Combinatoire et aléa autour des rectangulations

Éric Fusy

LIGM/CNRS, Université Gustave Eiffel

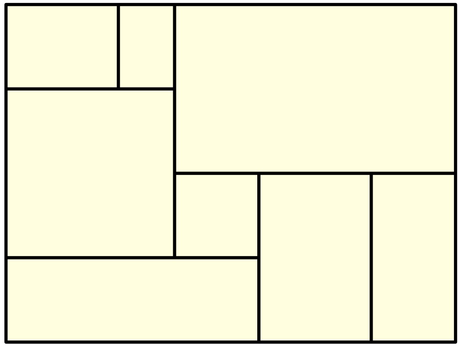
Alea 2026

Rectangulations

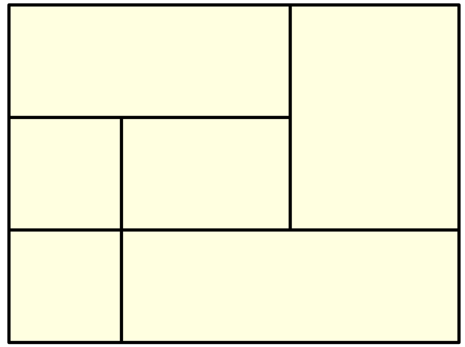
Rectangulation

||

tiling of a rectangle by rectangles



Generic (no +)



Not generic

Rectangulations

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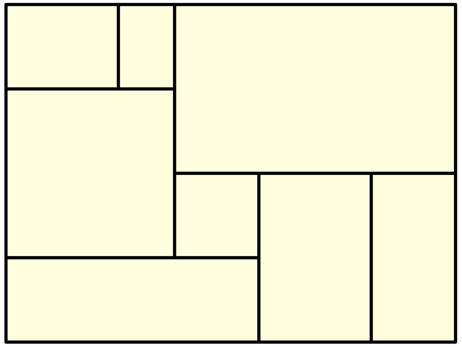

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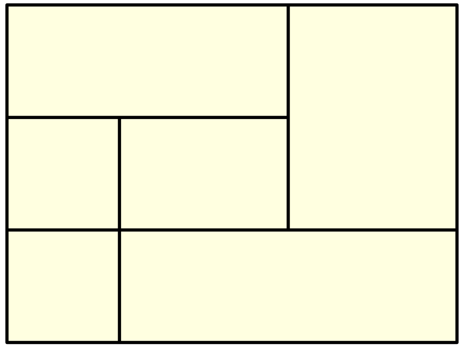
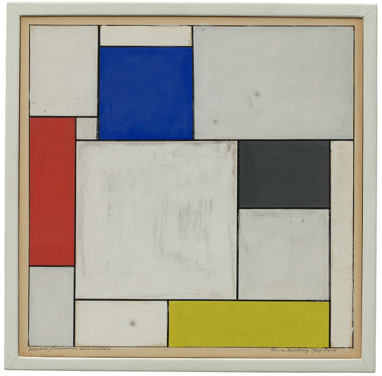


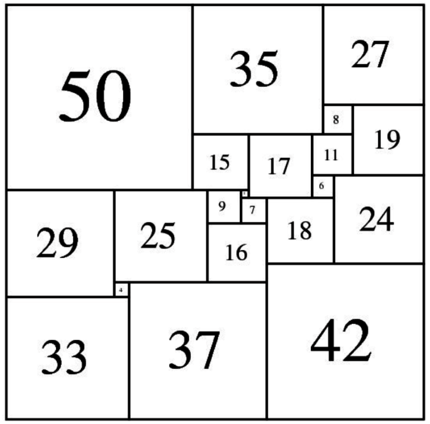
Not generic

Contexts where rectangulations appear:

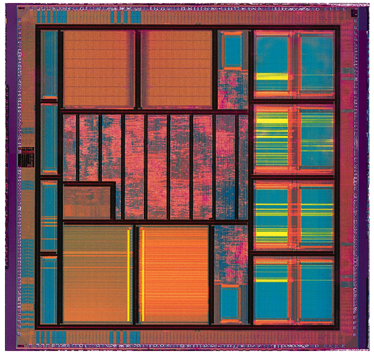
Theo van Doesburg'1924
"Composition décentralisée"



© squaring.net

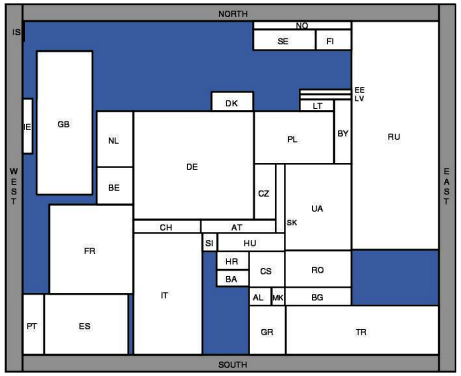


squaring the square



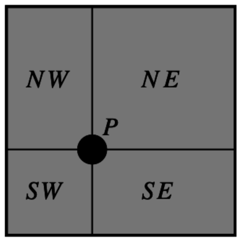
VLSI design

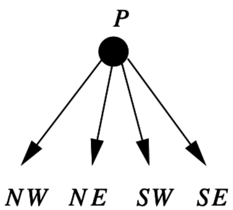
© Speckmann, Van Kreveld

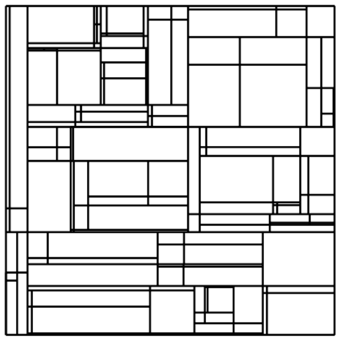


cartograms

© Flajolet-Sedgewick







quadtrees

Rectangulations

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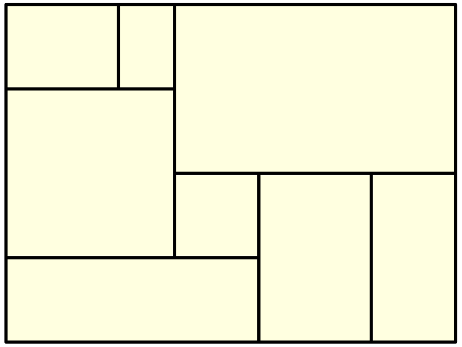

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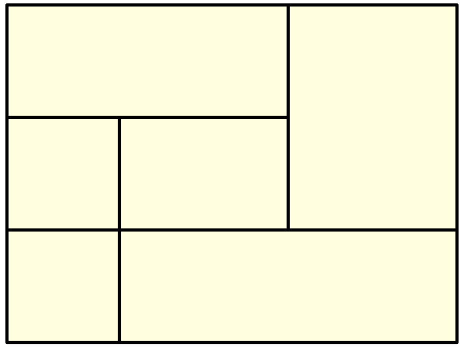

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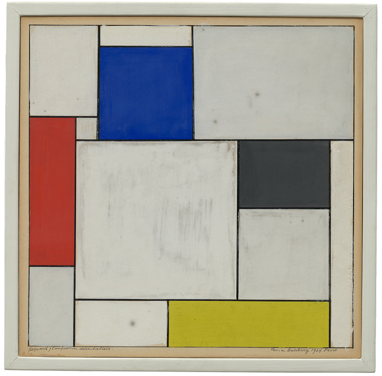

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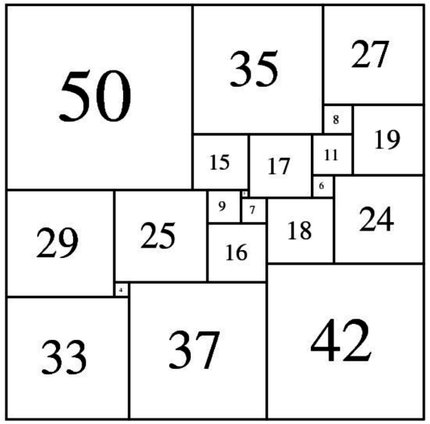
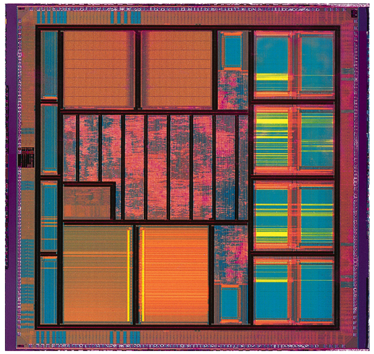
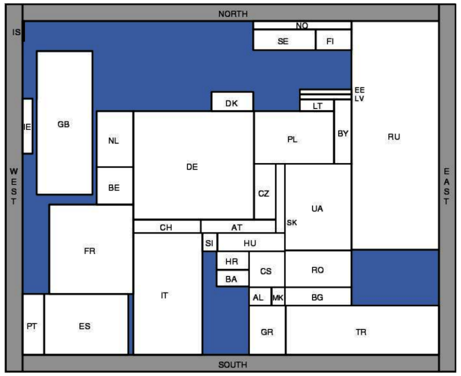


squaring the square



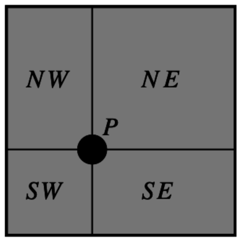
VLSI design

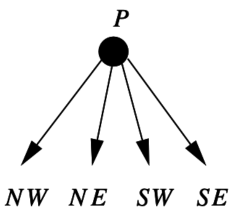
© Speckmann, Van Kreveld



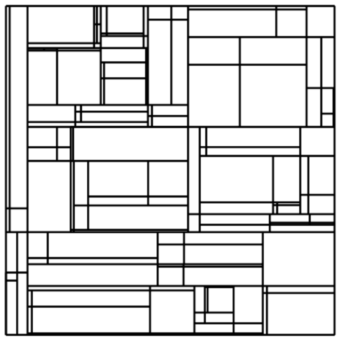
cartograms

© Flajolet-Sedgewick





quadtrees

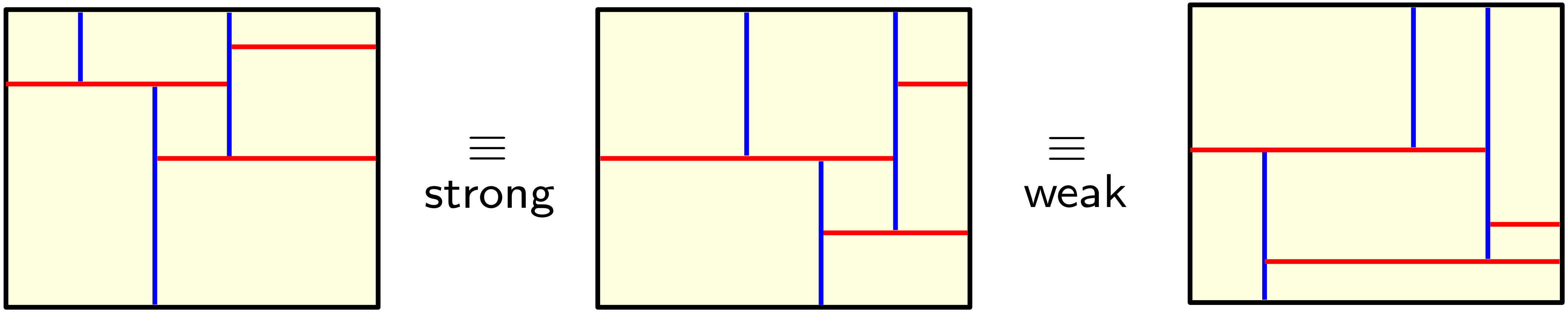


Combinatorics

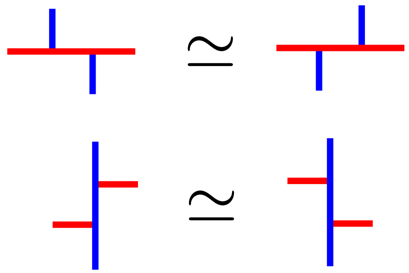
close links to:
- planar maps
- permutations
- quadrant walks

Combinatorial types of rectangulations

two equivalence relations: strong/weak



allows wall slides

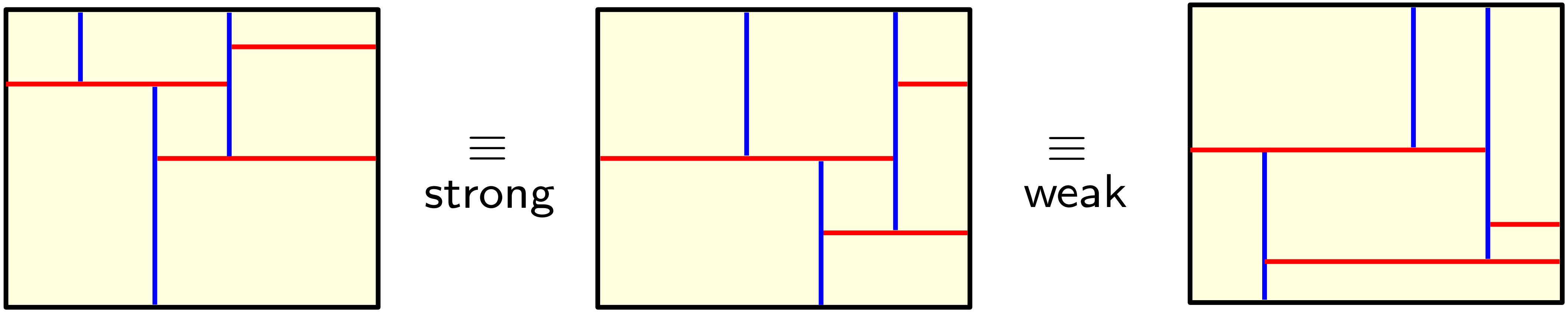


weak rectangulation = weak equivalence class (contact-system of segments)

strong rectangulation = strong equivalence class (contact-system of boxes)

Combinatorial types of rectangulations

two equivalence relations: strong/weak



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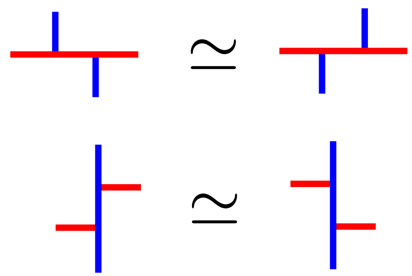


weak rectangulation = weak equivalence class (contact-system of segments)

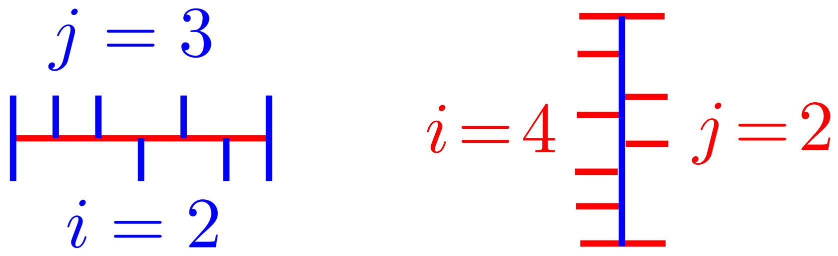
strong rectangulation = strong equivalence class (contact-system of boxes)

$w_n =$	# weak rectangulations of size n
$s_n =$	# strong rectangulations of size n

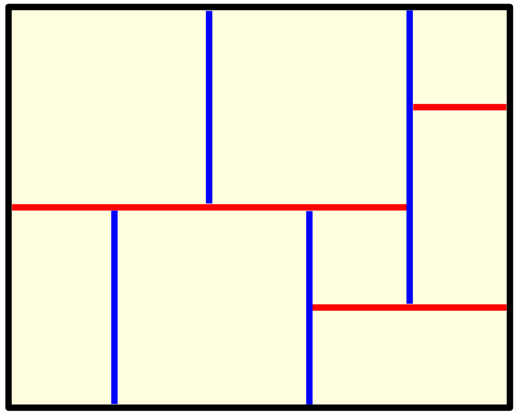
size n = # boxes

Strong representatives of a weak rectangulation

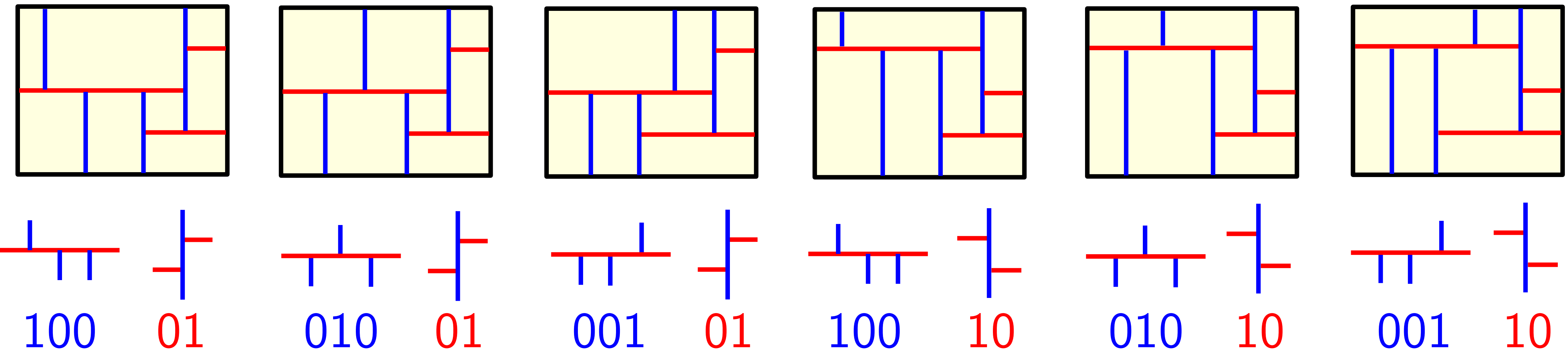
Rk: in strong rectangulation, contacts of a segment are totally ordered



$\#$ arrangements $= \binom{i+j}{i}$



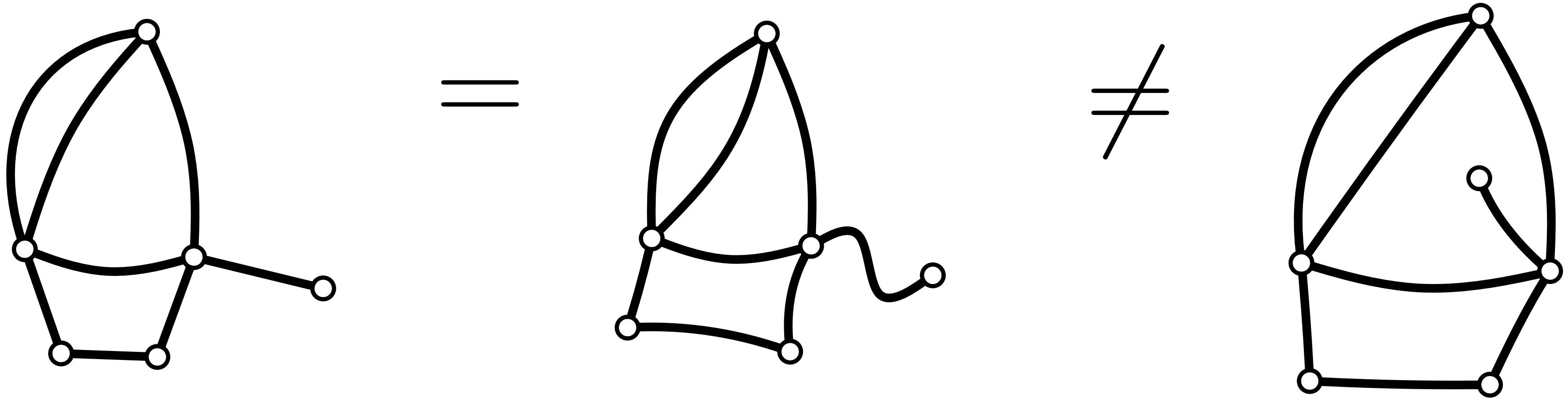
weak class



Rectangulations and planar maps

Planar maps

Def. Planar map = connected graph embedded in the plane up to isotopy



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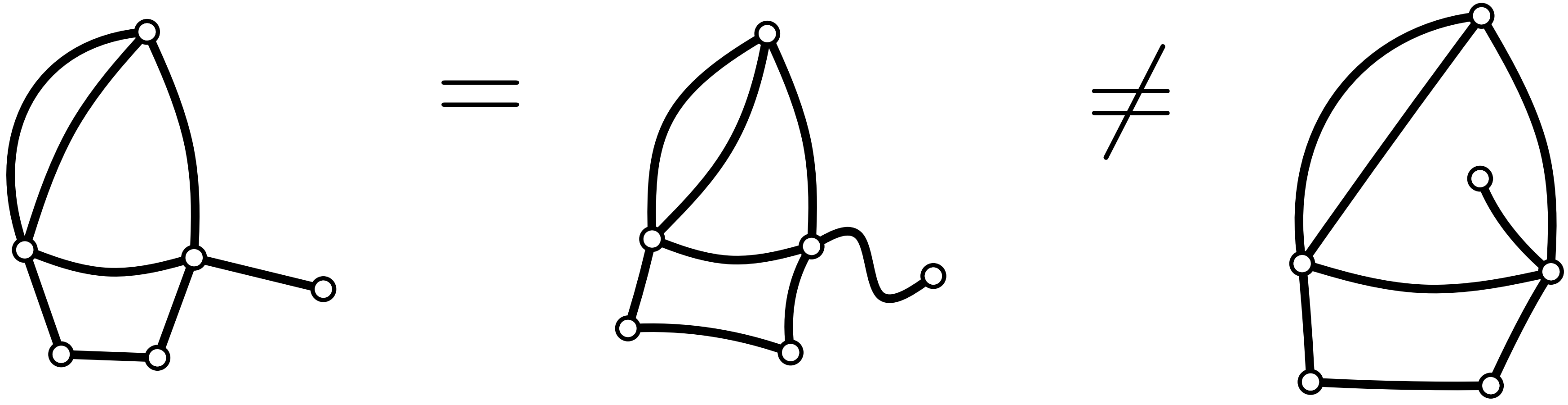
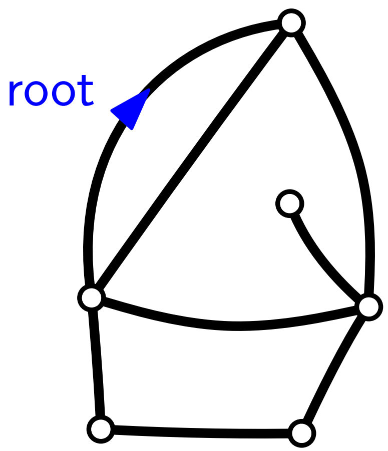
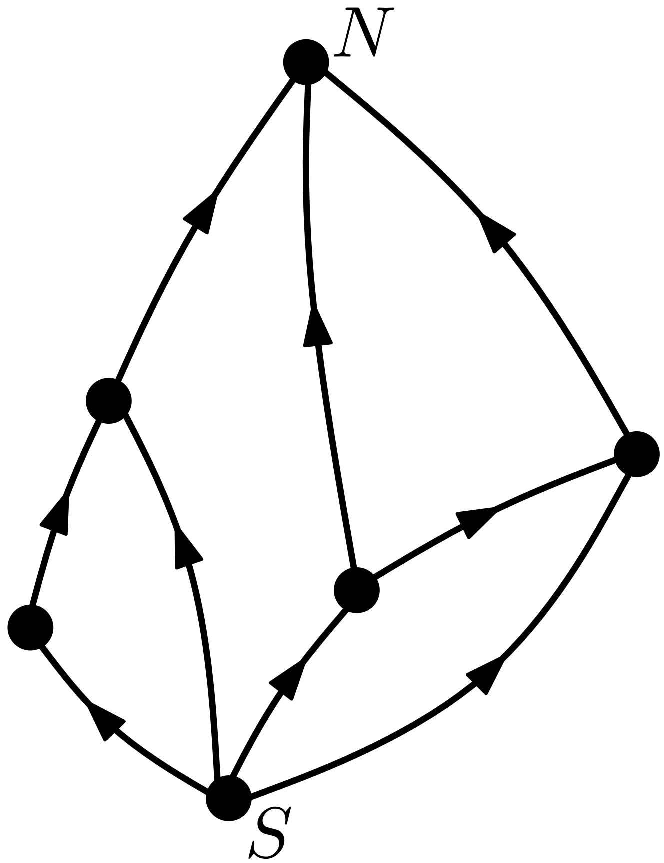


