Q = \sum_{i=0}^{r} p_i(x) D_x^i - D_y \sum_{u \in \mathfrak{U}} q_u(x, y)\, u$$

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This yields a coupled first-order linear system of differential equations for the q_u's with parameters $p_0, \ldots, p_r \in \mathbb{K}(x)$.

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$$\begin{aligned} P - D_y Q &= \sum_{i=0}^{r} p_i(x) D_x^i - D_y \sum_{u \in \mathfrak{U}} q_u(x,y)\, u \\ &= \sum_{i=0}^{r} p_i(x) D_x^i - \sum_{u \in \mathfrak{U}} \left(q_u(x,y) D_y + \tfrac{\mathrm{d}}{\mathrm{d}y} q_u(x,y) \right) u \end{aligned}$$

Since we want $P - D_y Q \in \operatorname{Ann}_{\mathbb{O}}(f)$ we **reduce** the above expression with a Gröbner basis of $\operatorname{Ann}_{\mathbb{O}}(f)$ and equate the (D_x, D_y)-coefficients to zero.

This yields a coupled first-order linear system of differential equations for the q_u's with parameters $p_0, \ldots, p_r \in \mathbb{K}(x)$.
$\longrightarrow$ There are algorithms to find **rational** solutions of such systems.

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$\longrightarrow$ There are algorithms to find **rational** solutions of such systems.

Finally: loop over the (a priori) unknown order r of the telescoper.
$\longrightarrow$ This is Chyzak's algorithm (analogously in other Ore algebras).

Application: Special Function Identities

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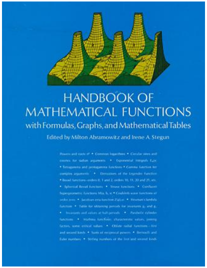
North-Holland

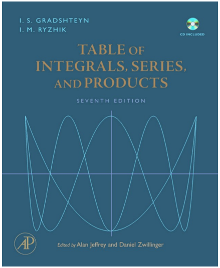
A holonomic systems approach to special functions identities *

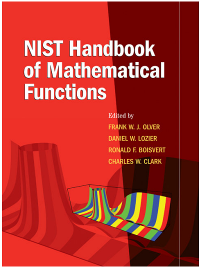
Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Received 14 November 1989

Abstract: We observe that many special functions are solutions of so-called holonomic systems. Bernstein's deep theory of holonomic systems is then invoked to show that any identity involving sums and integrals of products of these special functions can be verified in a finite number of steps. This is partially substantiated by an algorithm that proves terminating hypergeometric series identities, and that is given both in English and in MAPLE.







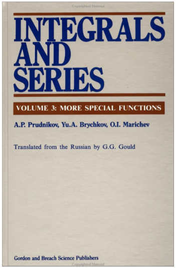
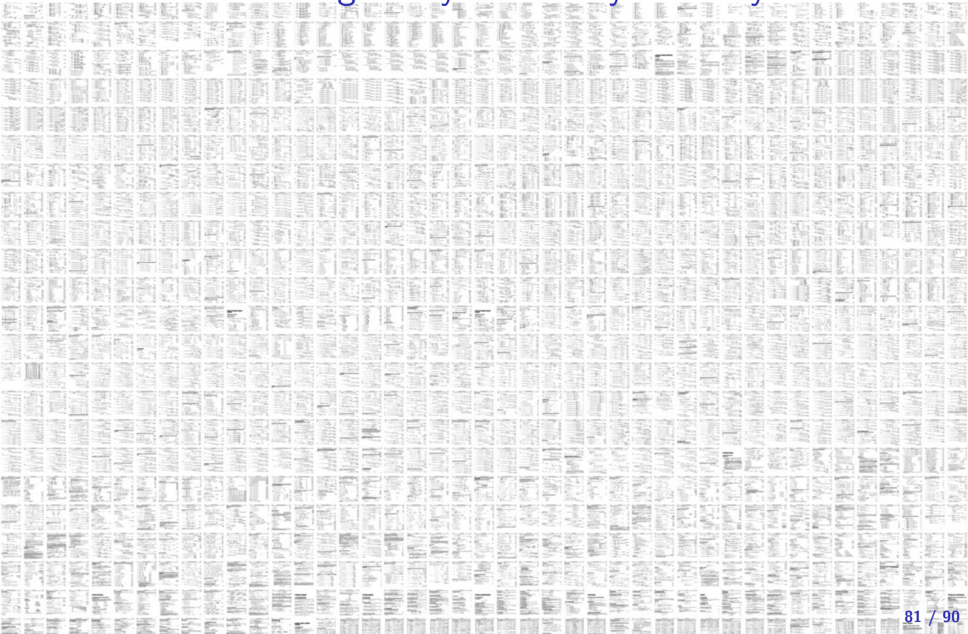