rooted planar map has marked oriented edge with outer face on its left

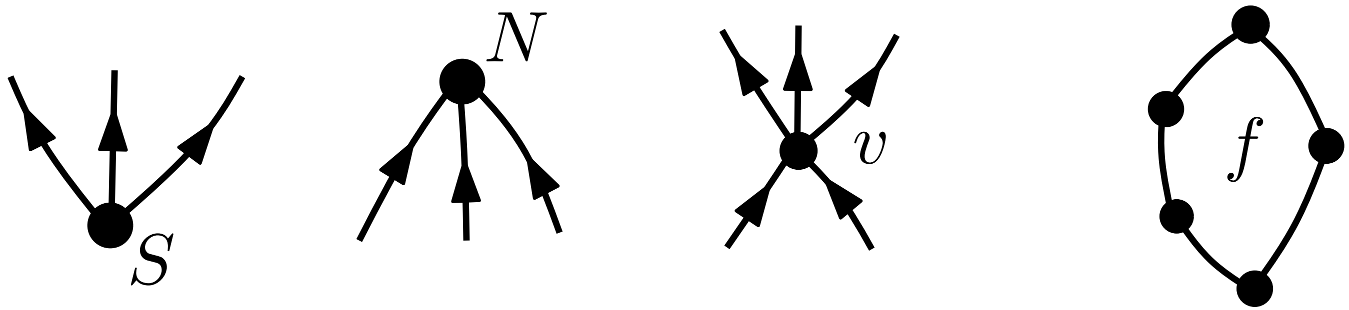


Plane bipolar orientations

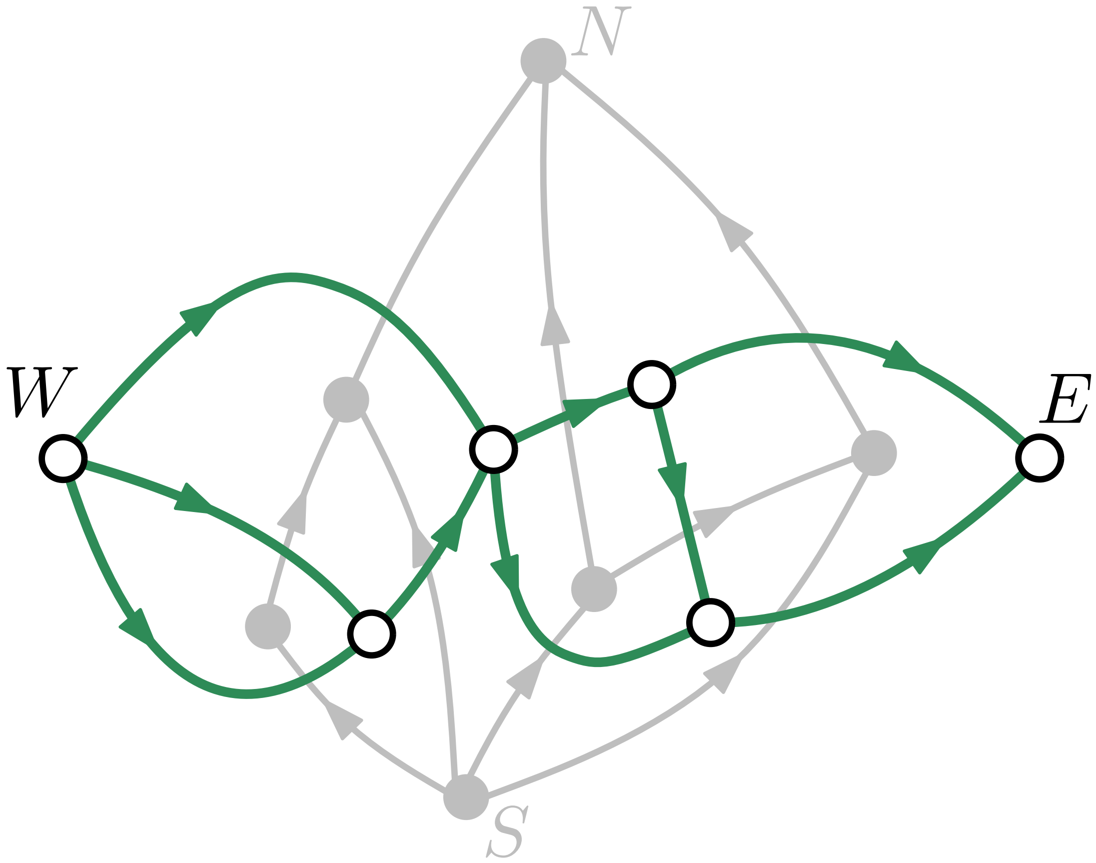


Acyclic orientation on planar map
with single min and single max
both incident to the outer face

Plane bipolar orientations $\Leftrightarrow$ local conditions [de Fraysseix et al'95]



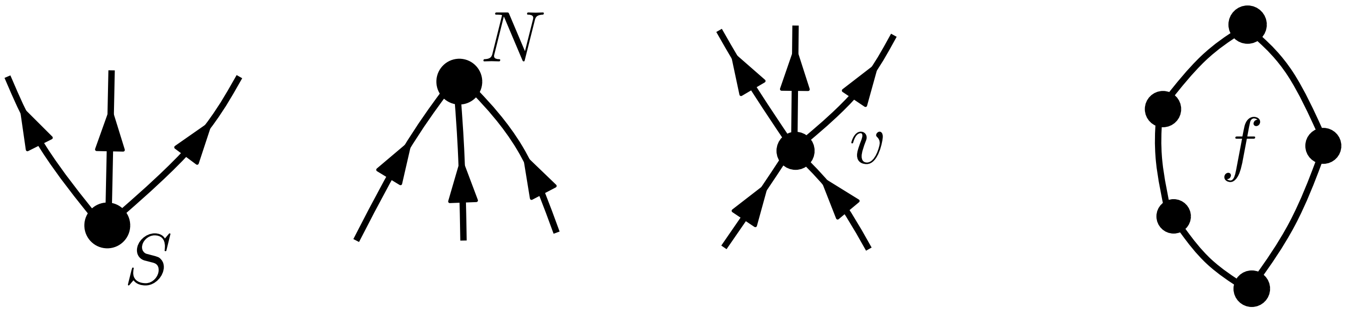
Plane bipolar orientations



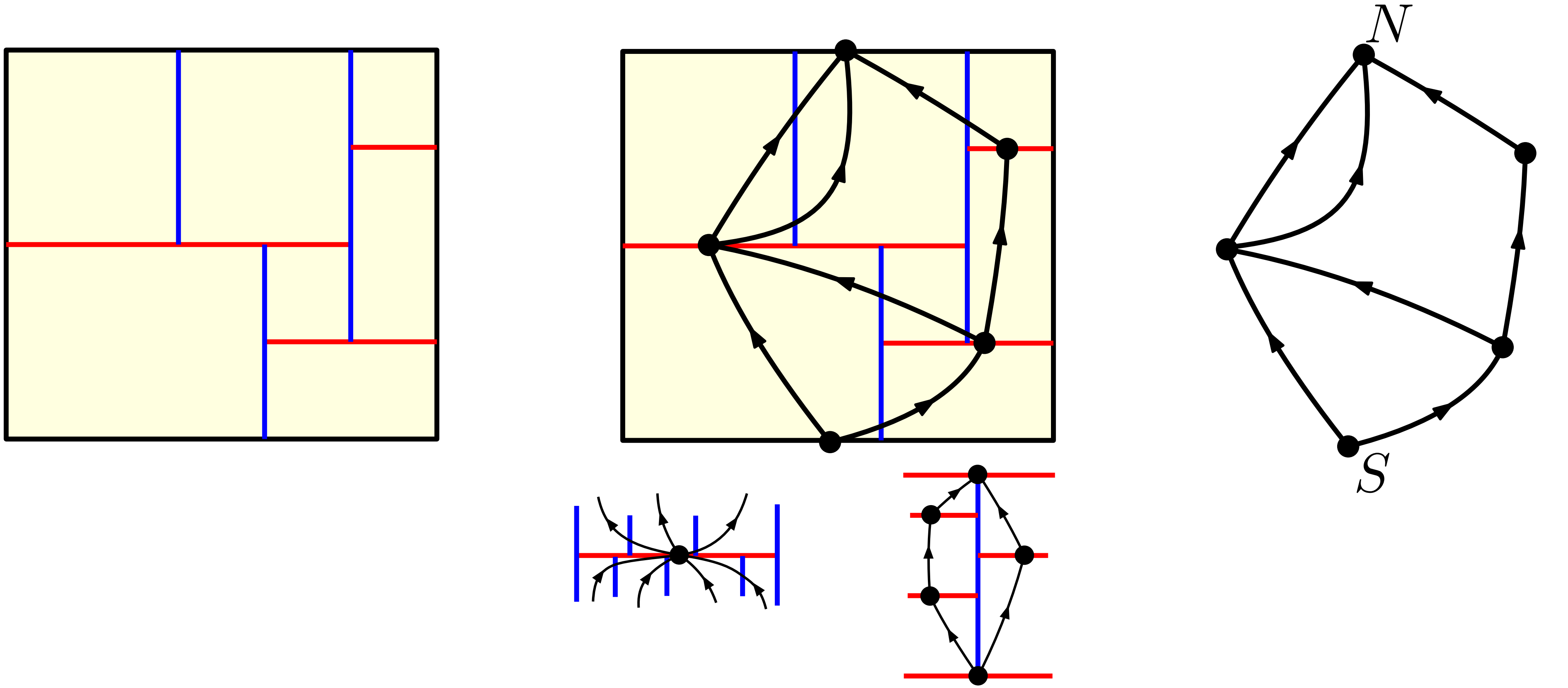
Acyclic orientation on planar map
with single min and single max
both incident to the outer face

dual bipolar orientation

Plane bipolar orientations ⇔ local conditions [de Fraysseix et al'95]



Link with weak rectangulations



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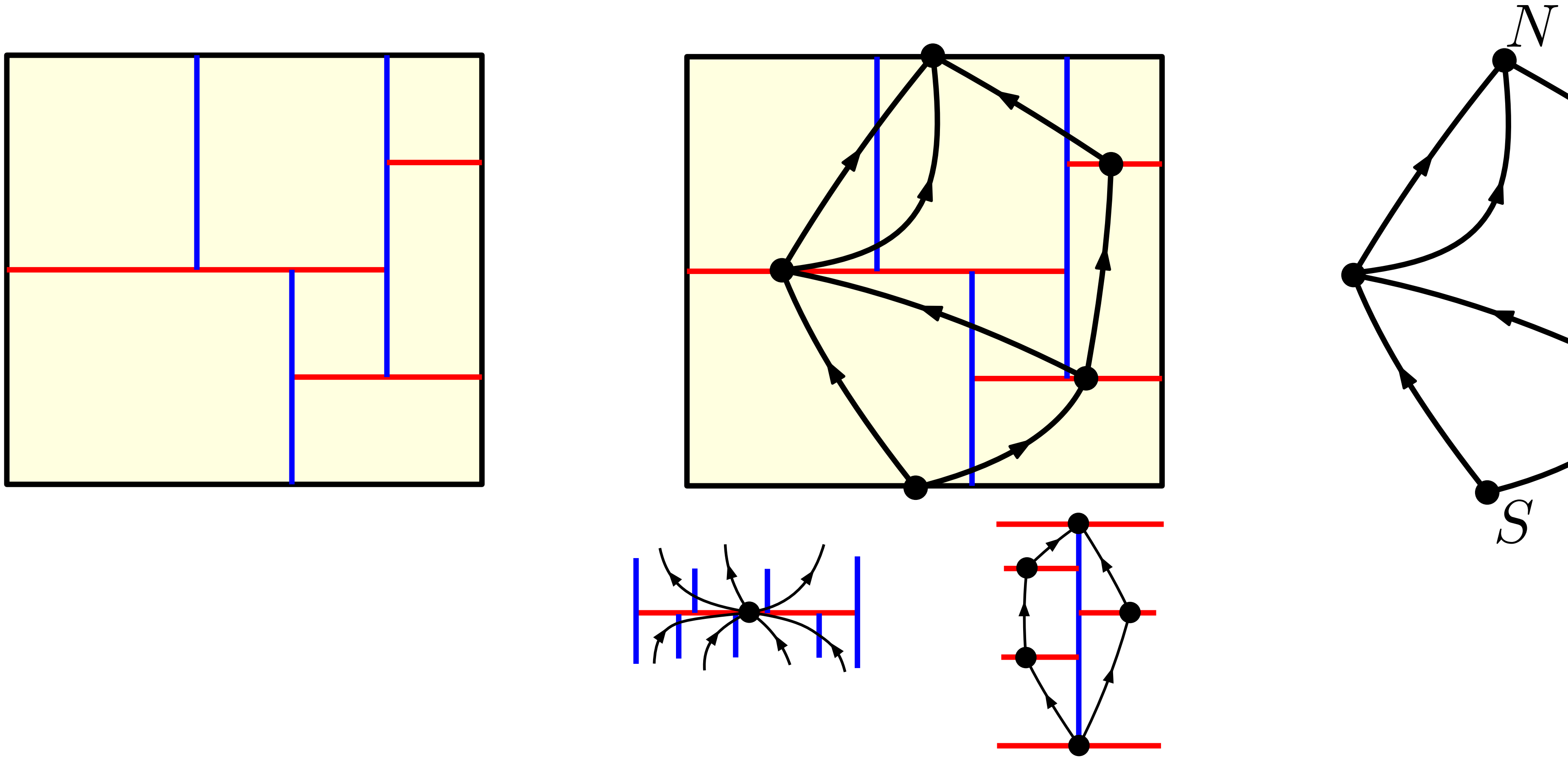
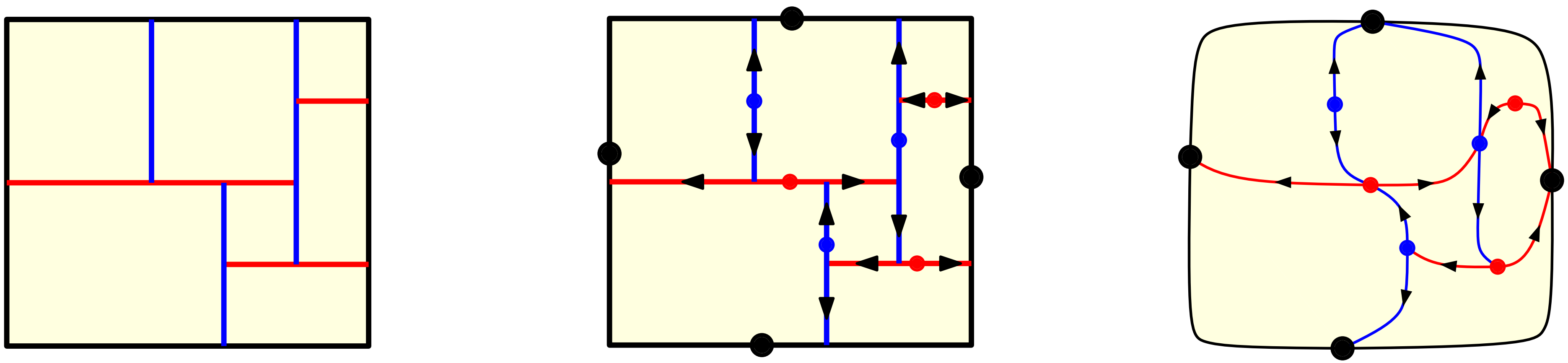
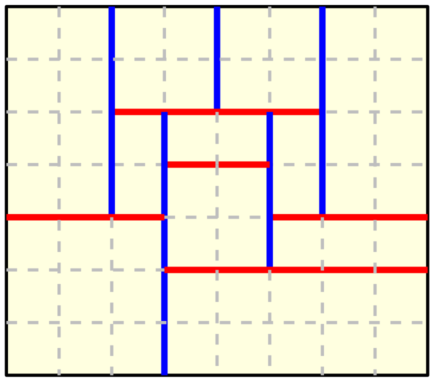


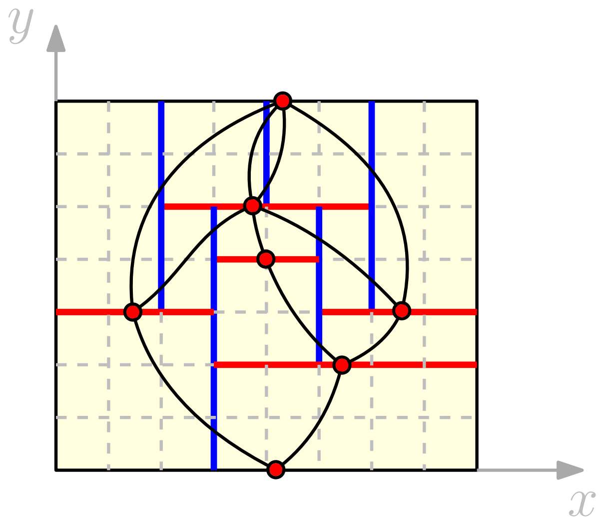
Rk: other oriented maps: 2-oriented quadrangulations [de Fraysseix et al'95]



Surjectivity/drawing algorithm

[Tamassia, Tollis'86]





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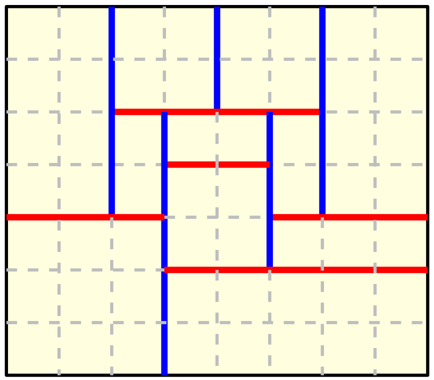

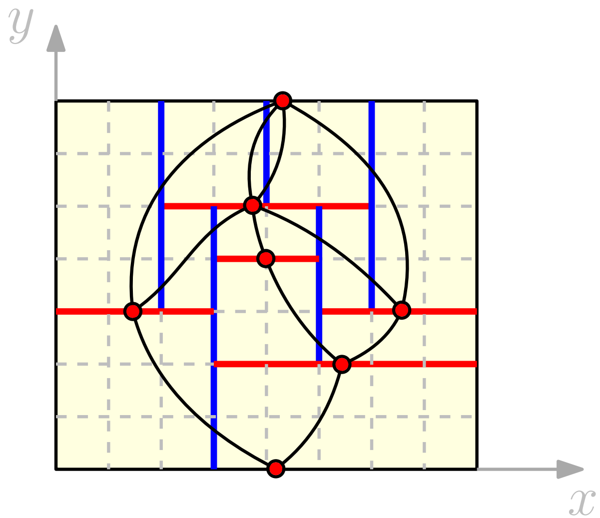

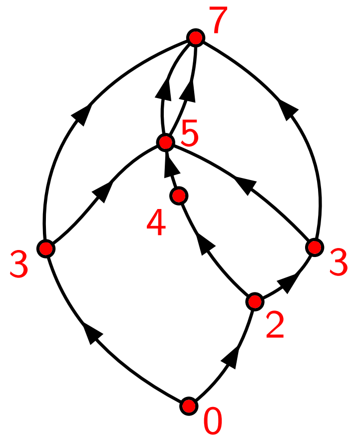

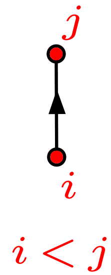

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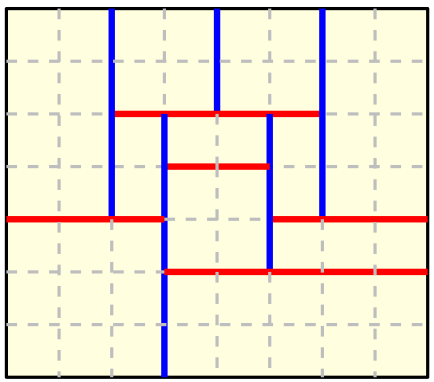

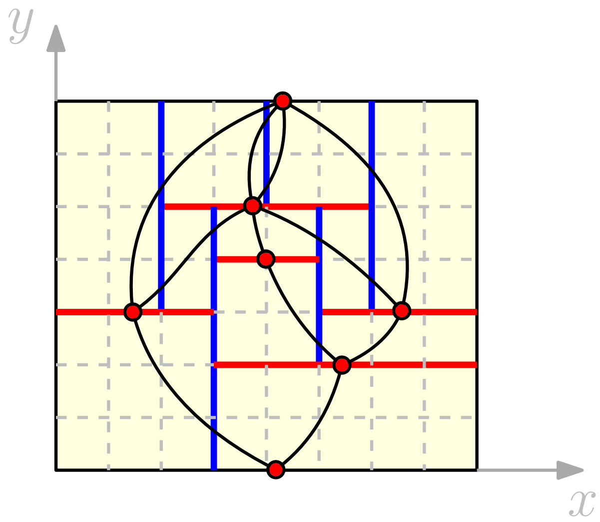

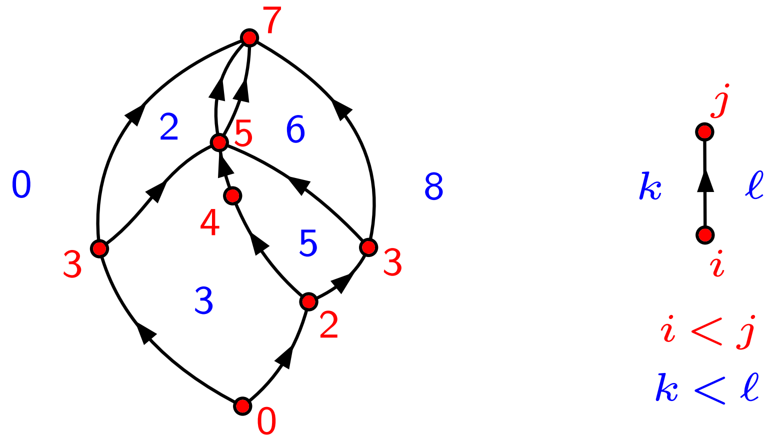

increasing red/blue labels

Surjectivity/drawing algorithm

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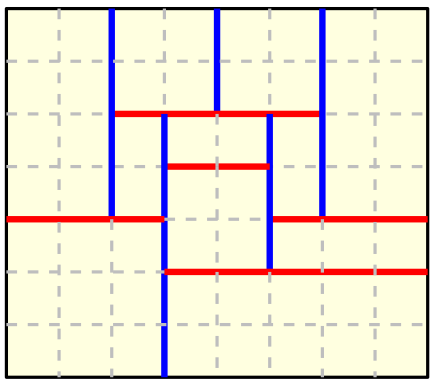

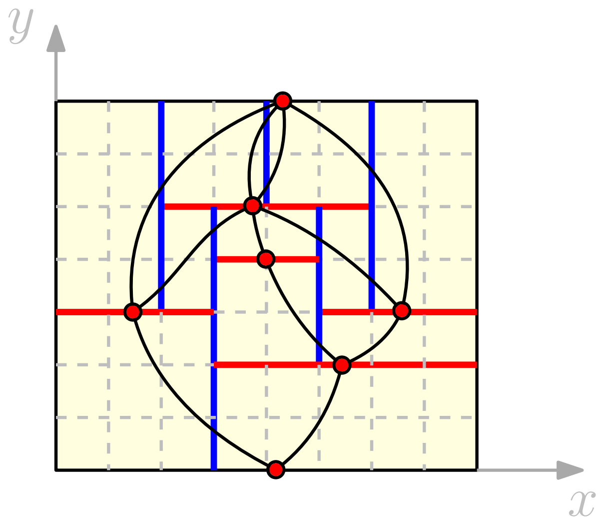

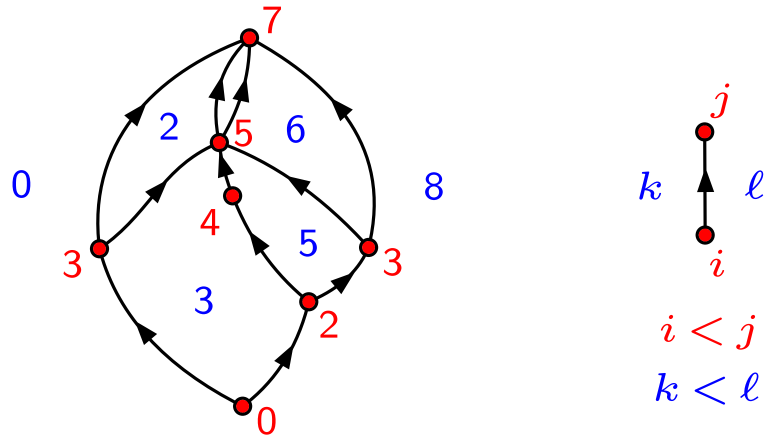

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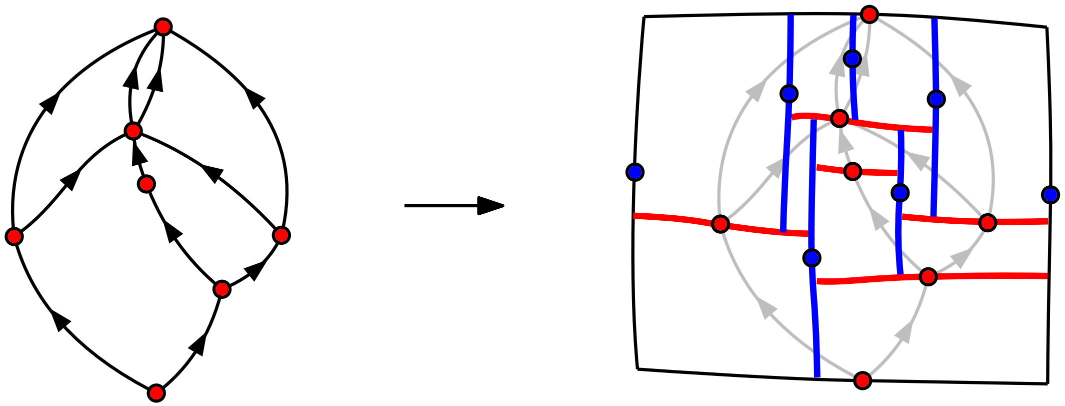
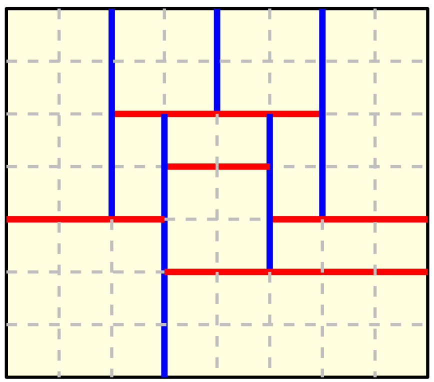
Conversely

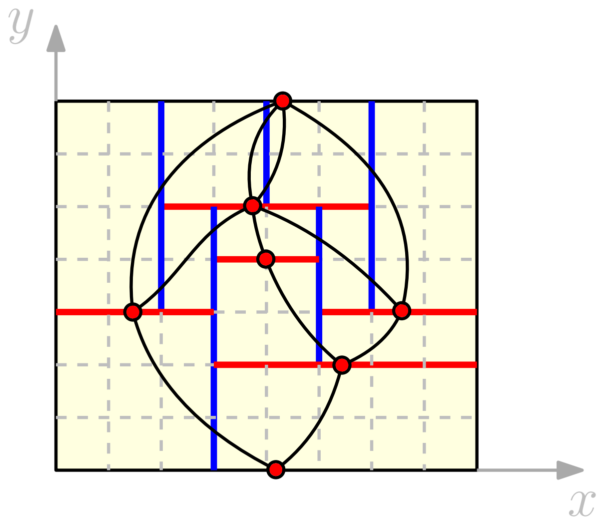


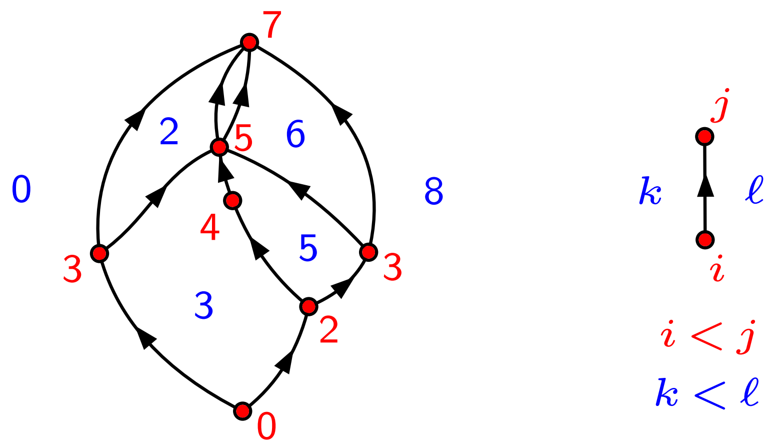
contact system
of curves

Surjectivity/drawing algorithm

[Tamassia, Tollis'86]







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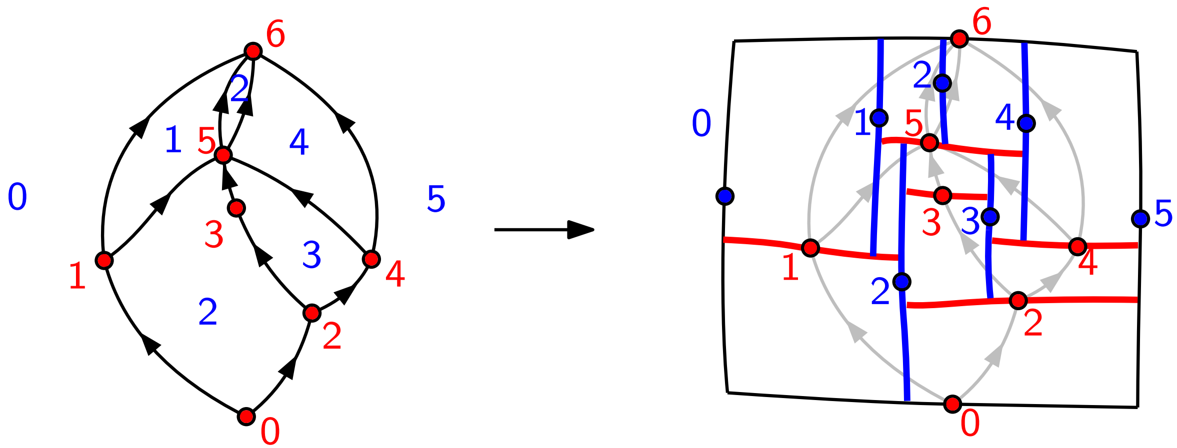

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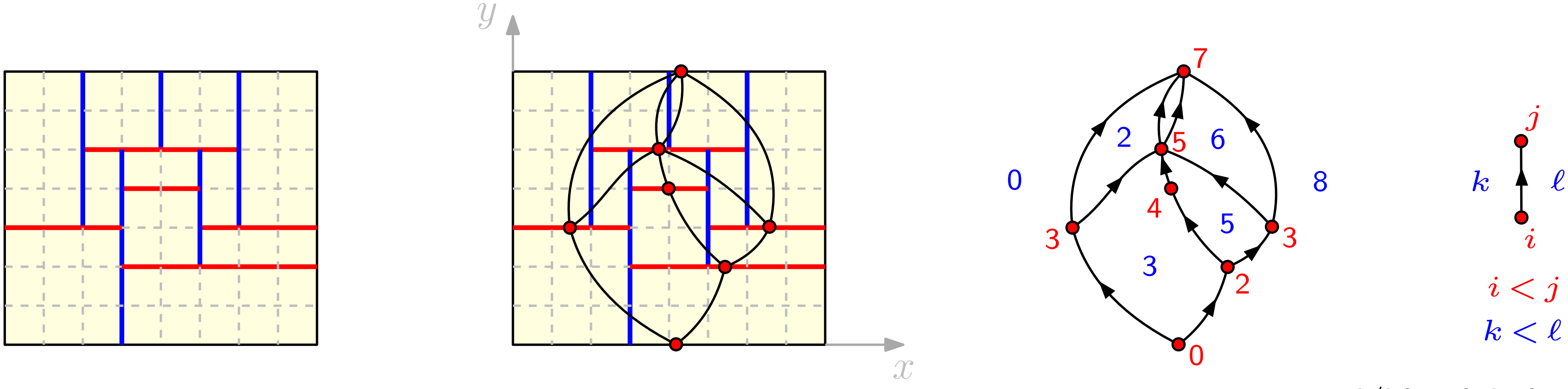

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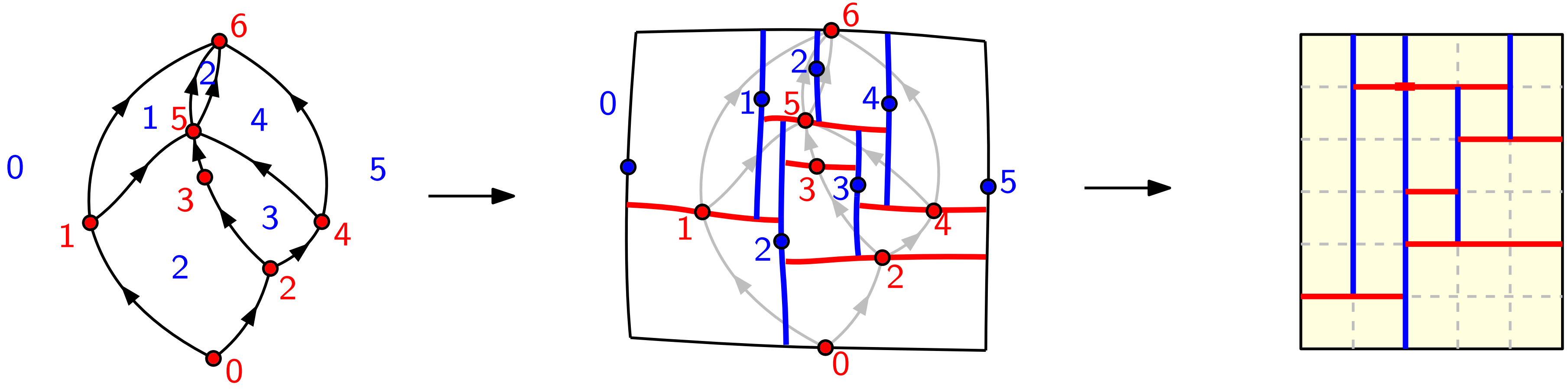
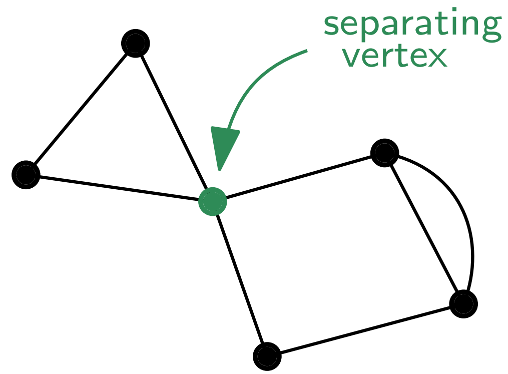


contact system of curves

Underlying maps

2-connected map
||
map with no separating vertex



not 2-connected

Underlying maps

2-connected map
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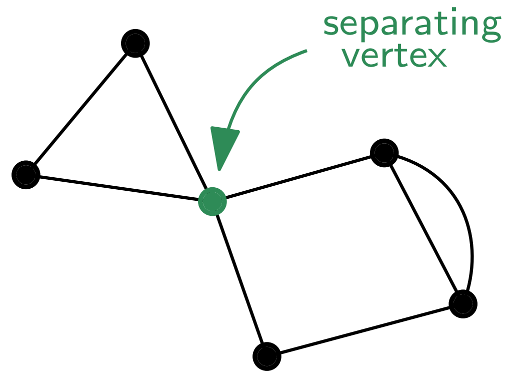
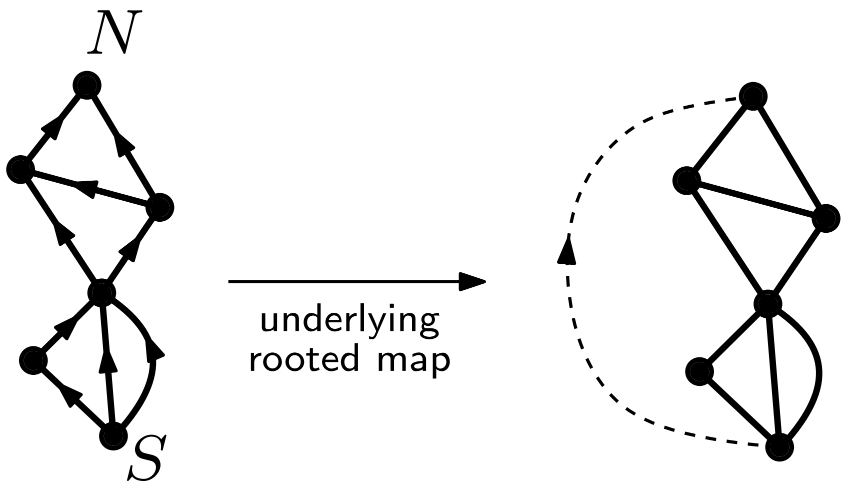
map with no separating vertex



not 2-connected



Underlying maps

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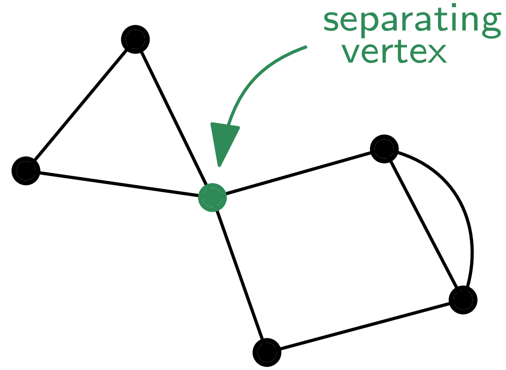

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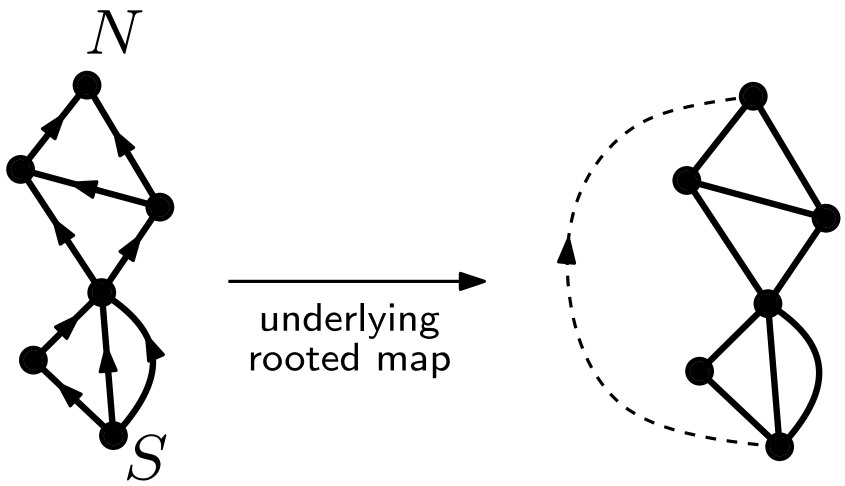
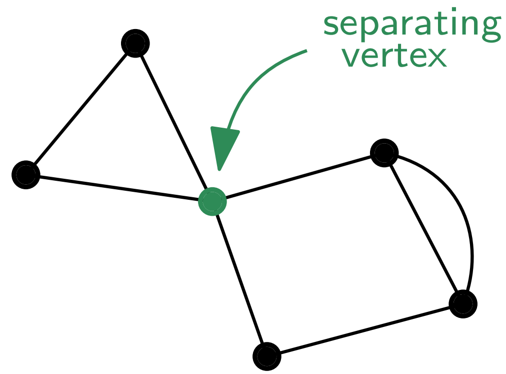


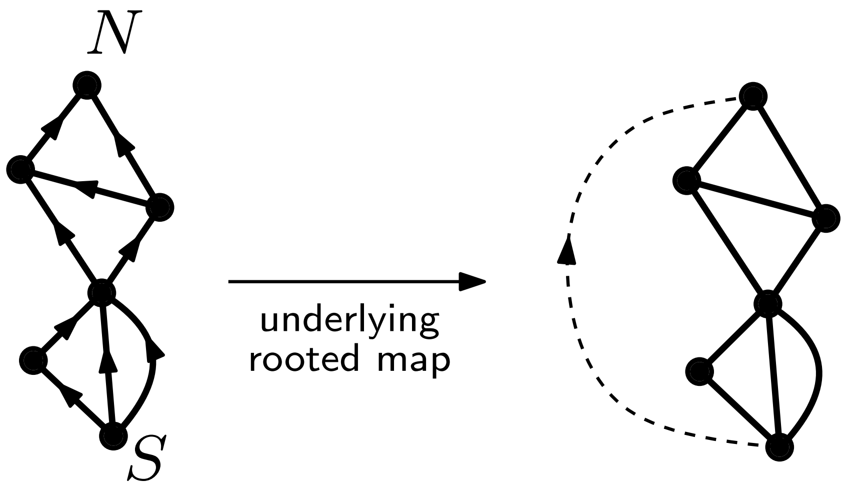
Theo: [Lempel et al.'67], [Rosenstiehl, Tarjan'86]
a rooted map admits a bipolar orientation iff it is 2-connected

Underlying maps

2-connected map
||
map with no separating vertex

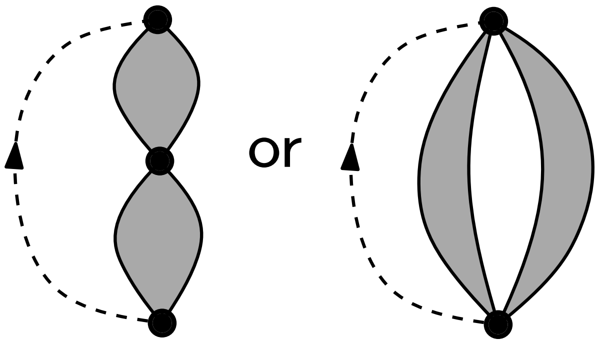


not 2-connected



Theo: [Lempel et al.'67], [Rosenstiehl, Tarjan'86]
a rooted map admits a bipolar orientation iff it is 2-connected

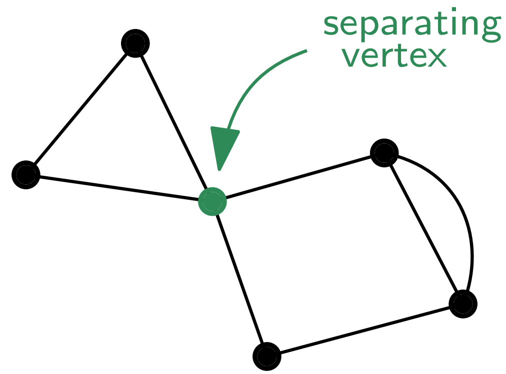
Rk: Is is not unique unless the map is series-parallel



(and recursively)

Underlying maps

2-connected map
||
map with no separating vertex



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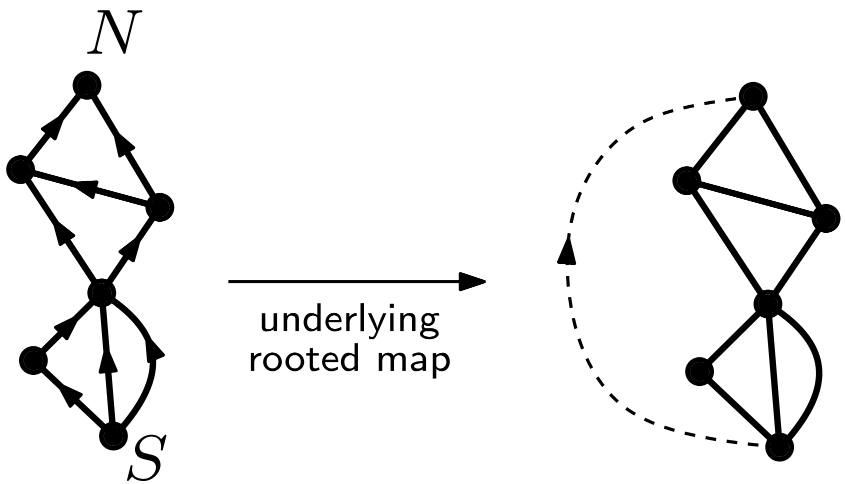
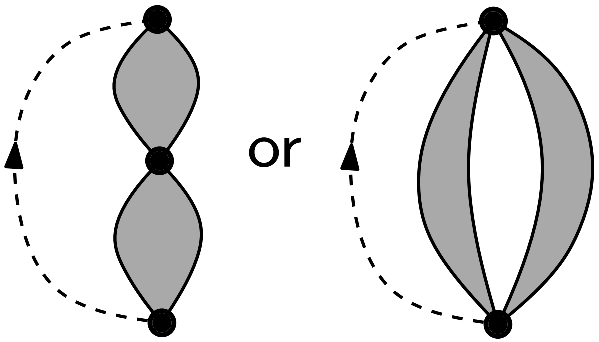
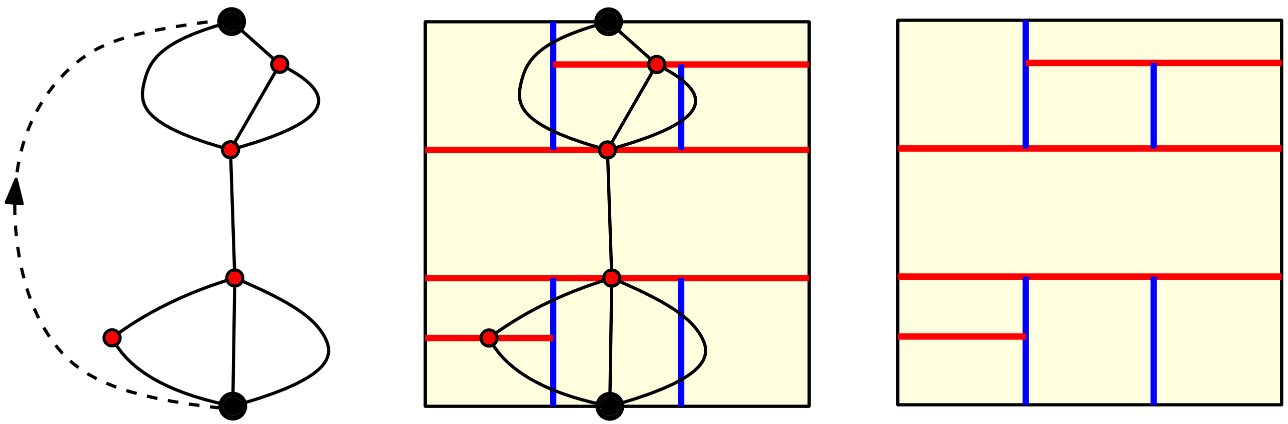


Theo: [Lempel et al.'67], [Rosenstiehl, Tarjan'86]
a rooted map admits a bipolar orientation iff it is 2-connected

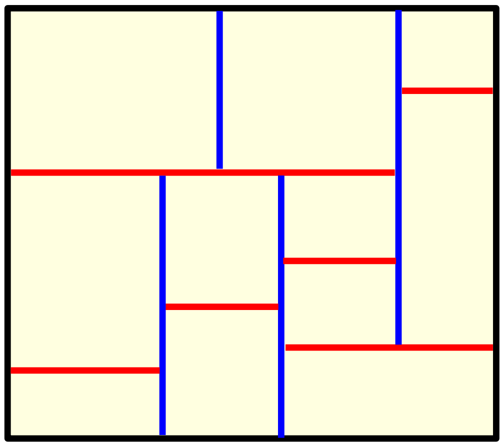
Rk: Is is not unique unless the map is series-parallel (guillotine rectangulation)