Table of Integrals by Gradshteyn and Ryzhik



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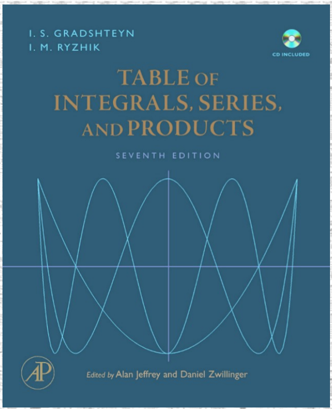

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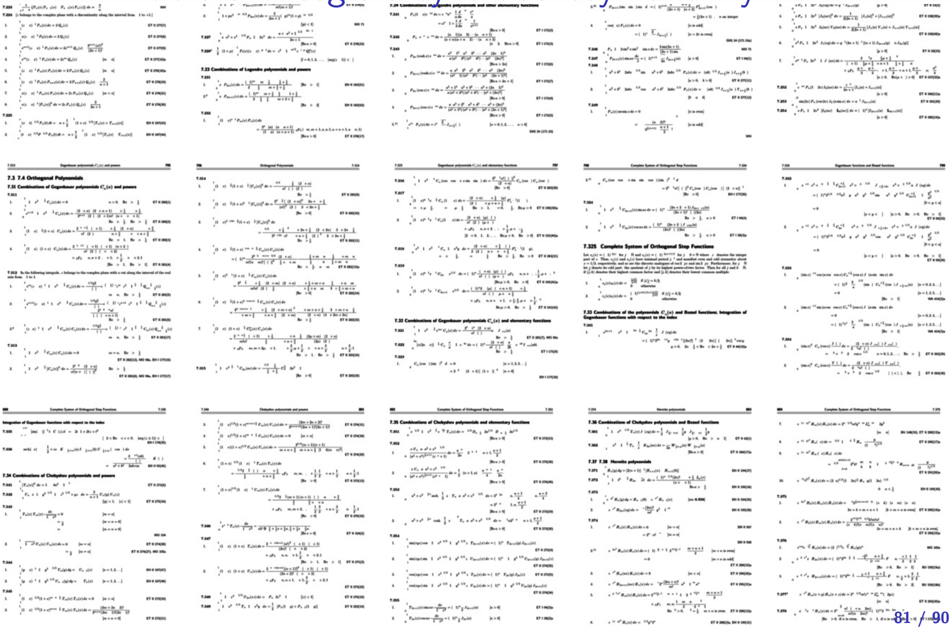

7.319

1. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n}^{\lambda}\left(\gamma x^{1/2}\right) dx = (-1)^n \frac{\Gamma(\lambda+n)\,\Gamma(\mu)\,\Gamma(\nu)}{n!\,\Gamma(\lambda)\,\Gamma(\mu+\nu)}\; {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right)$$
[$\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$] ET II 191(41)a

2. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n+1}^{\lambda}\left(\gamma x^{1/2}\right) dx = \frac{(-1)^n 2\gamma\,\Gamma(\mu)\,\Gamma(\lambda+n+1)\,\Gamma\left(\nu+\frac{1}{2}\right)}{n!\,\Gamma(\lambda)\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)} \times {}_3F_2\left(-n, n+\lambda+1, \nu+\tfrac{1}{2}; \tfrac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right)$$
[$\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}$] ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^{\nu}(x)$ and elementary functions

7.321 $$\int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$
[$\operatorname{Re}\nu > -\frac{1}{2}$] ET II 281(7), MO 99a

7.322 $$\int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}}\, C_n^{\nu}\left(\frac{x}{a}-1\right) e^{-bx}\, dx = (-1)^n \frac{\pi\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)} \left(\frac{a}{2b}\right)^{\nu} e^{-ab}\, I_{\nu+n}(ab)$$
[$\operatorname{Re}\nu > -\frac{1}{2}$] ET I 171(9)