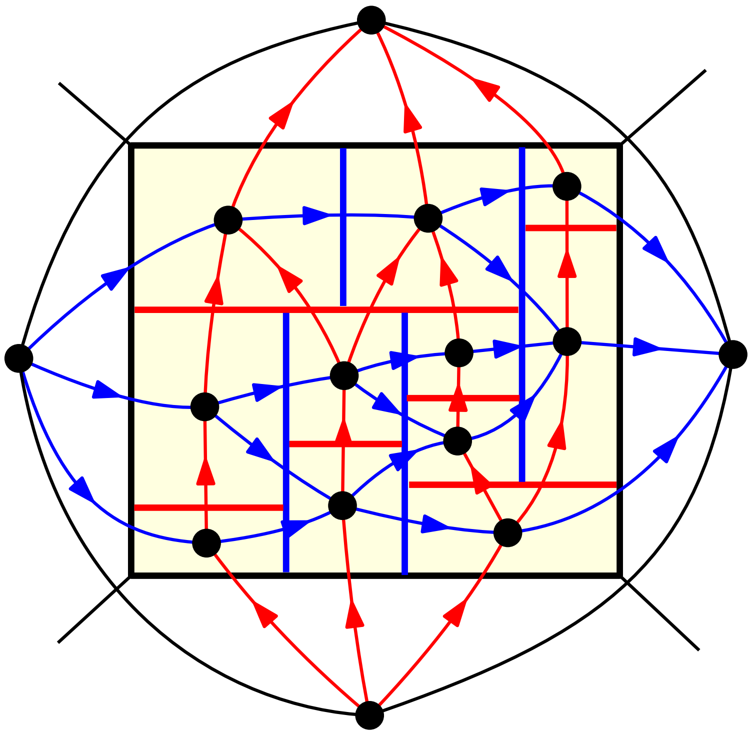


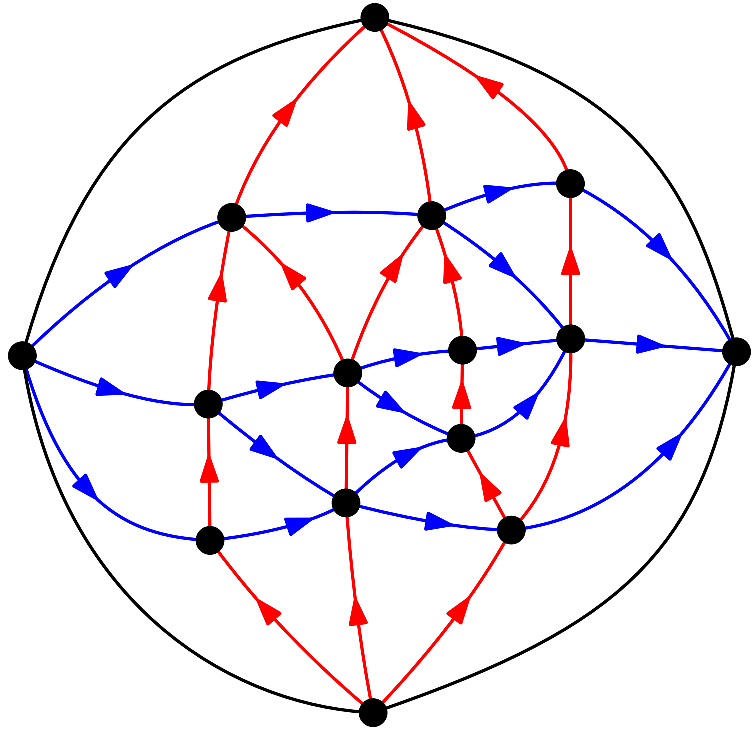
(and recursively)



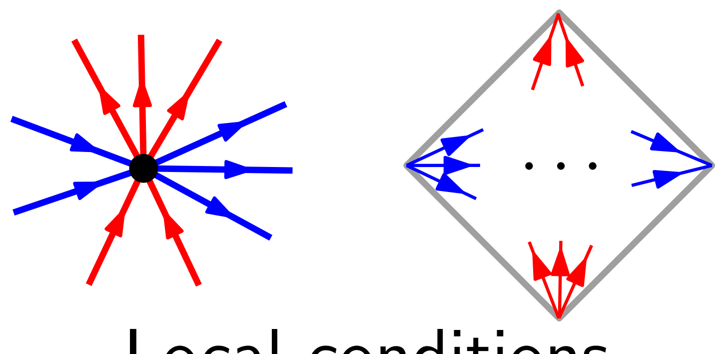
Oriented maps for strong rectangulations [He'93]





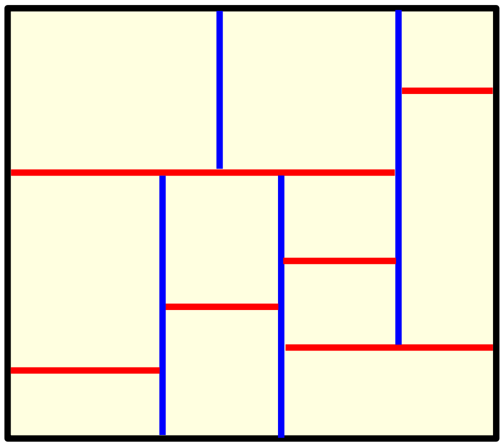


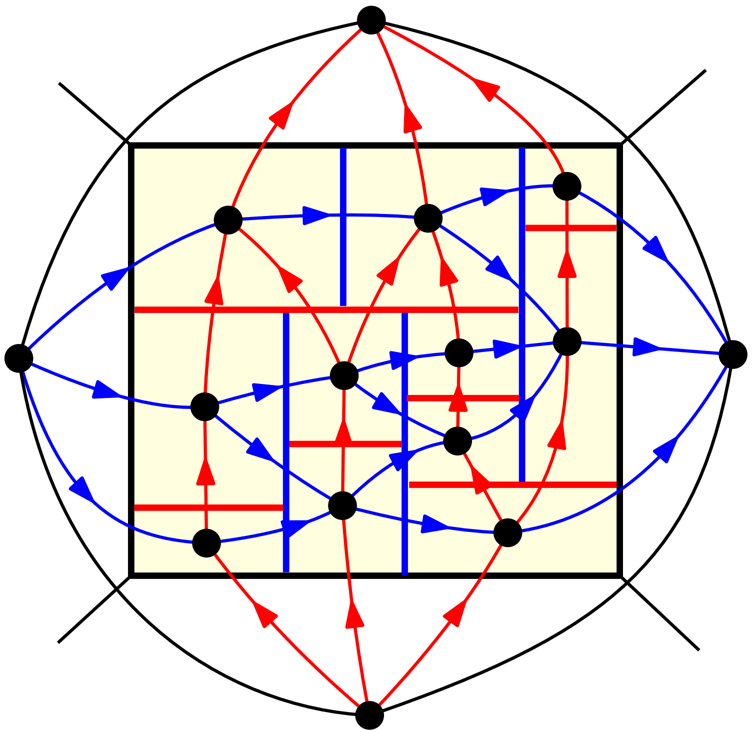
Pair of transversal
plane bipolar orientations

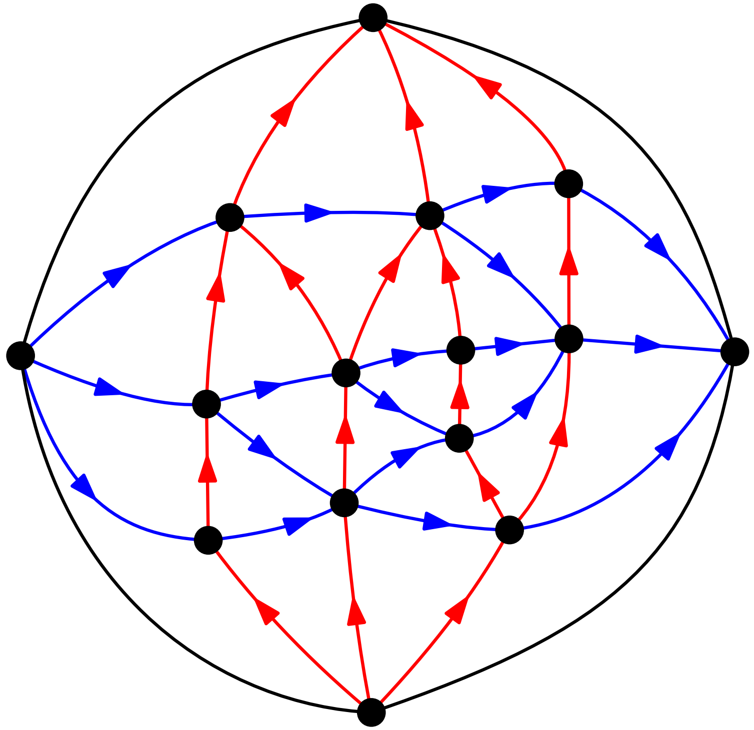


Local conditions

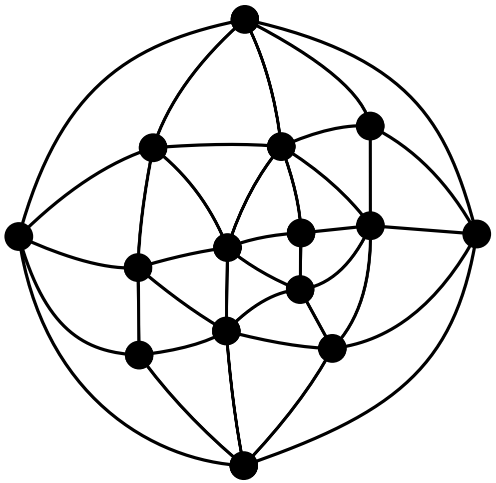
Oriented maps for strong rectangulations [He'93]





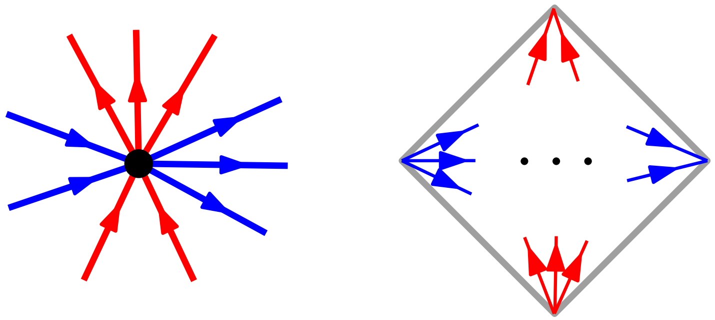


Pair of transversal plane bipolar orientations



Underlying maps:

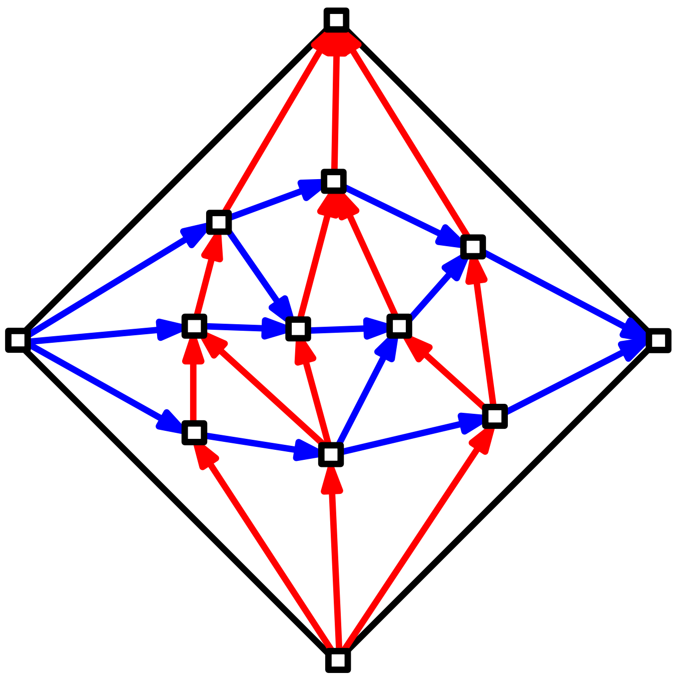
triangulations of 4-gon
every triangle bounds a face

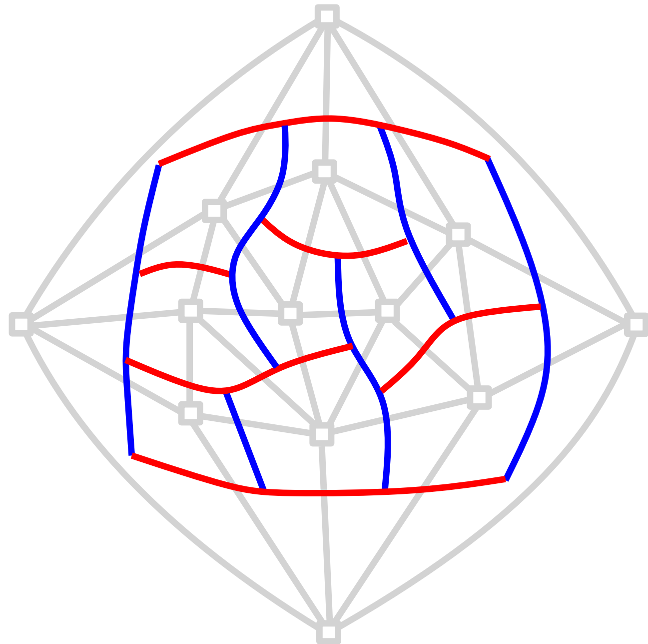


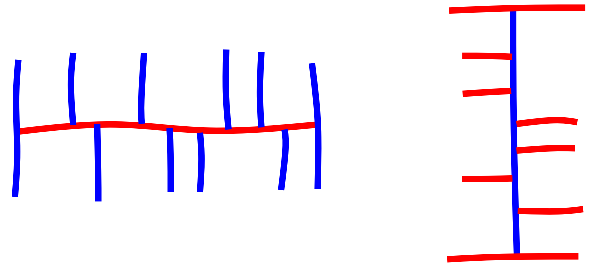
Local conditions

Surjectivity/drawing algorithm

[He'93]



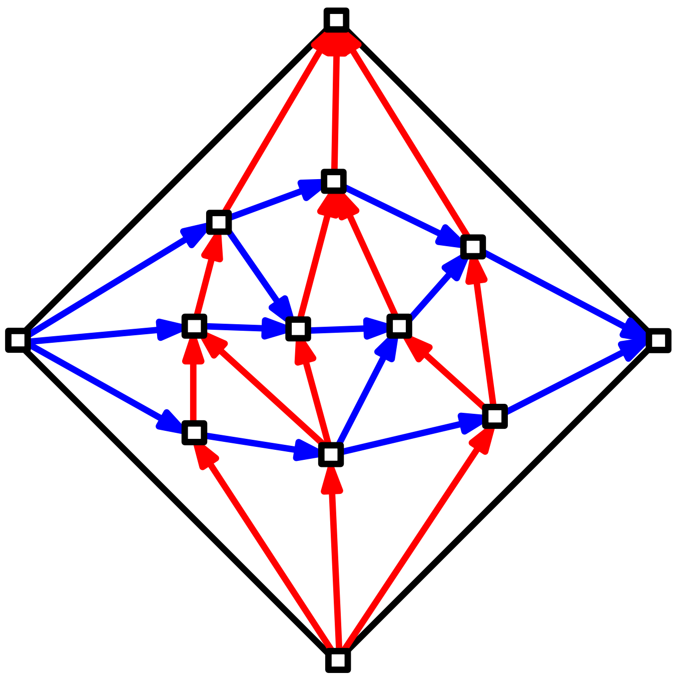


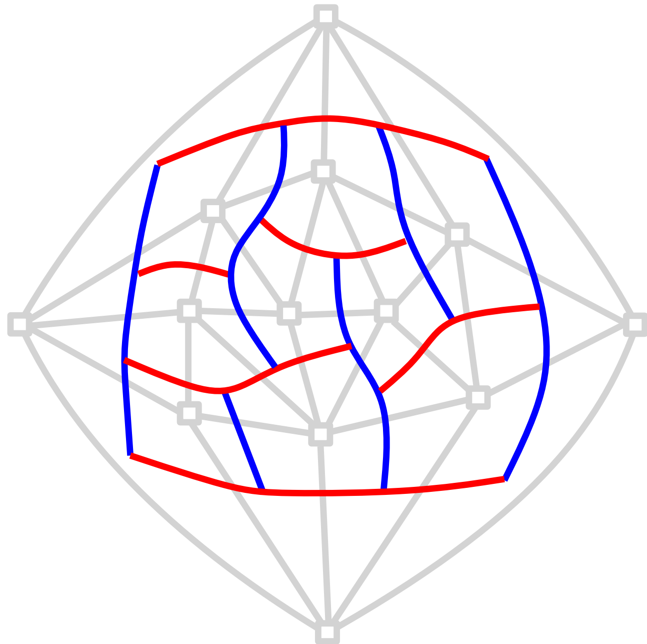


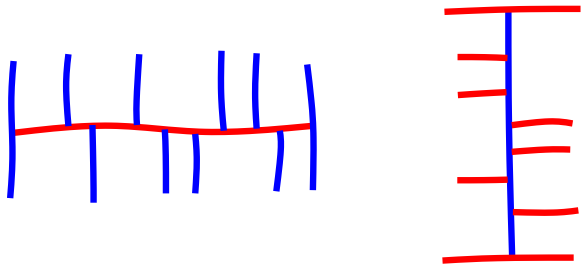
contact system
where order matters

Surjectivity/drawing algorithm

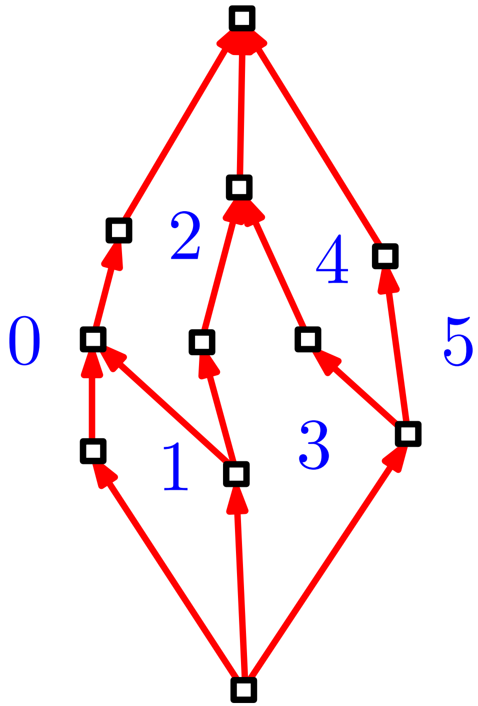
[He'93]

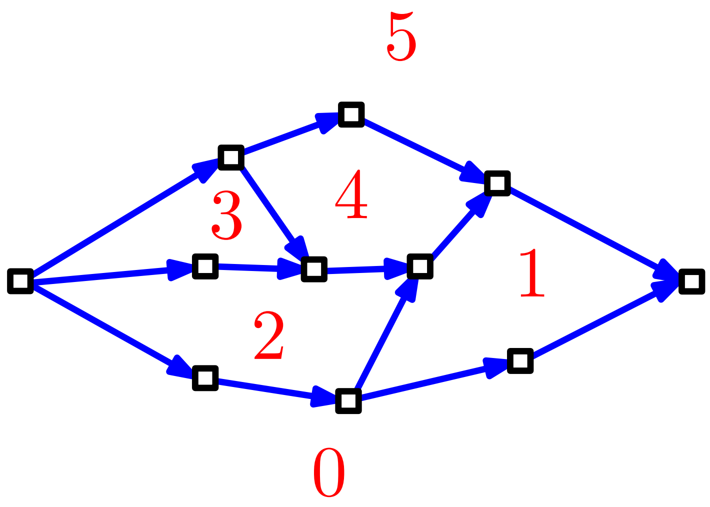


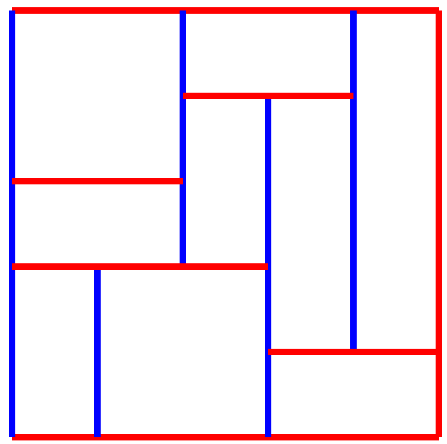




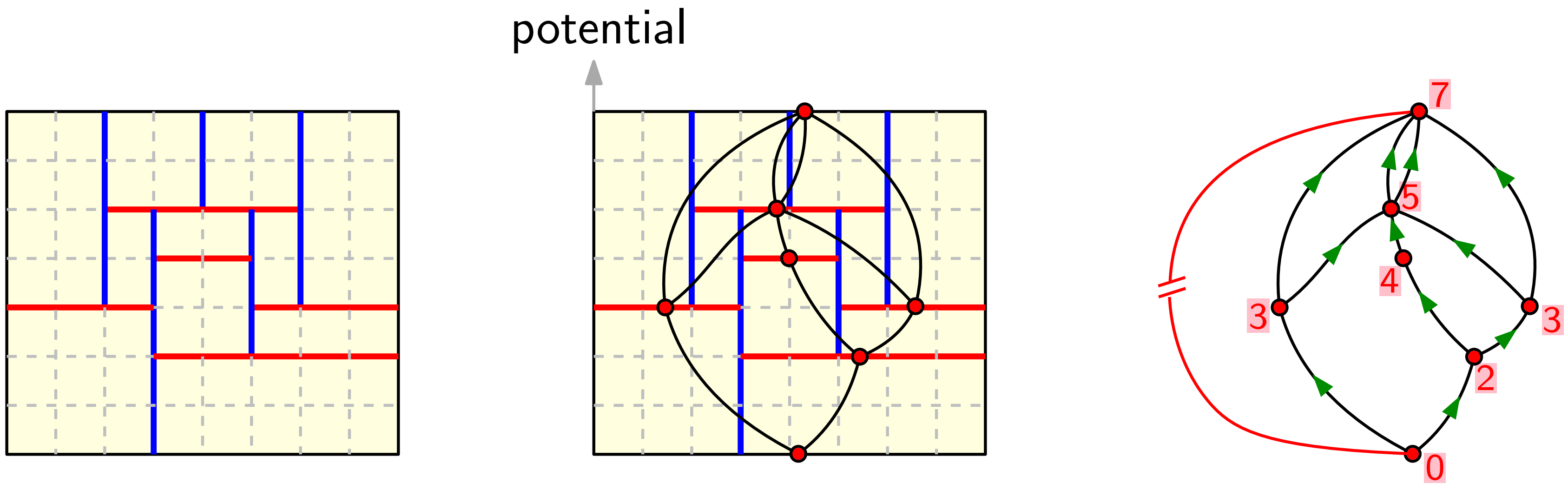
contact system
where order matters



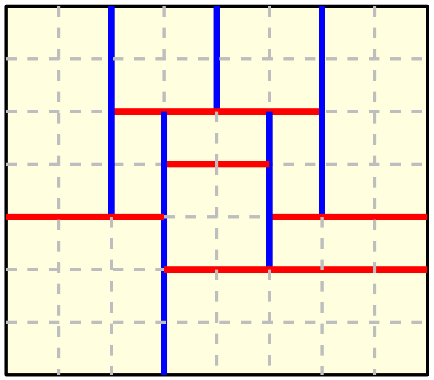


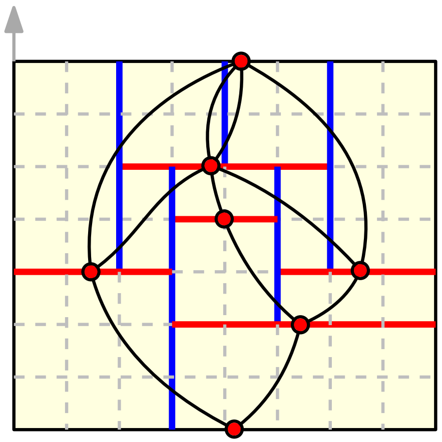


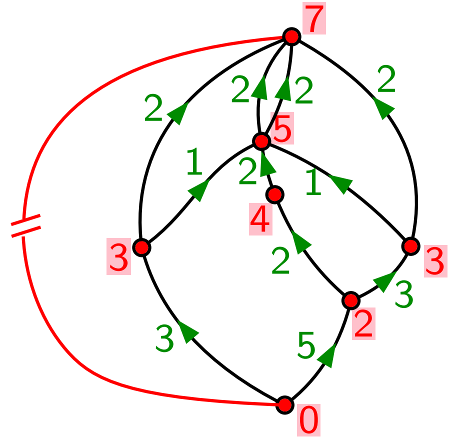
Electrical circuit interpretation (for weak rectangulations)