7.323

1. $$\int_0^{\pi} C_n^{\nu}(\cos\varphi)\,(\sin\varphi)^{2\nu}\, d\varphi = 0$$
[$n = 1, 2, 3, \ldots$]

Table of Integrals by Gradshteyn and Ryzhik

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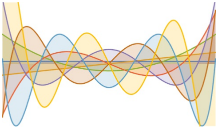

Gegenbauer polynomials $C_n^{(\alpha)}(x)$

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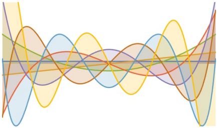

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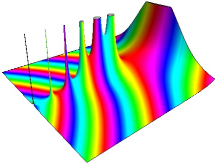

Gamma function $\Gamma(x)$

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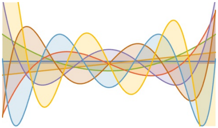

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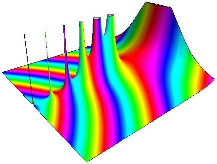

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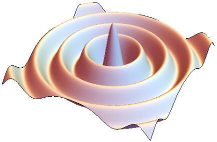

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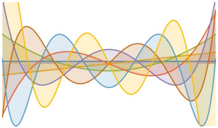
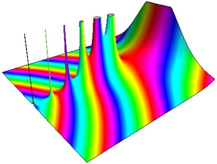


Gegenbauer polynomials $C_n^{(\alpha)}(x)$



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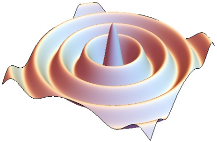

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$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax} \, C_n^{\nu}(x) \, dx = \frac{\pi 2^{1-\nu} i^n \, \Gamma(2\nu+n)}{n! \, \Gamma(\nu)} a^{-\nu} \, J_{\nu+n}(a)$$

Let's prove this identity with creative telescoping...

Example

$$\int_{-1}^{1} \left(1 - x^2\right)^{\nu - \frac{1}{2}} e^{iax} C_n^{\nu}(x)\,\mathrm{d}x = \frac{\pi\, 2^{1-\nu} i^n\, \Gamma(2\nu + n)}{n!\,\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

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```
CreativeTelescoping[
  (1-x^2)^(nu-1/2)*Exp[I*a*x]*GegenbauerC[n, nu, x],
  Der[x], {S[n], Der[a]}]
```

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$$\Big\{\big\{(a+an)S_n + (ian + 2ia\nu)D_a + (-in^2 - 2in\nu),$$
$$a^2 D_a^2 + (a + 2a\nu)D_a + (a^2 - n^2 - 2n\nu)\big\},$$
$$\big\{i(1+n)S_n - i(nx + 2\nu x),$$
$$(1+n)S_n - i(-a - inx - 2i\nu x + ax^2)\big\}\Big\}$$

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```
Annihilator[
  Pi*2^(1 - nu)*I^n*Gamma[2 nu + n]/n!/Gamma[nu]*
  a^(-nu)*BesselJ[nu + n, a], {S[n], Der[a]}]
```

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Diagonals of Rational Functions

Given a rational function in n variables

$$R(x_1, \ldots, x_n) = \frac{A(x_1, \ldots, x_n)}{B(x_1, \ldots, x_n)},$$

where $A, B \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $B(0, \ldots, 0) \neq 0$.

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Definition. The diagonal of R is defined through its multi-Taylor expansion around $(0, \dots, 0)$:

$$R(x_1, \dots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} r_{m_1,\dots,m_n} \cdot x_1^{m_1} \cdots x_n^{m_n},$$

as the power series in one variable:

$$\mathrm{Diag}\big(R(x_1, \dots, x_n)\big) := \sum_{m=0}^{\infty} r_{m,m,\dots,m} \cdot x^m.$$

Example of a Diagonal

Consider the Taylor expansion of the bivariate rational function

$$
\begin{aligned}
f(x,y) &= \frac{1}{1-x-y-2xy} \\
&= 1 + x + y + x^2 + 4xy + y^2 + x^3 + 7x^2y + 7xy^2 + \dots
\end{aligned}
$$