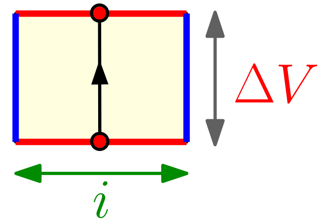


Electrical circuit interpretation (for weak rectangulations)

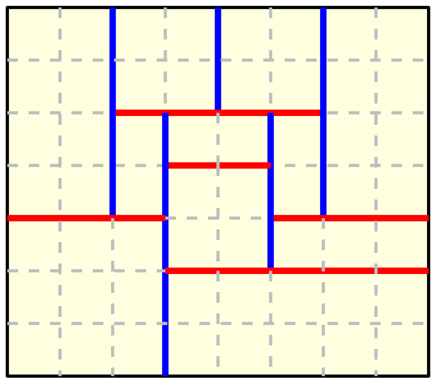


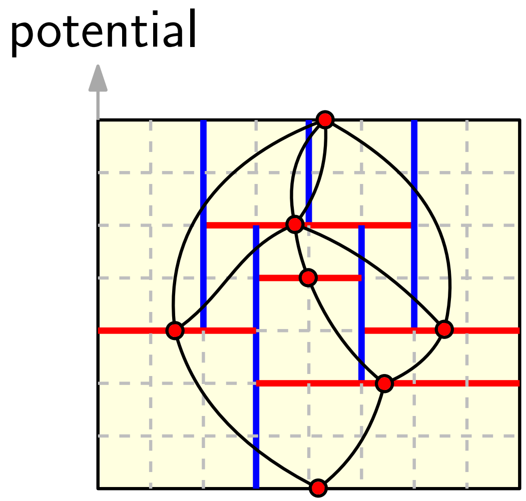


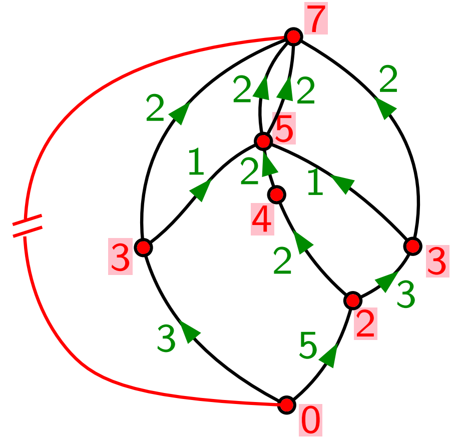


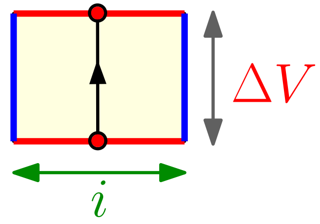


Electrical circuit interpretation (for weak rectangulations)



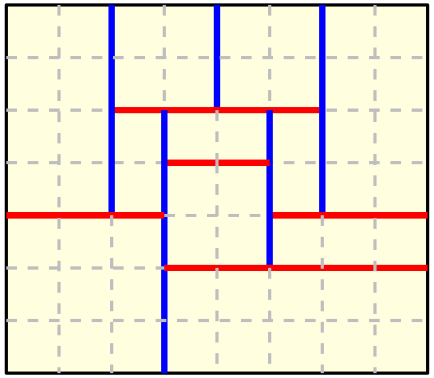


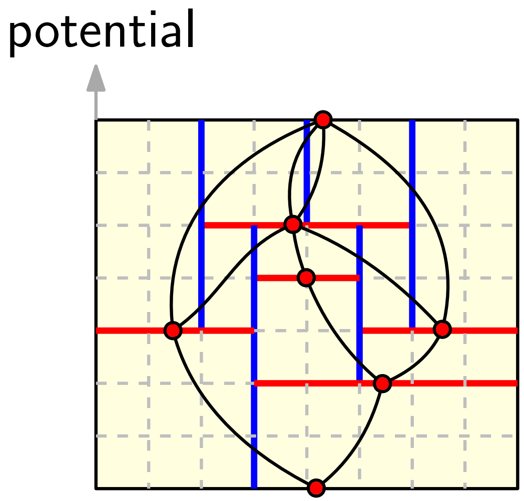


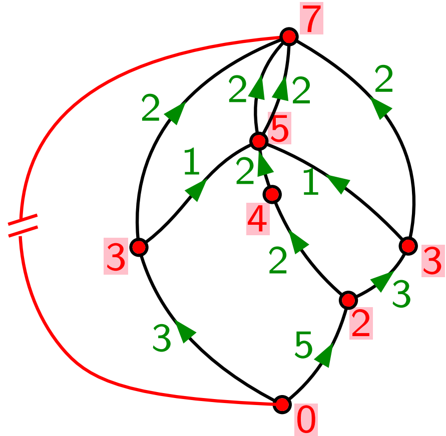


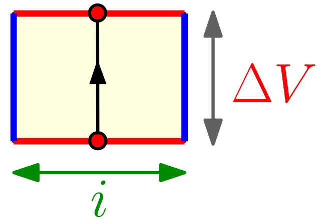
resistance
=
aspect ratio

Electrical circuit interpretation (for weak rectangulations)



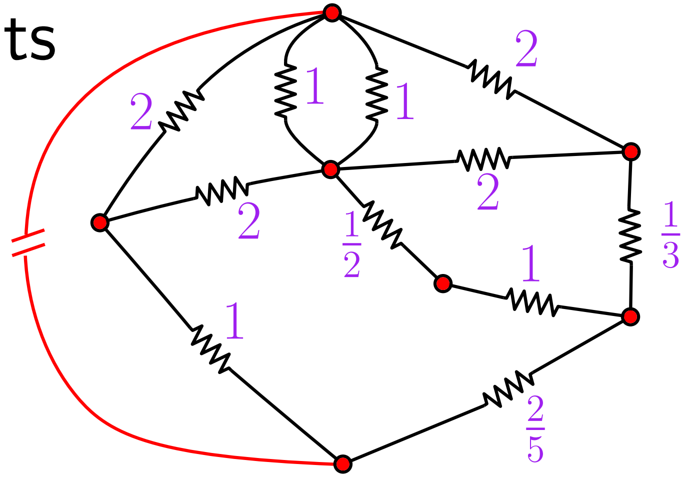




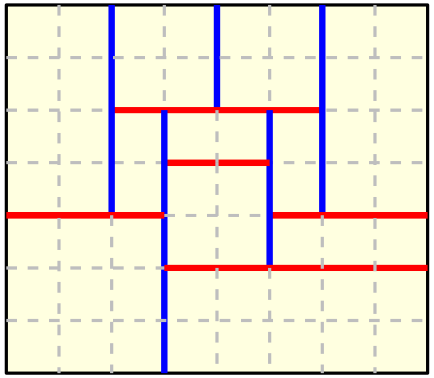


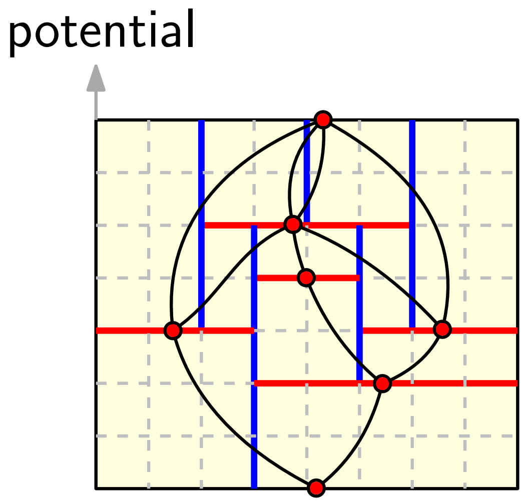
resistance = aspect ratio

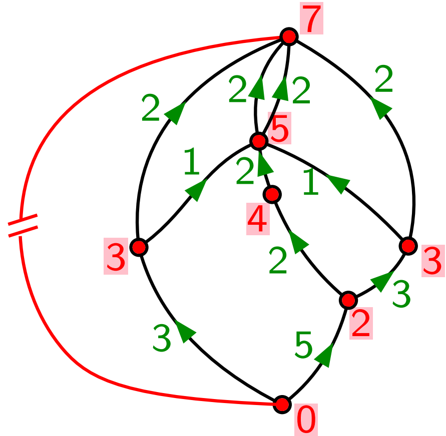
represents

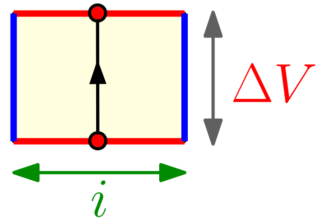


Electrical circuit interpretation (for weak rectangulations)



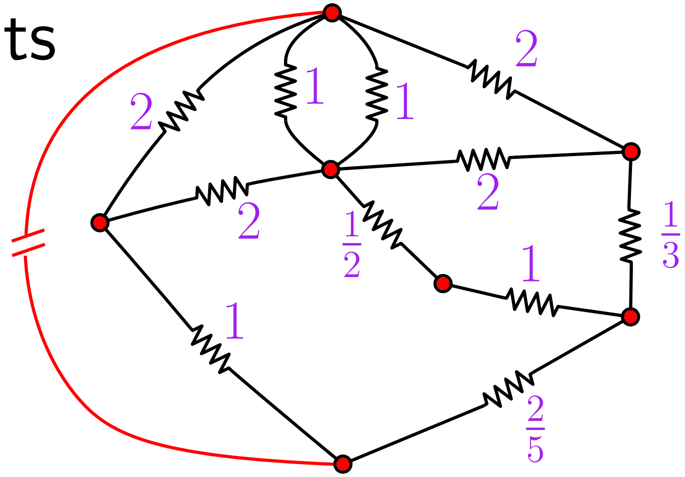






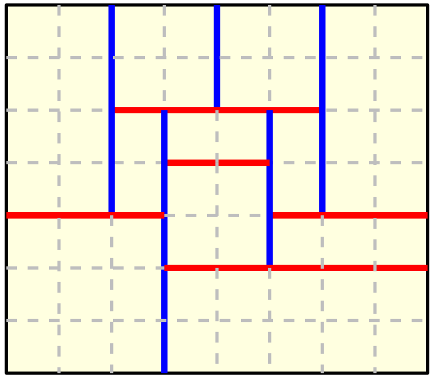
resistance
=
aspect ratio

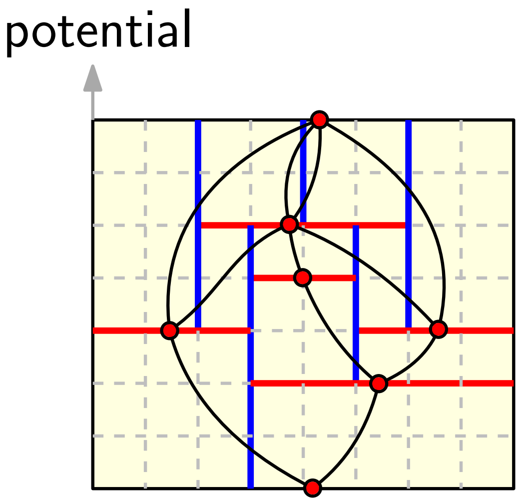
represents

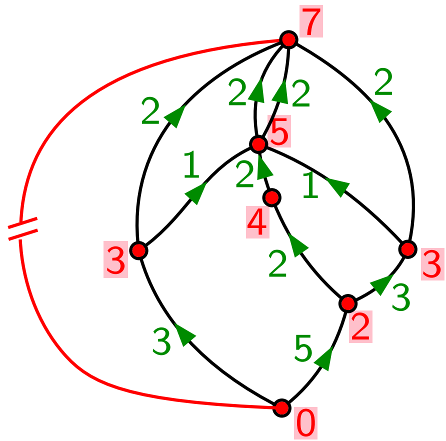


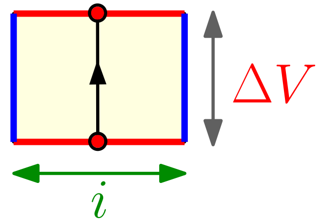
each **2-connected map** has **unique** rectangulation representative with **prescribed aspect ratios**

Electrical circuit interpretation (for weak rectangulations)



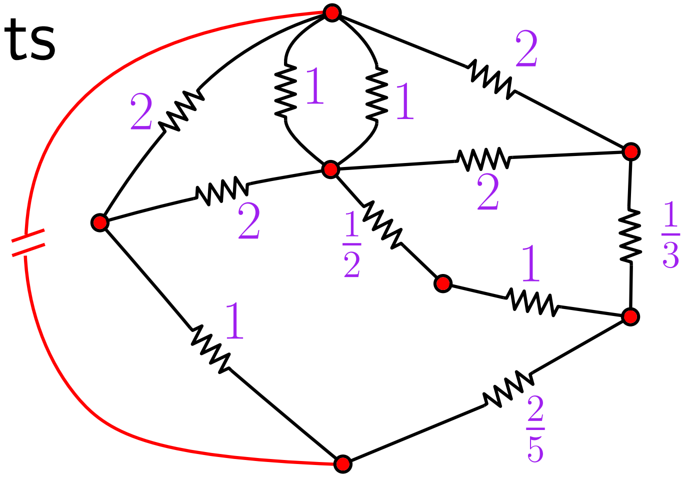






resistance
=
aspect ratio

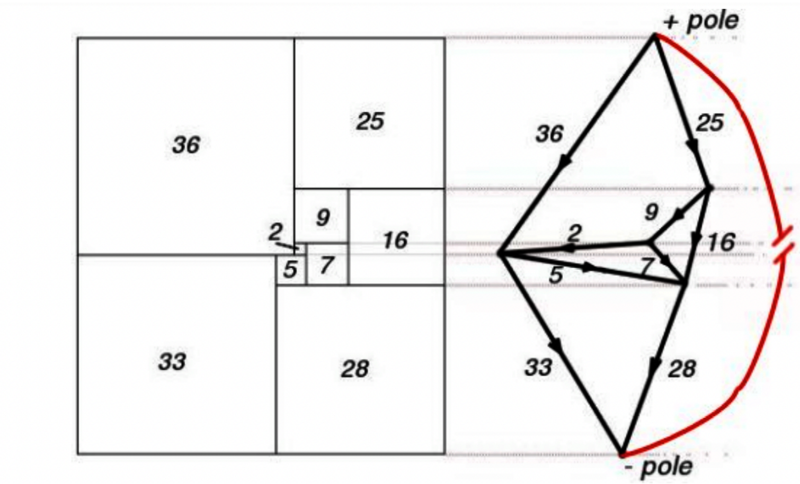
represents



each **2-connected map** has **unique** rectangulation representative with **prescribed aspect ratios**

application to "squaring the square"

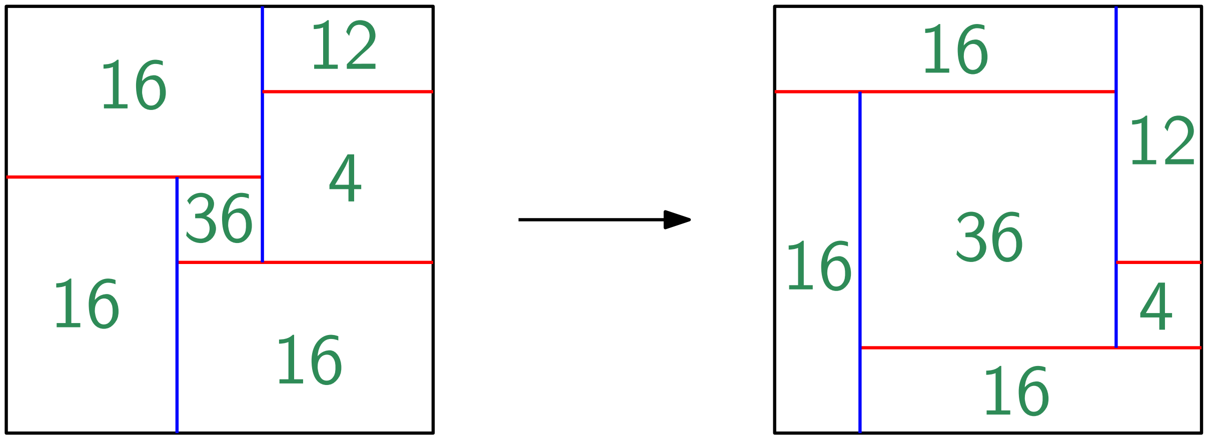
[Brooks, Smith, Stone, Tutte'40]



(www.squaring.net/history_theory/gfx/figure73.jpg)

Area universality (for weak rectangulations) [Eppstein et al.12]

each **weak rectangulation** has **unique** representative with **prescribed areas**

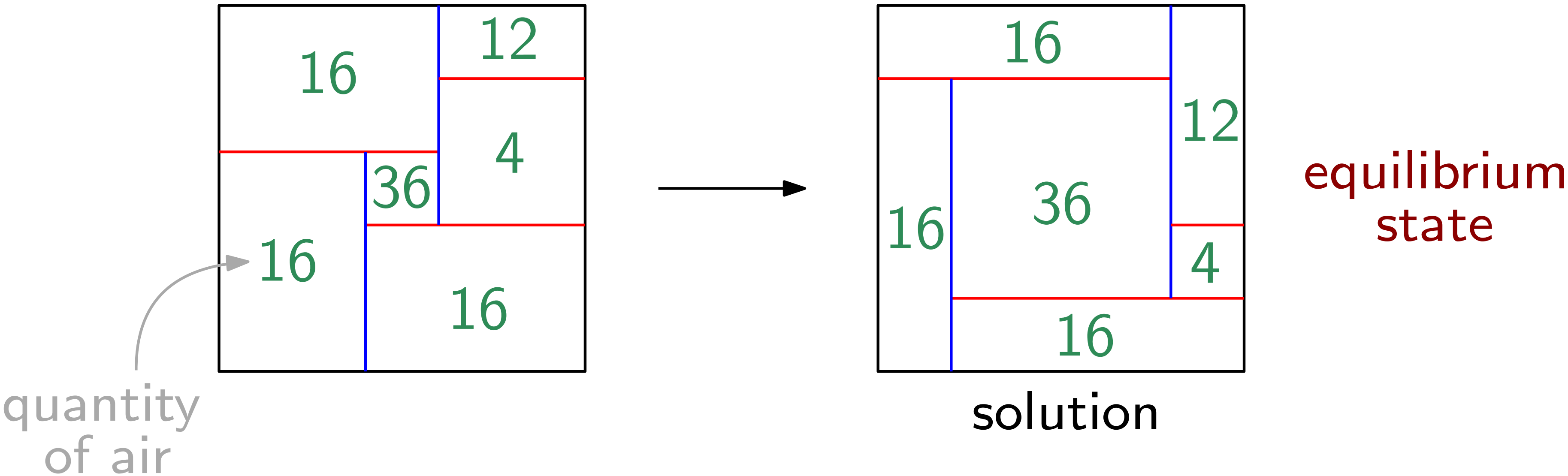


solution

Area universality (for weak rectangulations)

[Eppstein et al.12]

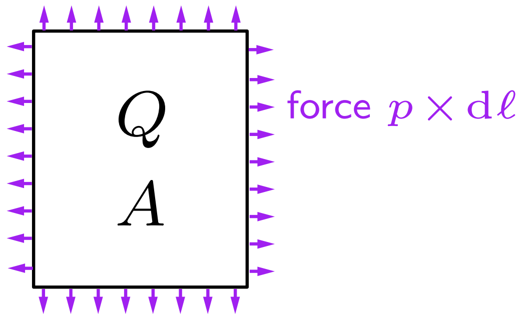
each **weak rectangulation** has **unique** representative with **prescribed areas**



$Q =$ quantity of air

$A =$ current area

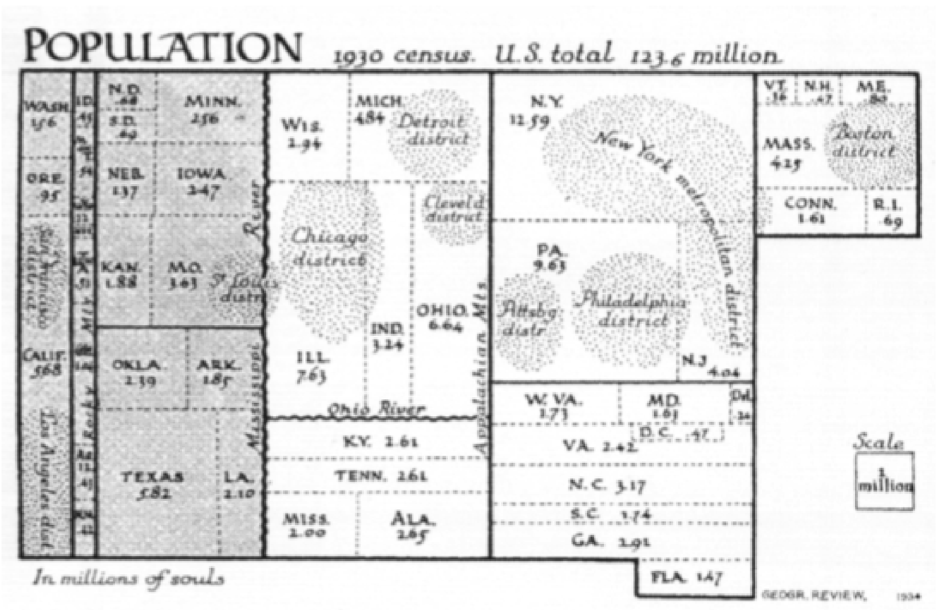
pressure $p = Q/A$

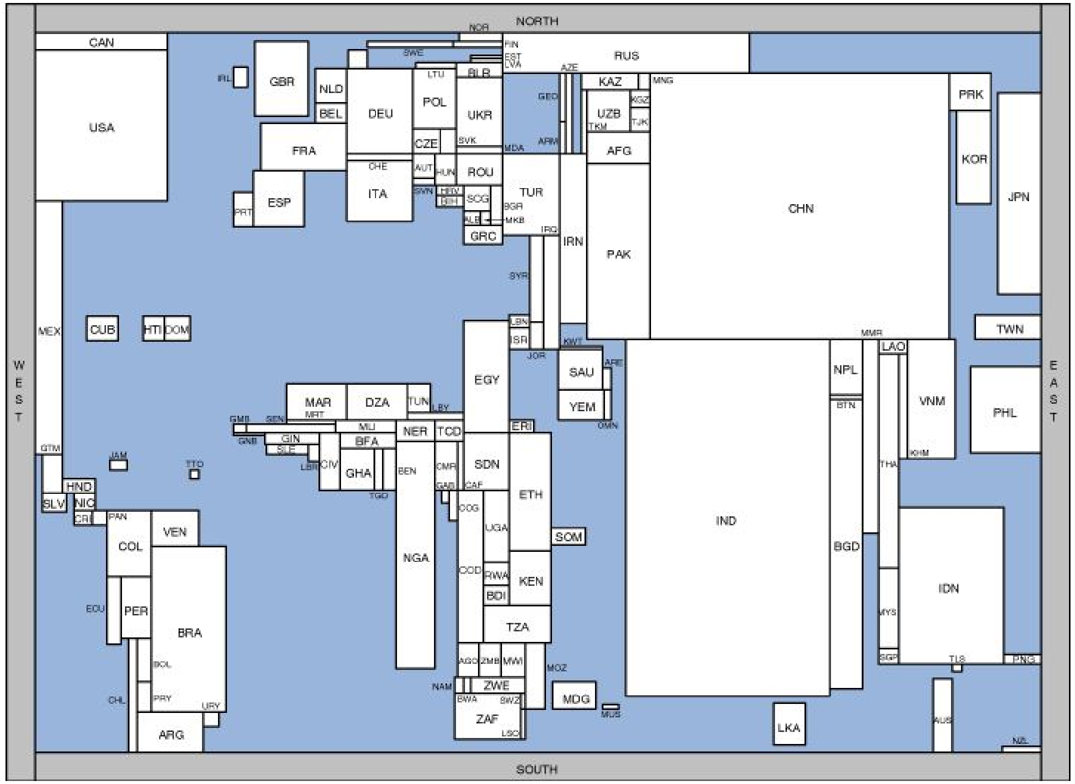


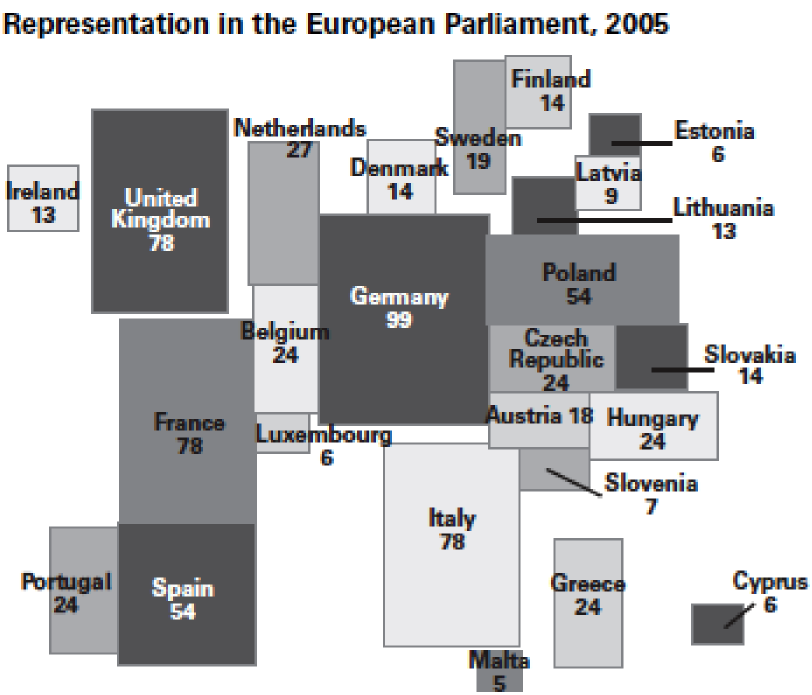
air pressure approach

[Izumi, Takahashi, Kajitani'98]
[Felsner'14]