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$$\begin{array}{llllllllllllll} = & 1 & + & y & + & y^2 & + & y^3 & + & y^4 & + & y^5 & + \dots \\ + & x & + & 4xy & + & 7xy^2 & + & 10xy^3 & + & 13xy^4 & + & 16xy^5 & + \dots \\ + & x^2 & + & 7x^2y & + & 22x^2y^2 & + & 46x^2y^3 & + & 79x^2y^4 & + & 121x^2y^5 & + \dots \\ + & x^3 & + & 10x^3y & + & 46x^3y^2 & + & 136x^3y^3 & + & 307x^3y^4 & + & 586x^3y^5 & + \dots \\ + & x^4 & + & 13x^4y & + & 79x^4y^2 & + & 307x^4y^3 & + & 886x^4y^4 & + & 2086x^4y^5 & + \dots \\ + & x^5 & + & 16x^5y & + & 121x^5y^2 & + & 586x^5y^3 & + & 2086x^5y^4 & + & 5944x^5y^5 & + \dots \end{array}$$

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$$\begin{array}{l} = \color{red}{1} + y + y^2 + y^3 + y^4 + y^5 + \dots \\ + x + \color{red}{4xy} + 7xy^2 + 10xy^3 + 13xy^4 + 16xy^5 + \dots \\ + x^2 + 7x^2y + \color{red}{22x^2y^2} + 46x^2y^3 + 79x^2y^4 + 121x^2y^5 + \dots \\ + x^3 + 10x^3y + 46x^3y^2 + \color{red}{136x^3y^3} + 307x^3y^4 + 586x^3y^5 + \dots \\ + x^4 + 13x^4y + 79x^4y^2 + 307x^4y^3 + \color{red}{886x^4y^4} + 2086x^4y^5 + \dots \\ + x^5 + 16x^5y + 121x^5y^2 + 586x^5y^3 + 2086x^5y^4 + \color{red}{5944x^5y^5} + \dots \end{array}$$

Then the diagonal of f is

$$\mathrm{Diag}(f) = 1 + 4x + 22x^2 + 136x^3 + 886x^4 + 5944x^5 + \dots$$

Diagonals as Integrals

Note that a diagonal $\operatorname{Diag}\big(R(x,y,z)\big)$ can also be expressed as

$$\langle y^0 z^0\rangle\, R\Big(\frac{x}{y},\frac{y}{z},z\Big) = \operatorname{res}_{y,z}\frac{1}{yz}R\Big(\frac{x}{y},\frac{y}{z},z\Big) = \oint \frac{1}{yz}R\Big(\frac{x}{y},\frac{y}{z},z\Big)\mathrm{d}y\,\mathrm{d}z.$$

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Indeed, writing

$$R(x,y,z) = \sum_{l\geqslant 0}\sum_{m\geqslant 0}\sum_{n\geqslant 0} r_{l,m,n}\, x^l y^m z^n$$

one obtains

$$R\Big(\frac{x}{y},\frac{y}{z},z\Big) = \sum_{l\geqslant 0}\sum_{m\geqslant 0}\sum_{n\geqslant 0} a_{l,m,n}\, x^l y^{m-l} z^{n-m}.$$

Back to Balanced Binary Matrices

Theorem. For any fixed k the sequence $b_k(n)$ of balanced $2k \times 2n$ binary matrices is P-recursive.

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Proof. Consider the elementary symmetric function of degree k:

$$e_k(x_1, \ldots, x_{2k}) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant 2k} x_{i_1} \cdots x_{i_k}.$$

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- Extracting the coefficient of $x_1^n \cdots x_{2k}^n$ in $e_k(x_1, \ldots, x_{2k})^{2n}$ collects those that are also row-balanced.

$$b_k(n) = \langle x_1^n \cdots x_{2k}^n \rangle e_k(x_1, \ldots, x_{2k})^{2n}.$$

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- Then $e_k(x_1, \ldots, x_{2k})^{2n}$ is the weight enumerator of all column-balanced $2k \times 2n$ matrices.
- Extracting the coefficient of $x_1^n \cdots x_{2k}^n$ in $e_k(x_1, \ldots, x_{2k})^{2n}$ collects those that are also row-balanced.

$$b_k(n) = \langle x_1^0 \cdots x_{2k}^0 \rangle \left(\frac{e_k(x_1, \ldots, x_{2k})^2}{x_1 \cdots x_{2k}} \right)^n.$$

Back to Balanced Binary Matrices

Theorem. For any fixed k the sequence $b_k(n)$ of balanced $2k \times 2n$ binary matrices is P-recursive.