Application to cartogram representations

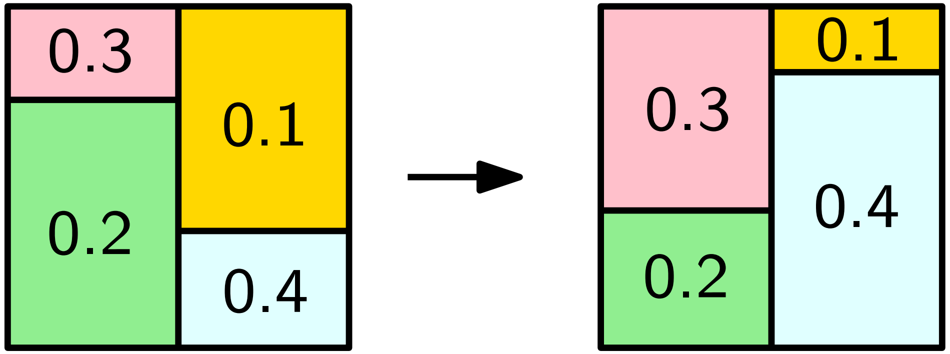






Area universality and 1-sidedness

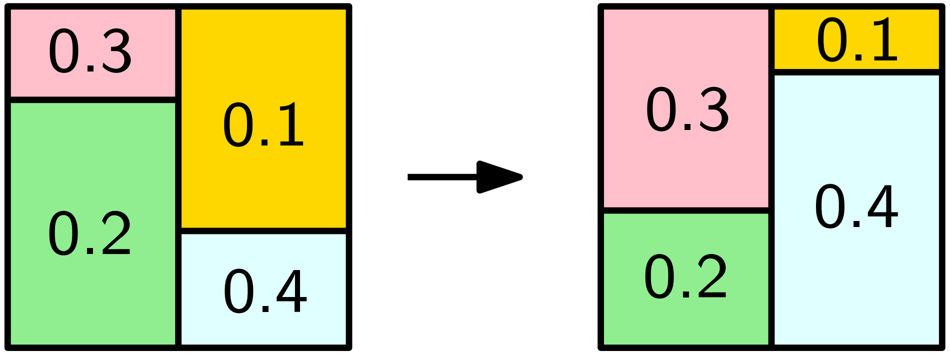
[Eppstein et al.12]



region adjacencies change
(due to 2-sided segment)

Area universality and 1-sidedness

[Eppstein et al.12]



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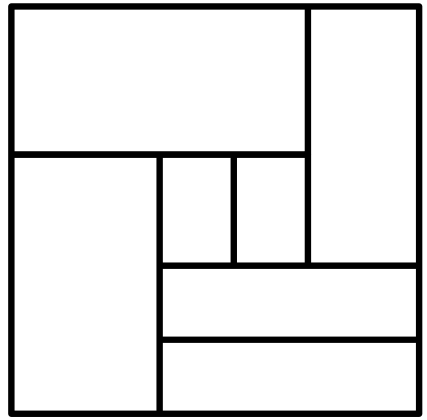
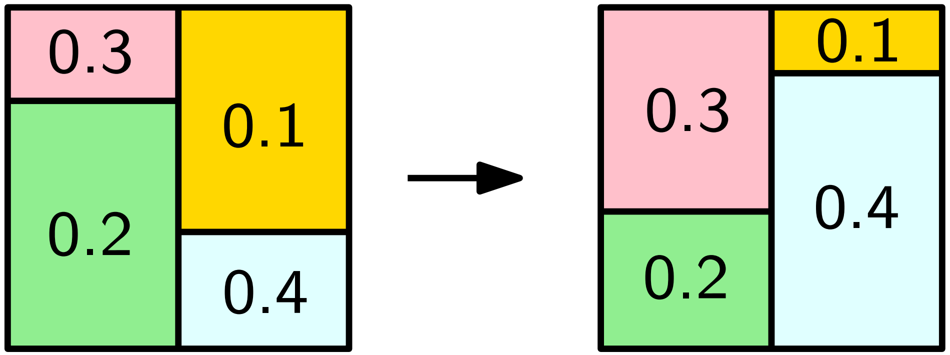


1-sided rectangulations
are area-universal

Area universality and 1-sidedness

[Eppstein et al.12]



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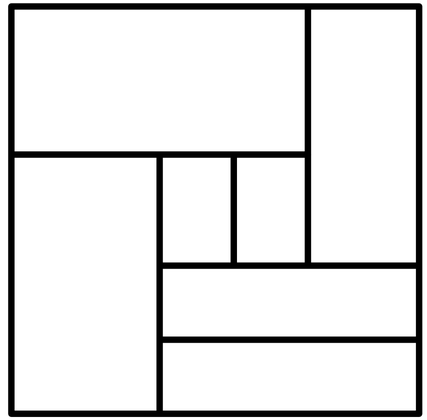

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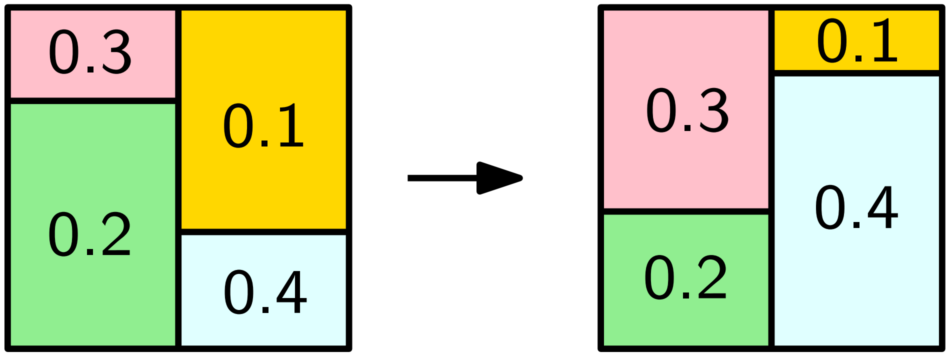

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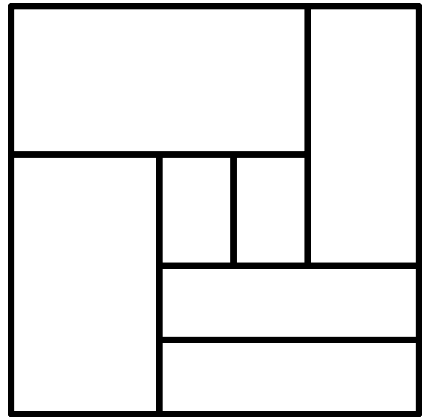

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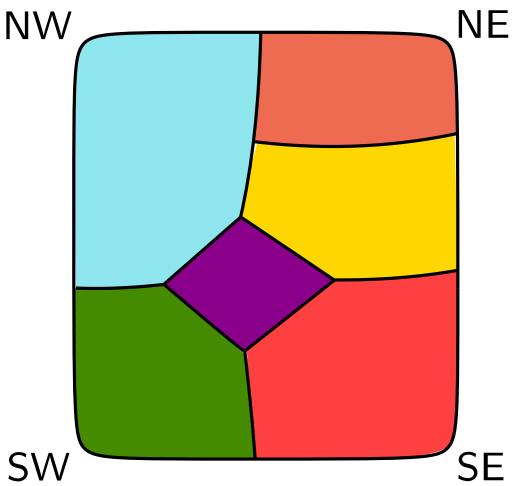
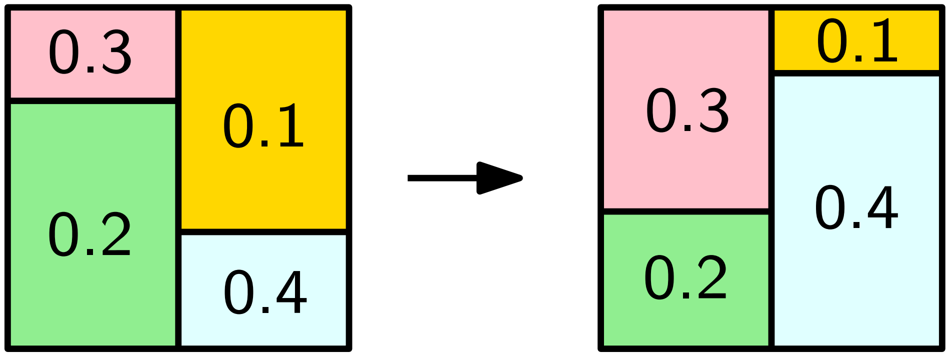


Area universality and 1-sidedness

[Eppstein et al.12]



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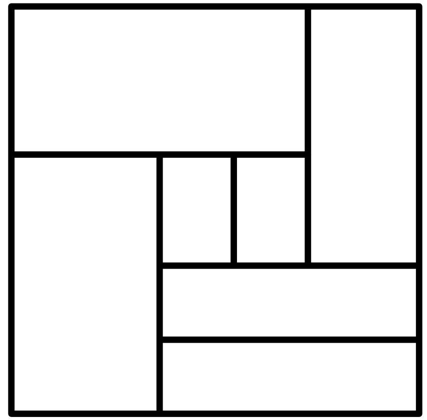

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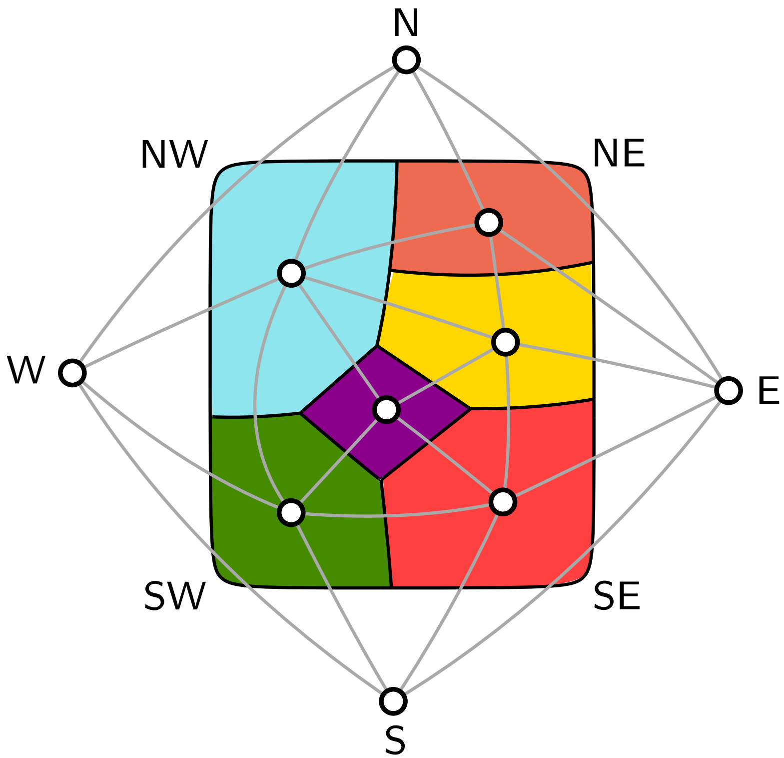
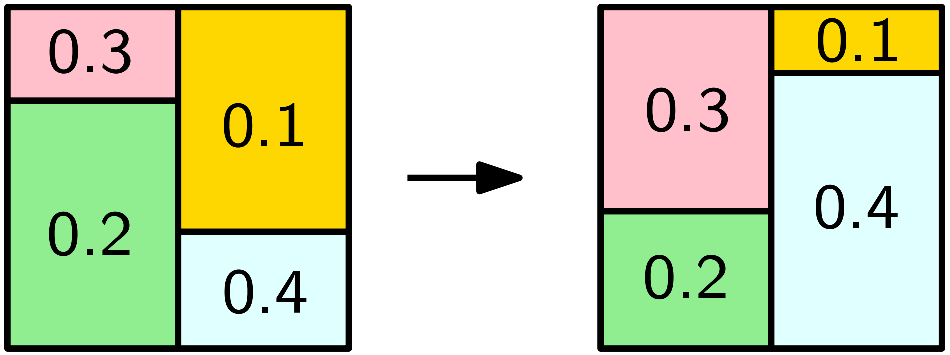


Area universality and 1-sidedness

[Eppstein et al.12]



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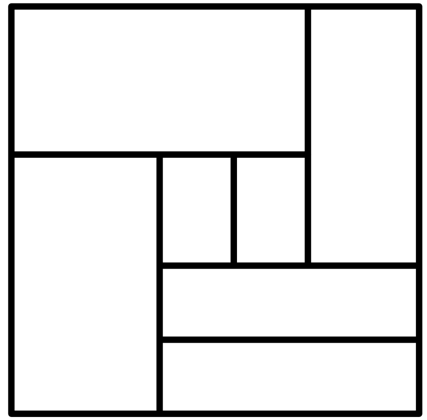

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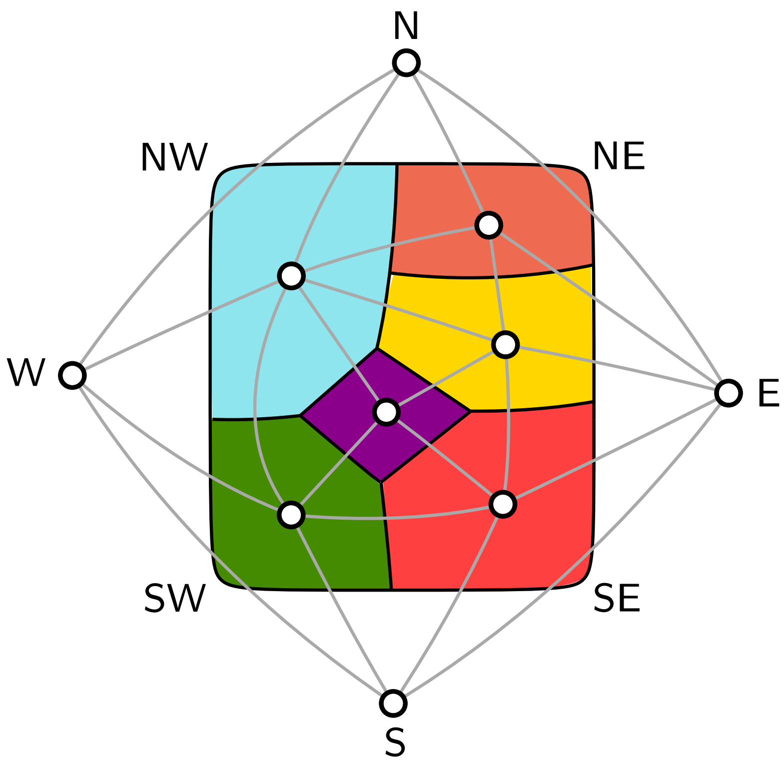
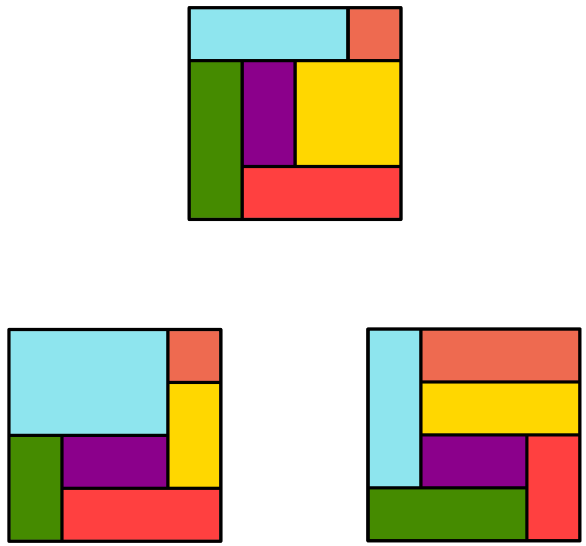
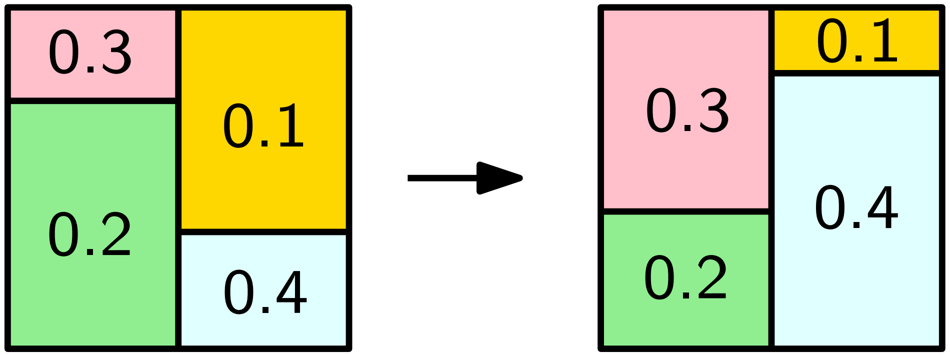




Area universality and 1-sidedness

[Eppstein et al.12]



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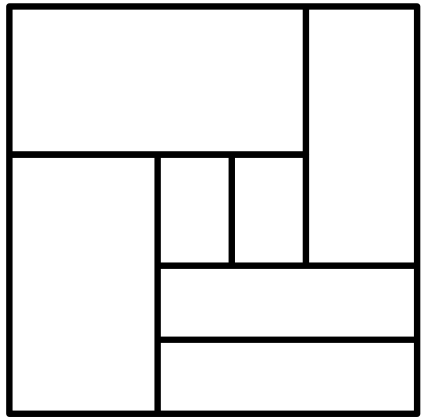

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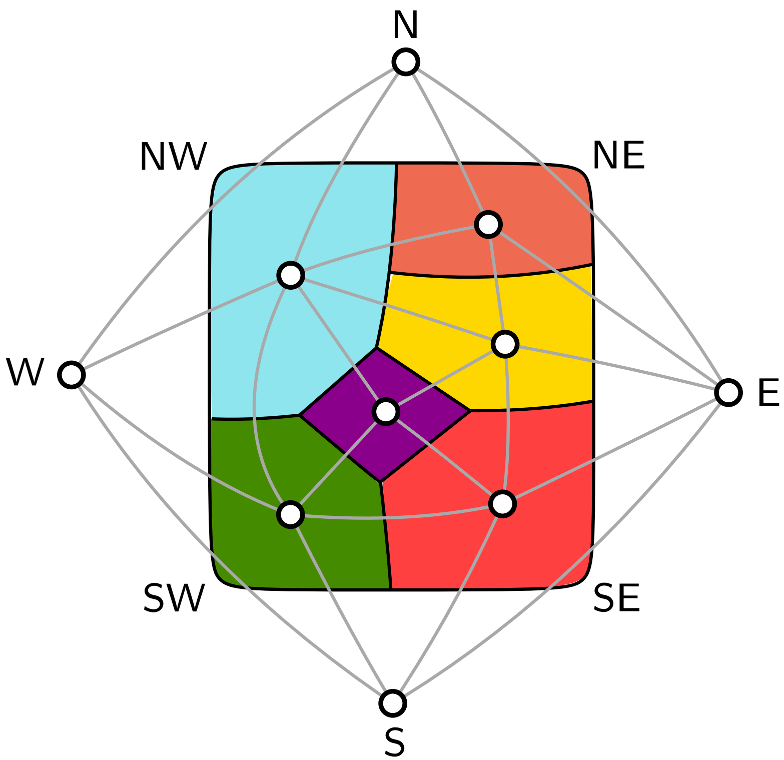
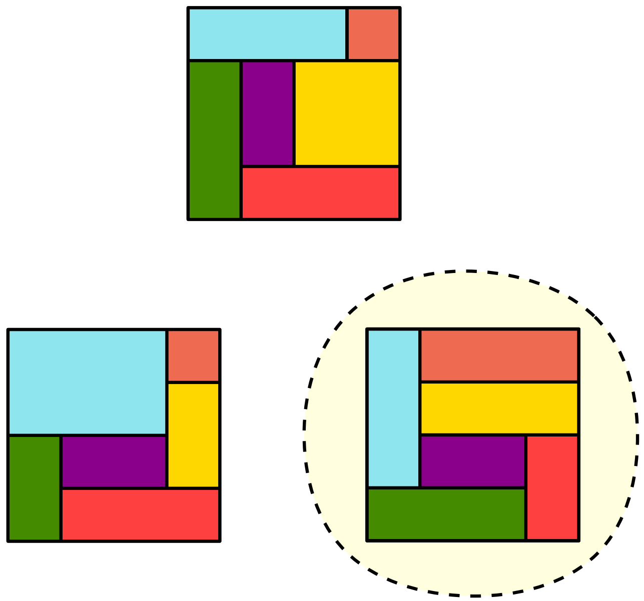
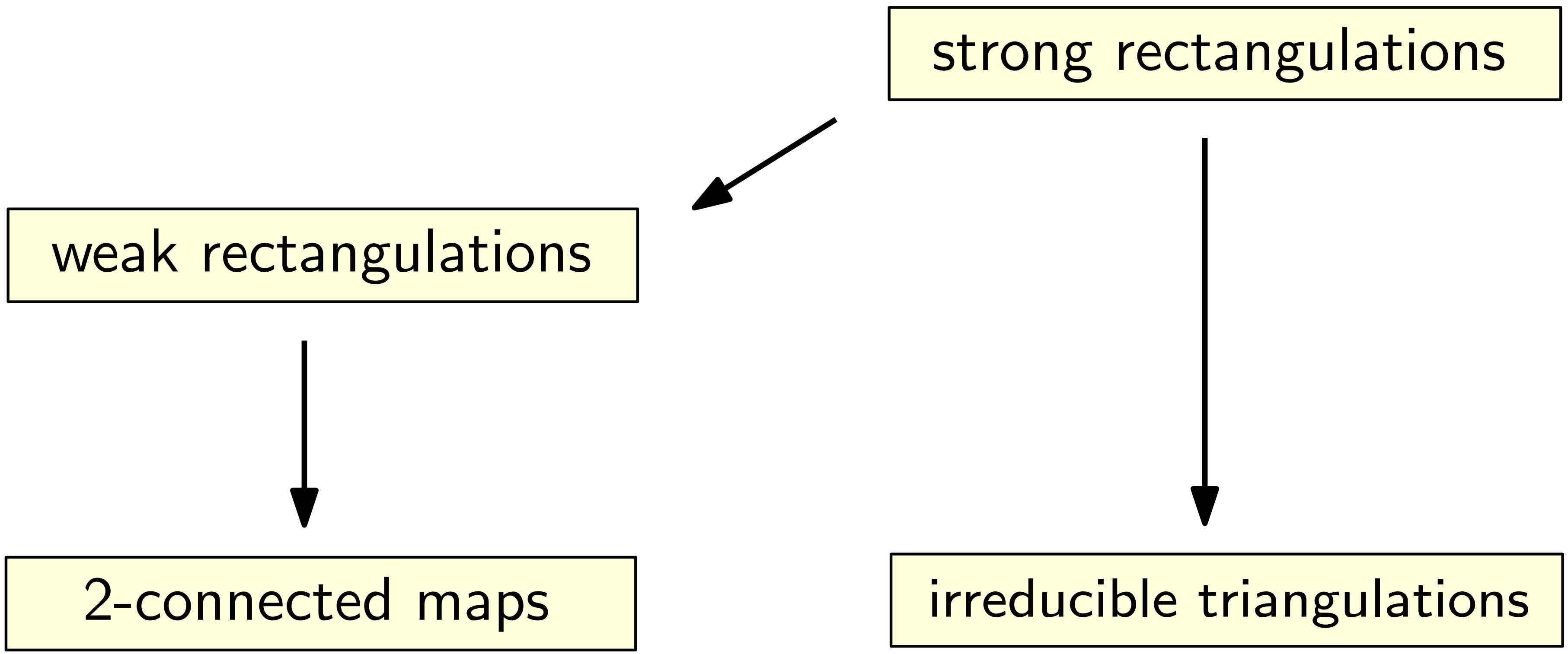


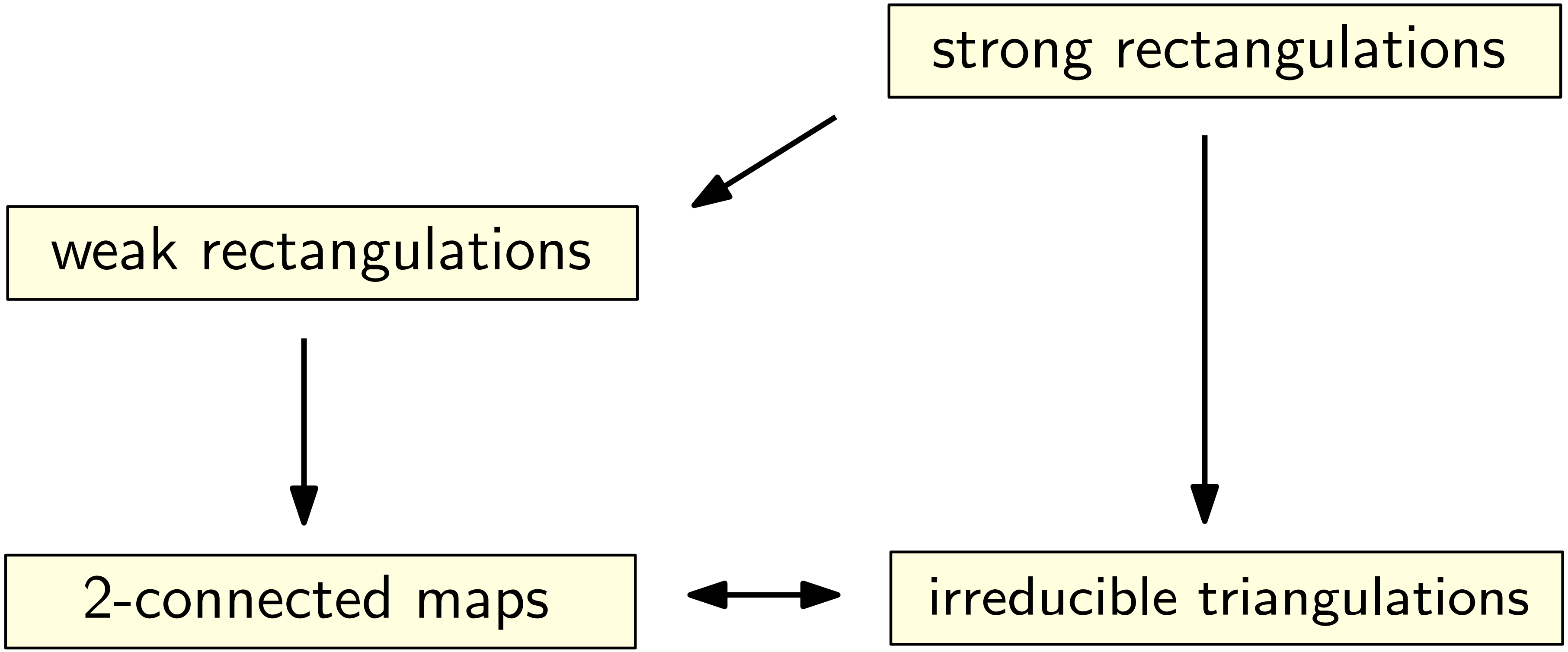


1-sided

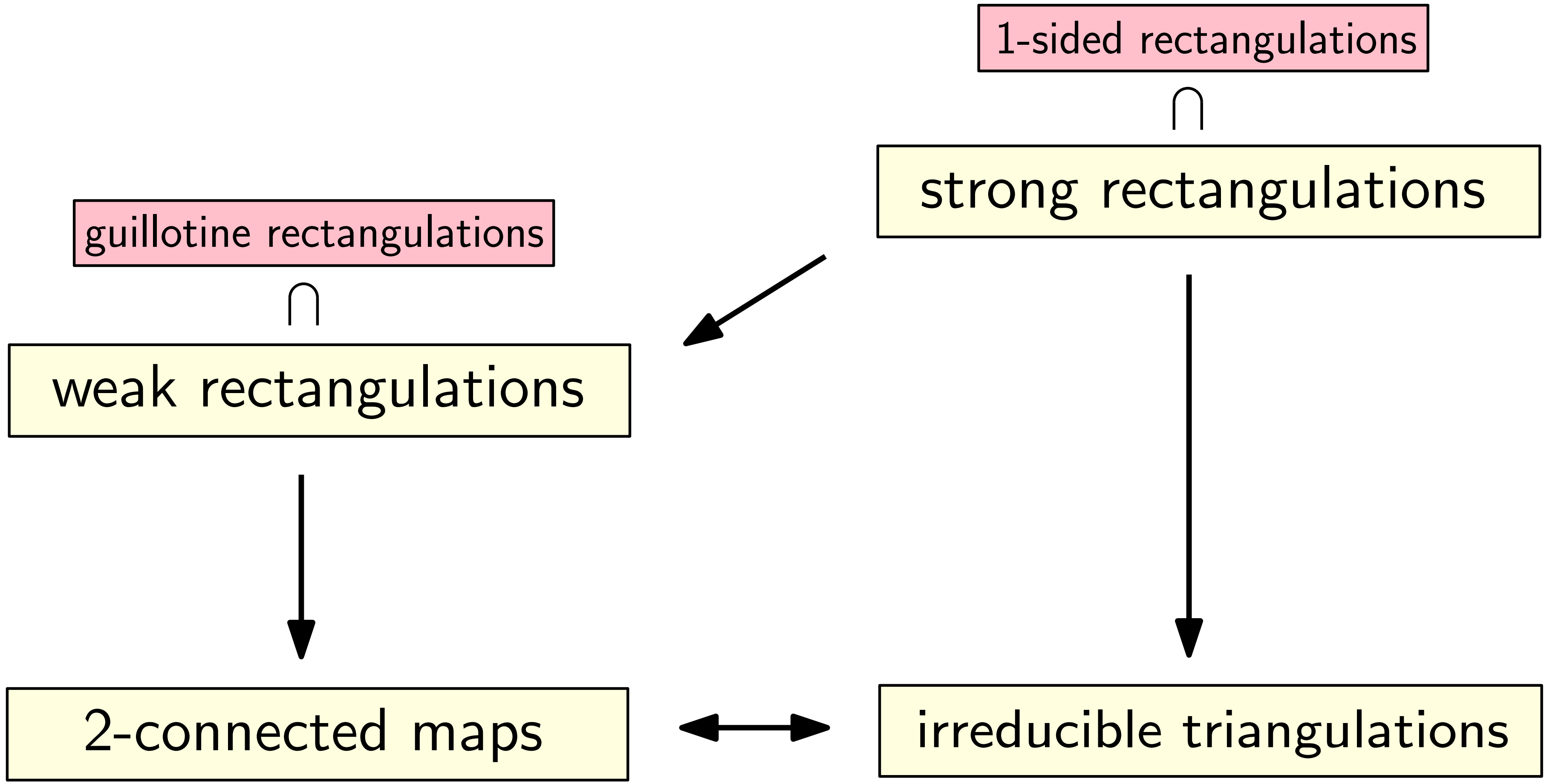
Families related to rectangulations



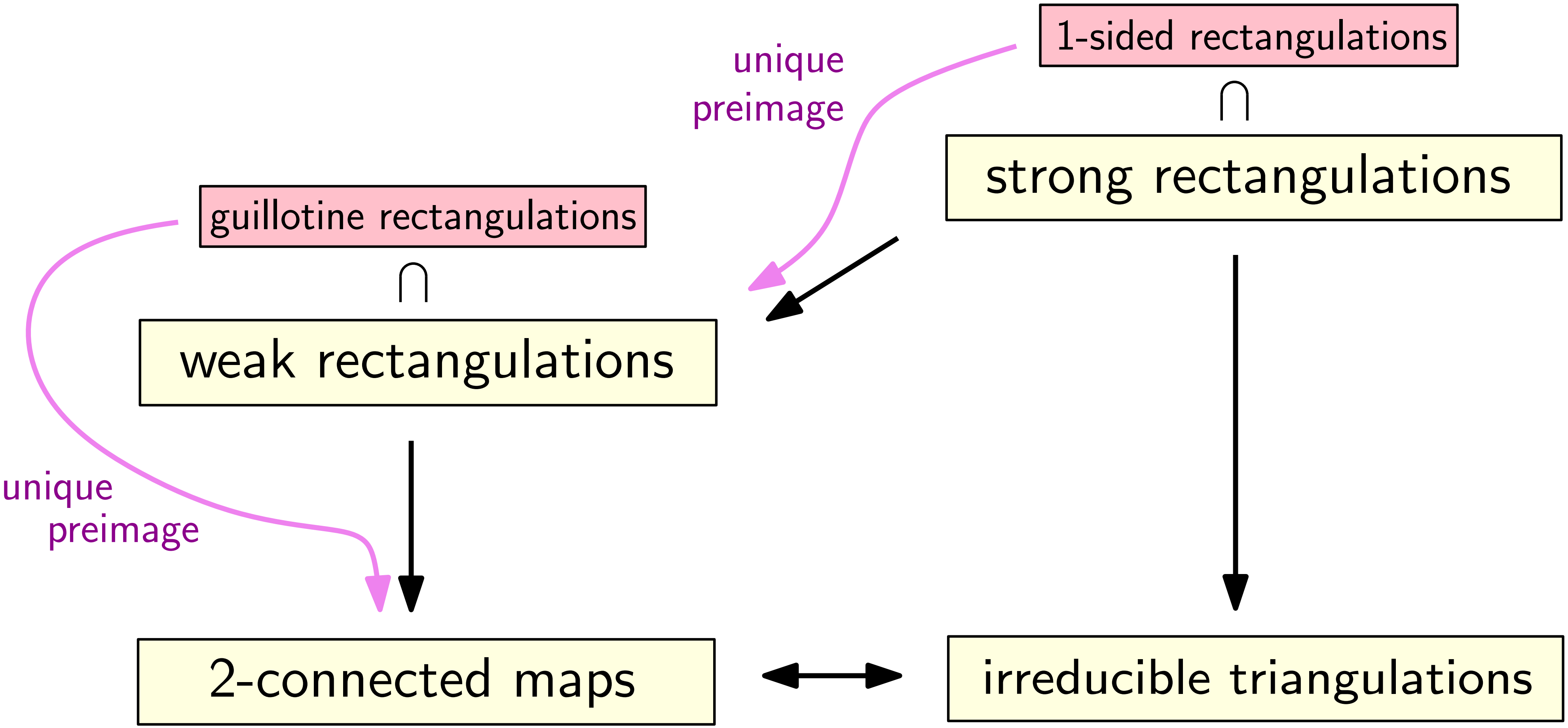
Families related to rectangulations



Families related to rectangulations

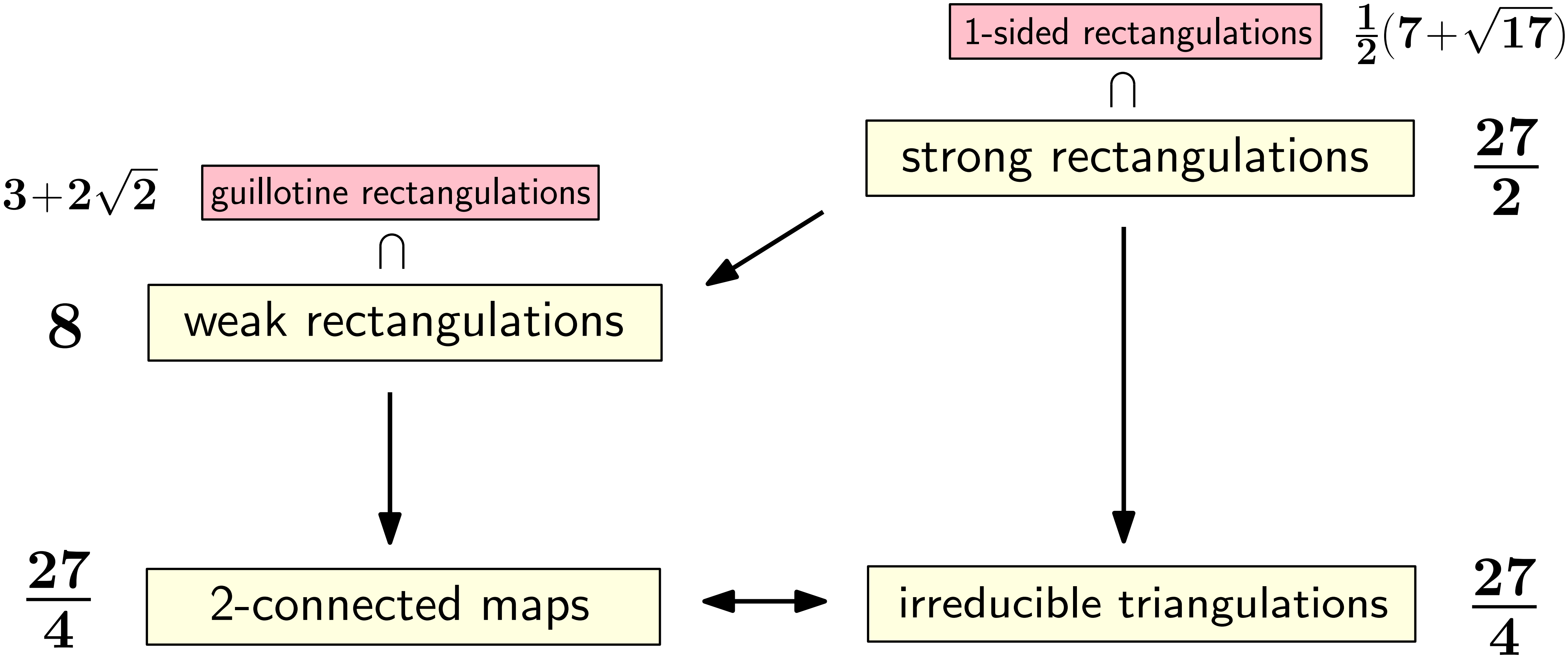


Families related to rectangulations



Families related to rectangulations

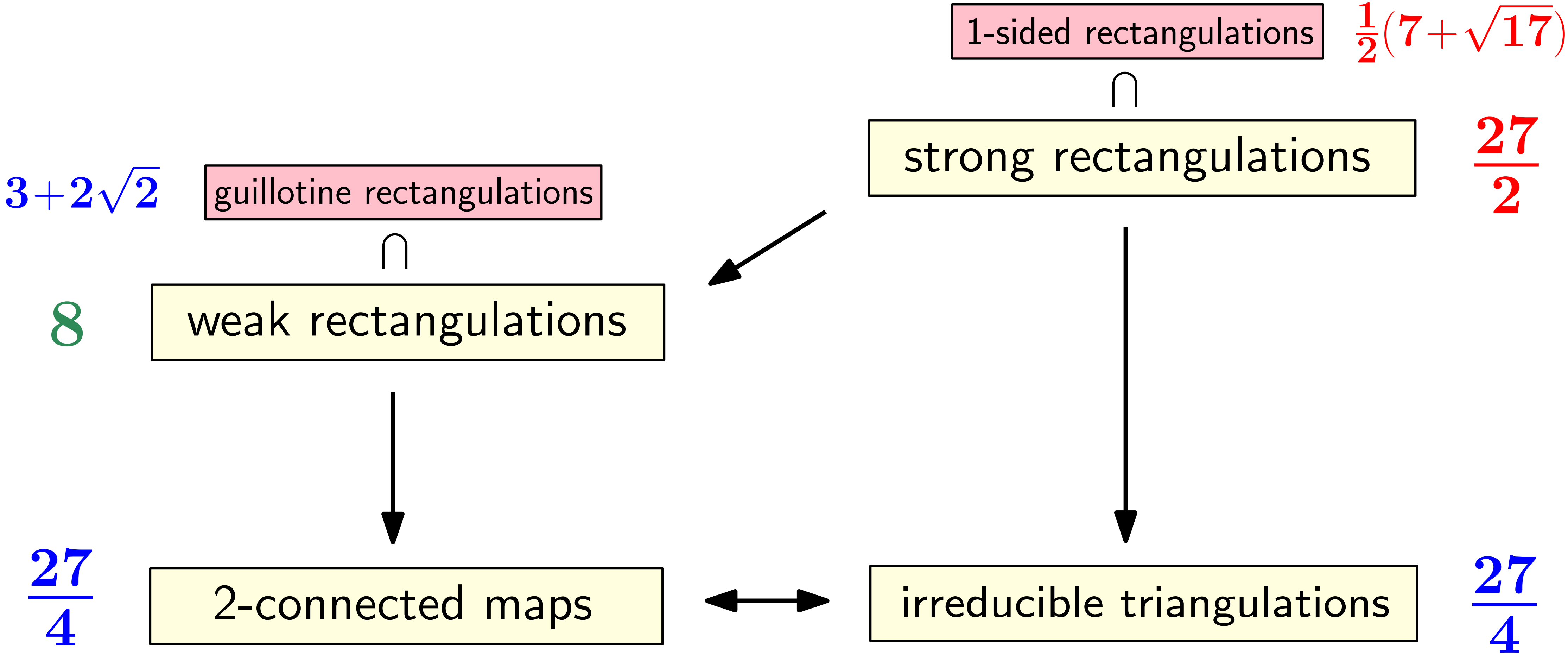
exponential growth



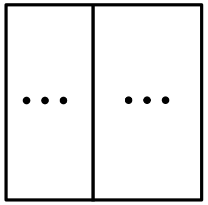
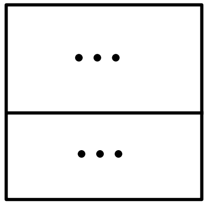
Families related to rectangulations

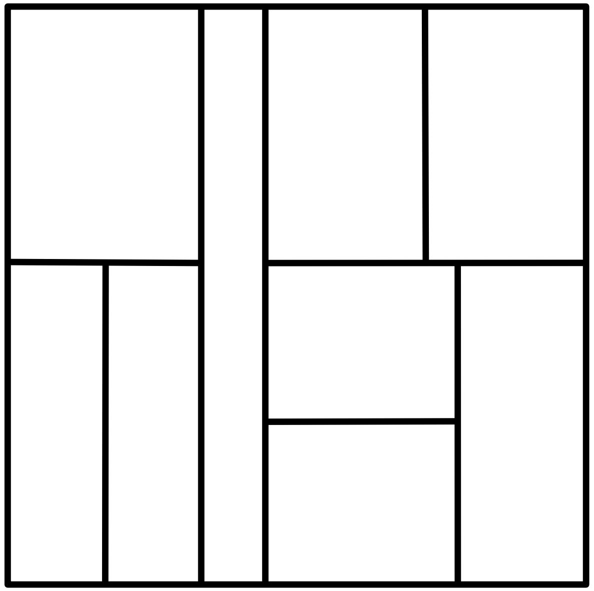
exponential growth and status of generating function:

algebraic / D-finite / not D-finite

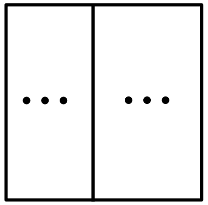
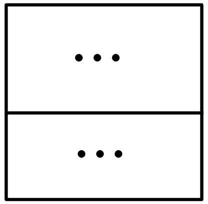


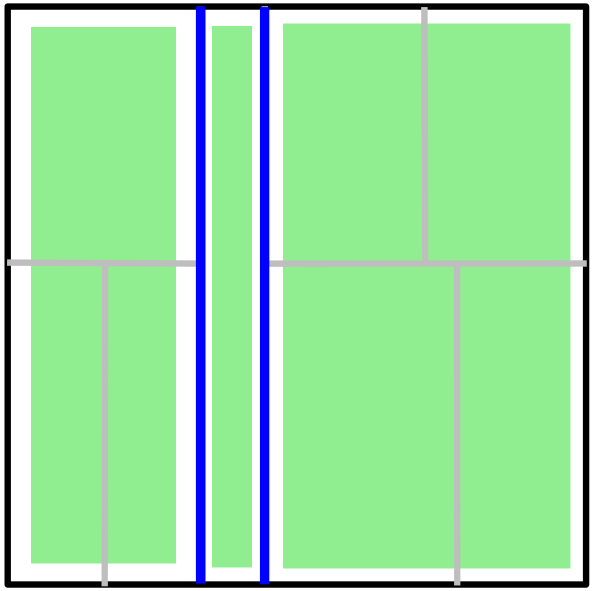
Guillotine rectangulations

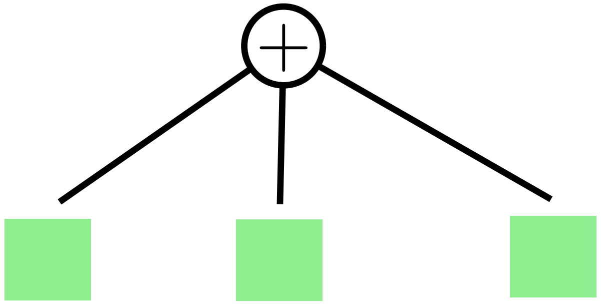
of the form  or  (and recursively)



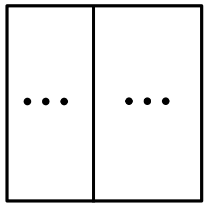
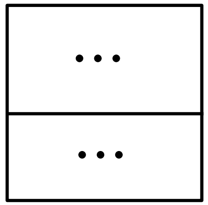
Guillotine rectangulations

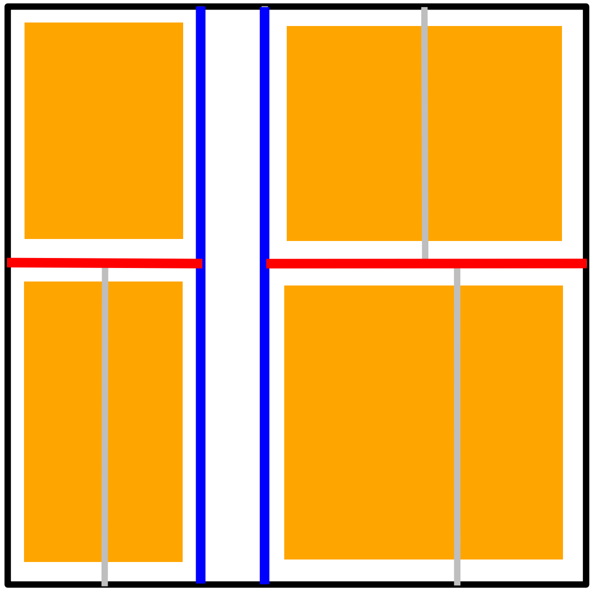
of the form  or  (and recursively)