Proof. Consider the elementary symmetric function of degree k:

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$$\sum_{n=0}^{\infty} b_k(n) x^{2n} = \mathrm{Diag}\left(\frac{1}{1 - e_k(x_1, \ldots, x_{2k})}\right).$$

First Result

(joint work with Robert Dougherty-Bliss, Natalya Ter-Saakov, Doron Zeilberger)

Theorem. Let $b_2(n)$ be the number of $4 \times 2n$ balanced matrices. Then

$$
\begin{aligned}
&36(2n+3)(2n+1)(n+1)\, b_2(n) \\
&- 2(2n+3)(10n^2+30n+23)\, b_2(n+1) \\
&+ (n+2)^3\, b_2(n+2) = 0
\end{aligned}
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for all $n \geqslant 0$.

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for all $n \geqslant 0$.

Exercise 10. Use creative telescoping to rigorously derive this recurrence. If you have some time, do the same for the theorem on the next slide.

Second Result

Theorem. Let $b_3(n)$ be the number of $6 \times 2n$ balanced matrices. Then

$$\begin{aligned}
&51200(2n+7)(2n+5)(2n+3)(2n+1)(n+2)(n+1) \\
&\qquad \times \left(33n^2+242n+445\right) b_3(n) \\
&-128(2n+7)(2n+5)(2n+3)(n+2)\big(7491n^4+84898n^3 \\
&\qquad +351364n^2+628997n+414370\big)\, b_3(n+1) \\
&+16(2n+5)(2n+7)\big(2772n^6+48048n^5+344379n^4 \\
&\qquad +1307394n^3+2775099n^2+3125336n+1460132\big)\, b_3(n+2) \\
&+2(2n+7)(n+3)\big(3201n^6+61886n^5+497179n^4+2124170n^3 \\
&\qquad +5089654n^2+6484024n+3431096\big)\, b_3(n+3) \\
&-(n+3)(n+4)^5\left(33n^2+176n+236\right) b_3(n+4) = 0
\end{aligned}$$

for all $n \geqslant 0$.

Balanced Matrices Avoiding Some Patterns

Theorem. Let H and V be finite sets of words (patterns) in $\{0, 1\}$. Let $b_{H,V,k}(n)$ be the number of balanced $2k \times 2n$ binary matrices, that avoid the patterns of H in every row and the patterns of V in every column. Then $b_{H,V,k}(n)$ is P-recursive for any fixed k.

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Proof idea.

- Assign a weight of a matrix $A = (a_{ij}, 1 \leqslant i \leqslant 2k, 1 \leqslant j \leqslant n)$ to be $t^n x_1^{a_1} \cdots x_{2k}^{a_{2k}}$, where a_i is the number of ones in row i.

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- Use the transfer matrix method to find the weight-enumerator of the set of all matrices avoiding H horizontally and V vertically.
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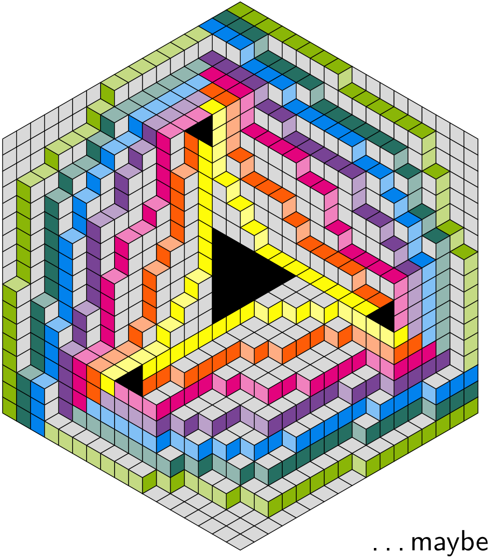
- In order to count balanced such matrices with $2n$ columns, we have to extract the coefficient of $t^{2n} x_1^n \cdots x_{2k}^n$.

Exercise 11. Work out the details and derive a recurrence for balanced $4 \times 2n$ matrices that avoid the patterns 010 and 101 both horizontally and vertically.

Encore: Symbolic Determinants



. . . maybe another time!