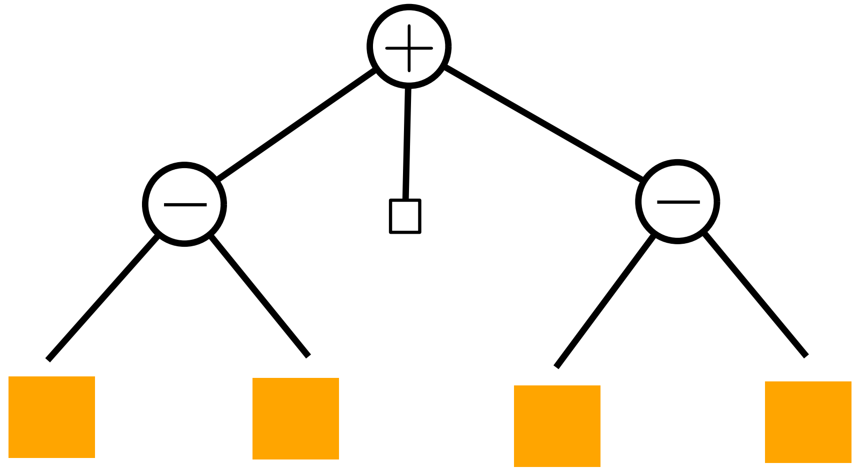




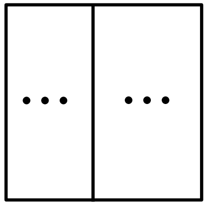
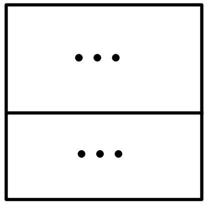
Guillotine rectangulations

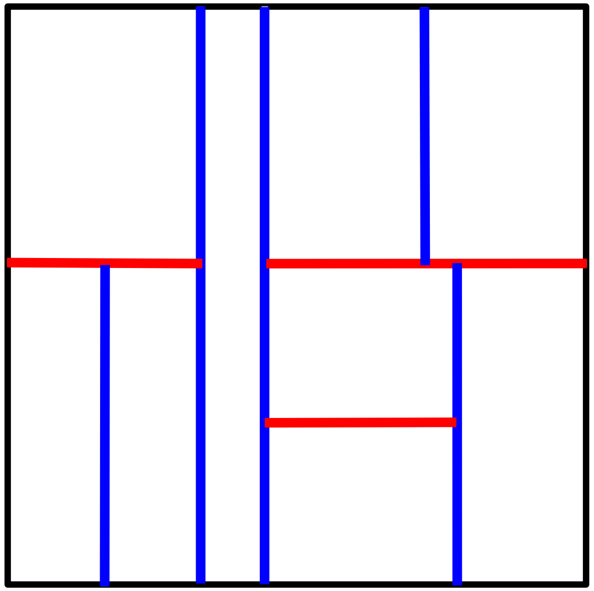
of the form  or  (and recursively)

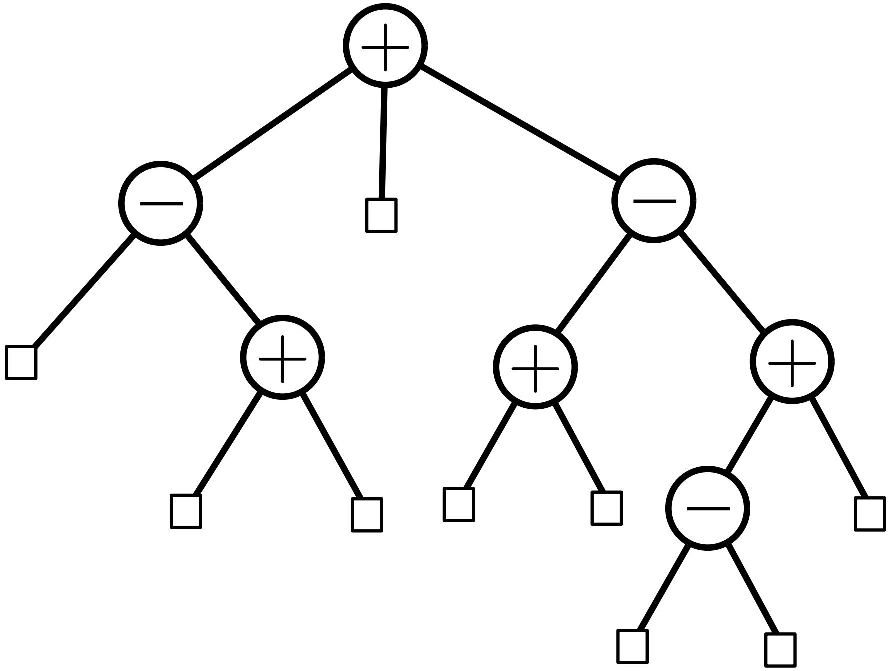




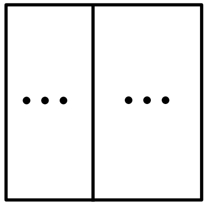
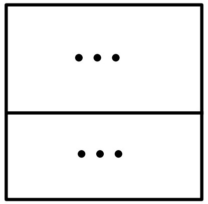
Guillotine rectangulations

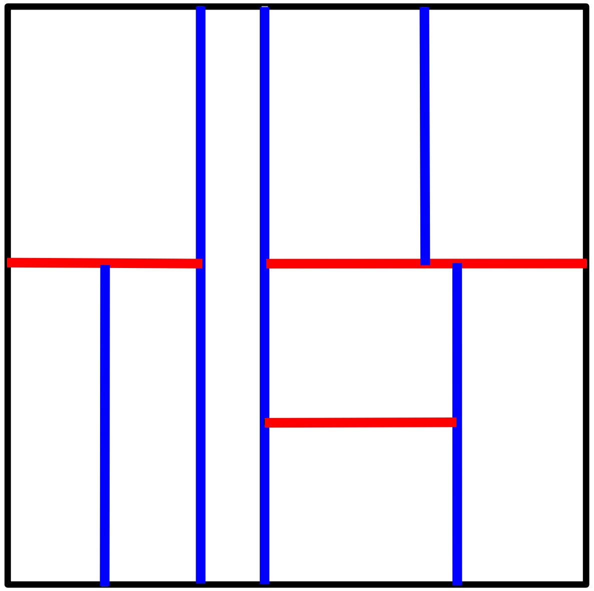
of the form  or  (and recursively)

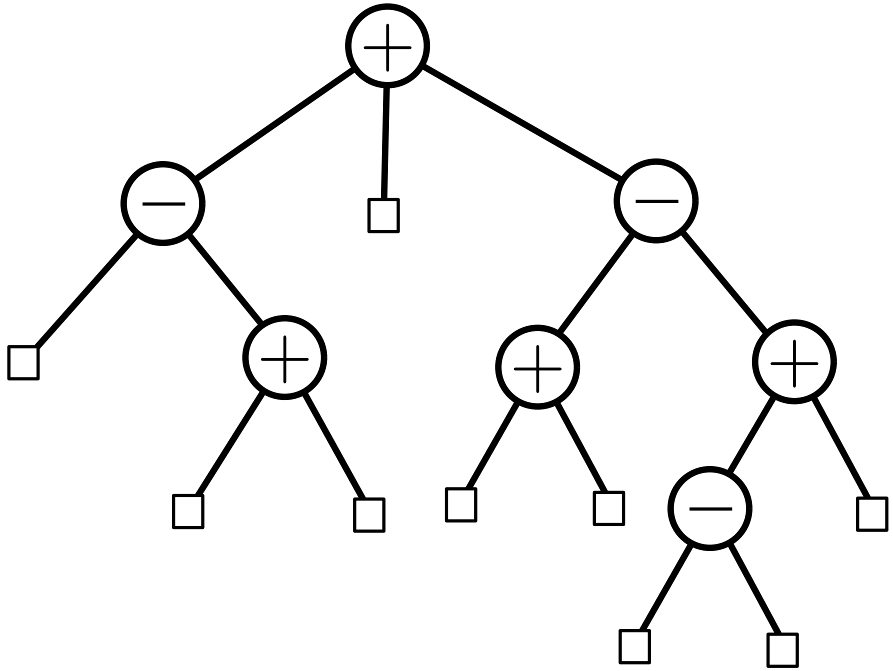




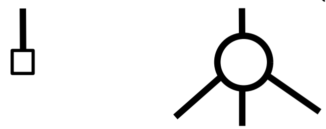
Guillotine rectangulations

of the form  or  (and recursively)

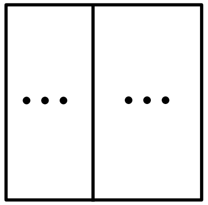
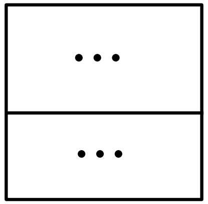


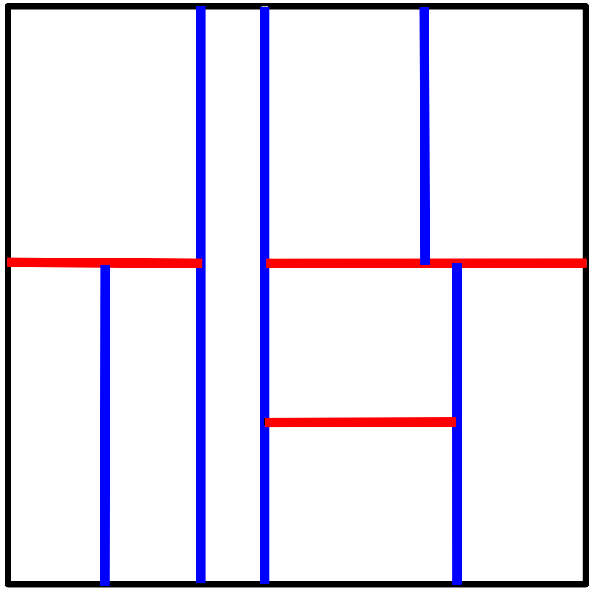


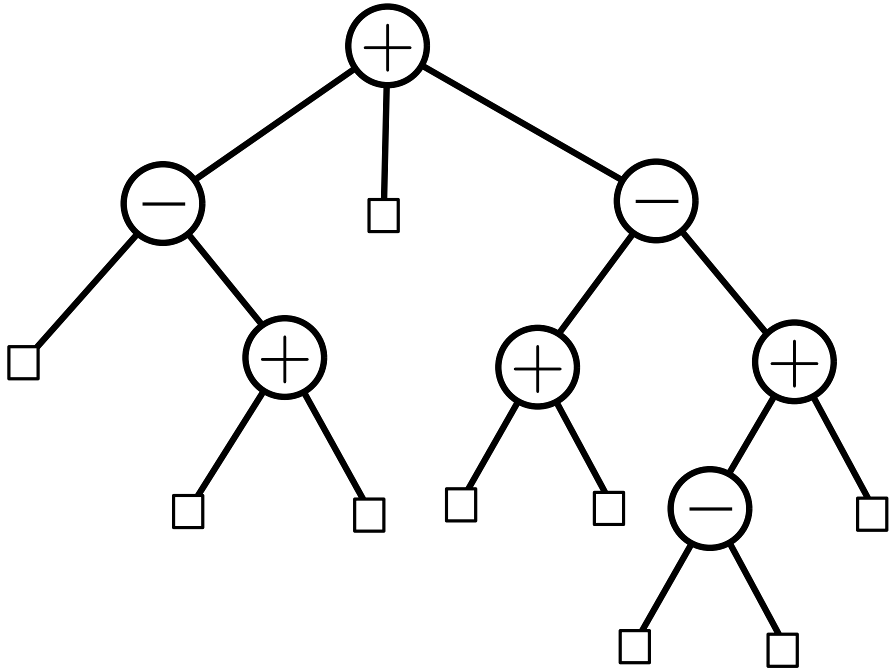
Generating function: $G(t) = 2S(t) - t$ where $S(t) = t + \dfrac{S(t)^2}{1 - S(t)}$



Guillotine rectangulations

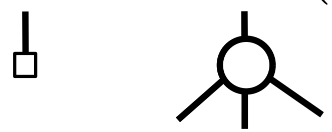
of the form  or  (and recursively)



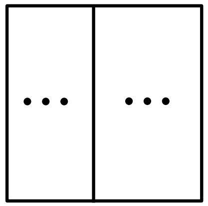
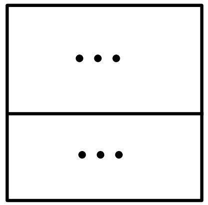


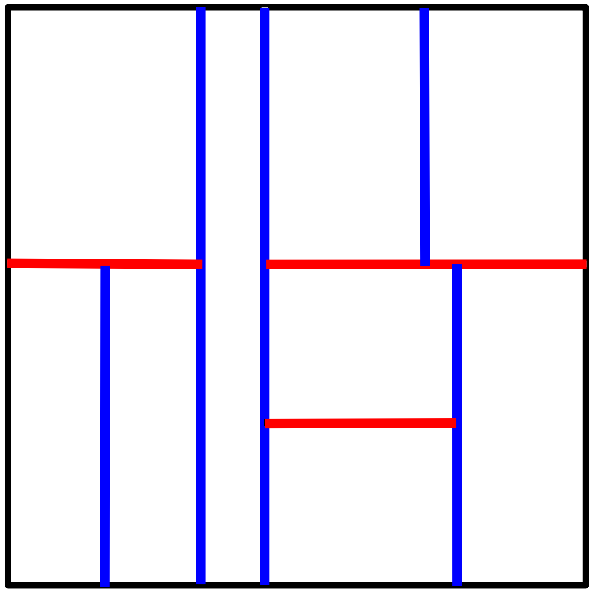
Generating function: $G(t) = 2S(t) - t$ where $S(t) = t + \dfrac{S(t)^2}{1 - S(t)}$

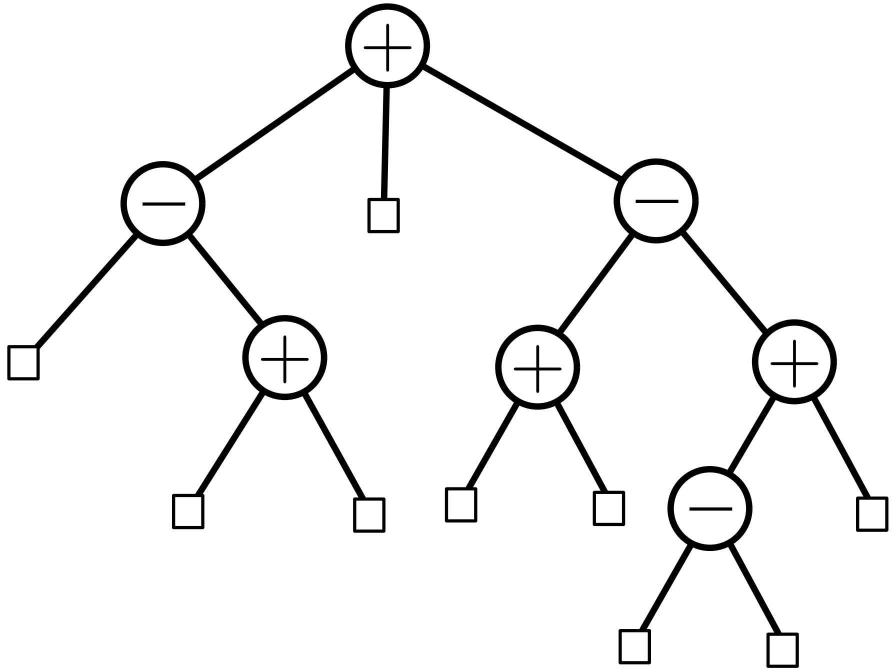
$$\Rightarrow \quad \boxed{G = t(1 + G) + G^2}$$



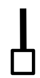
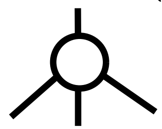
Guillotine rectangulations

of the form  or  (and recursively)





Generating function: $G(t) = 2S(t) - t$ where $S(t) = t + \dfrac{S(t)^2}{1 - S(t)}$

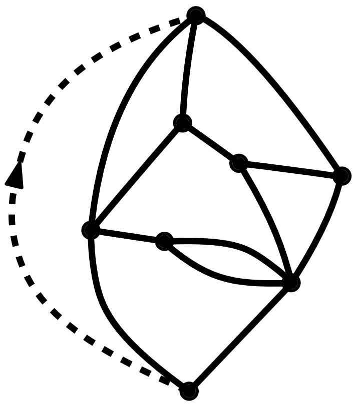
$$\Rightarrow \boxed{G = t(1+G) + G^2}$$

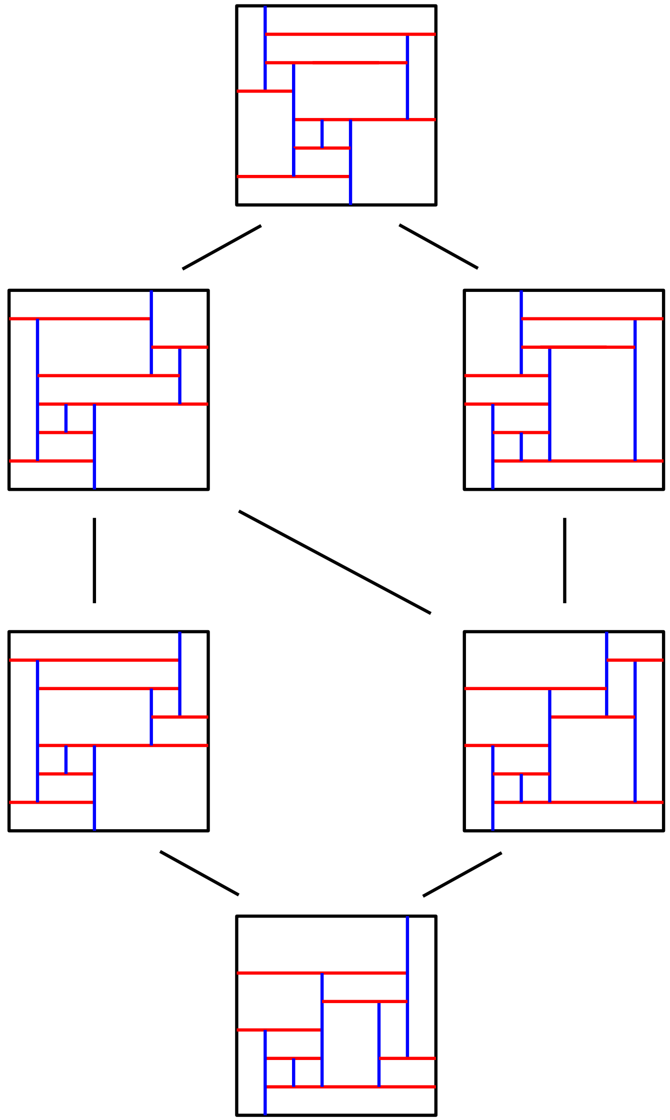
$$\Rightarrow \; G(t) = \frac{1}{2}\left(1 - t - \sqrt{1 - 6t + t^2}\right) = t + 2t^2 + 6t^3 + 22t^4 + 90t^5 + \cdots$$

$[t^n]G =$ $(n-1)$th Schröder number $\sim$ $\boxed{c\,(3+2\sqrt{2})^n\, n^{-3/2}}$

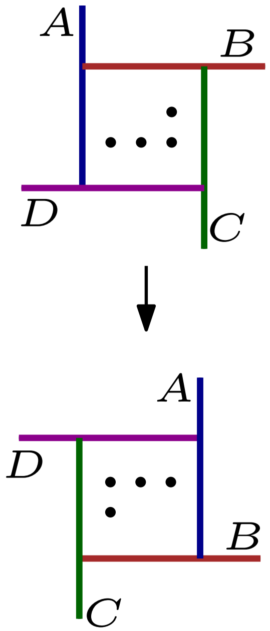
Set of weak rectangulations of a 2-conn. map

[Rosenstiehl, Tarjan'86]
[Ossona de Mendez'94]





Covering relations:



Set of weak rectangulations of a 2-conn. map

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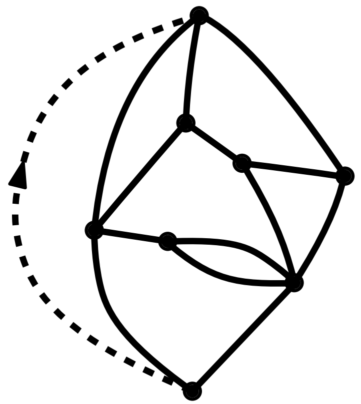

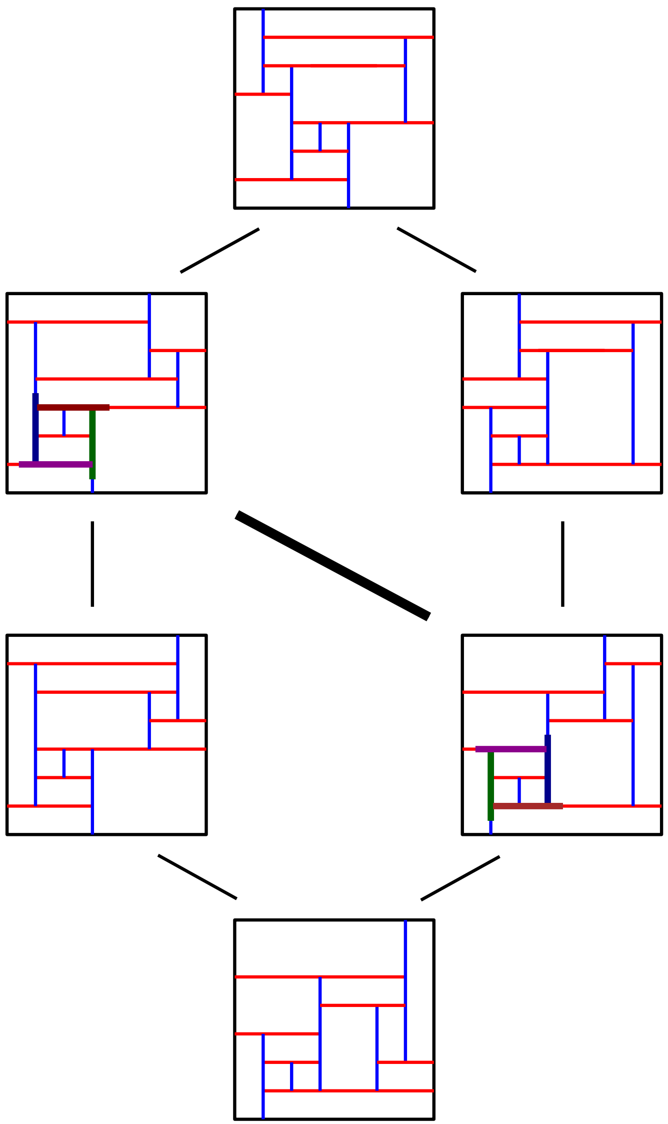

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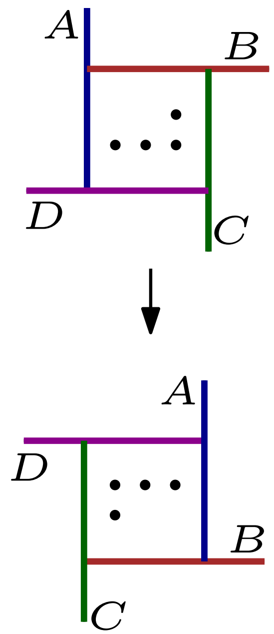

Set of weak rectangulations of a 2-conn. map

[Rosenstiehl, Tarjan'86]
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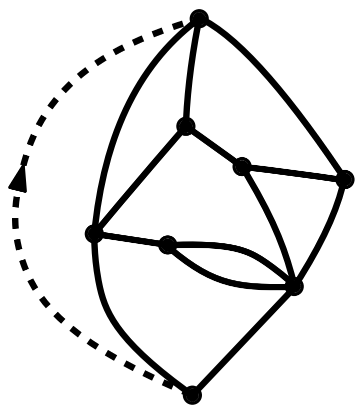

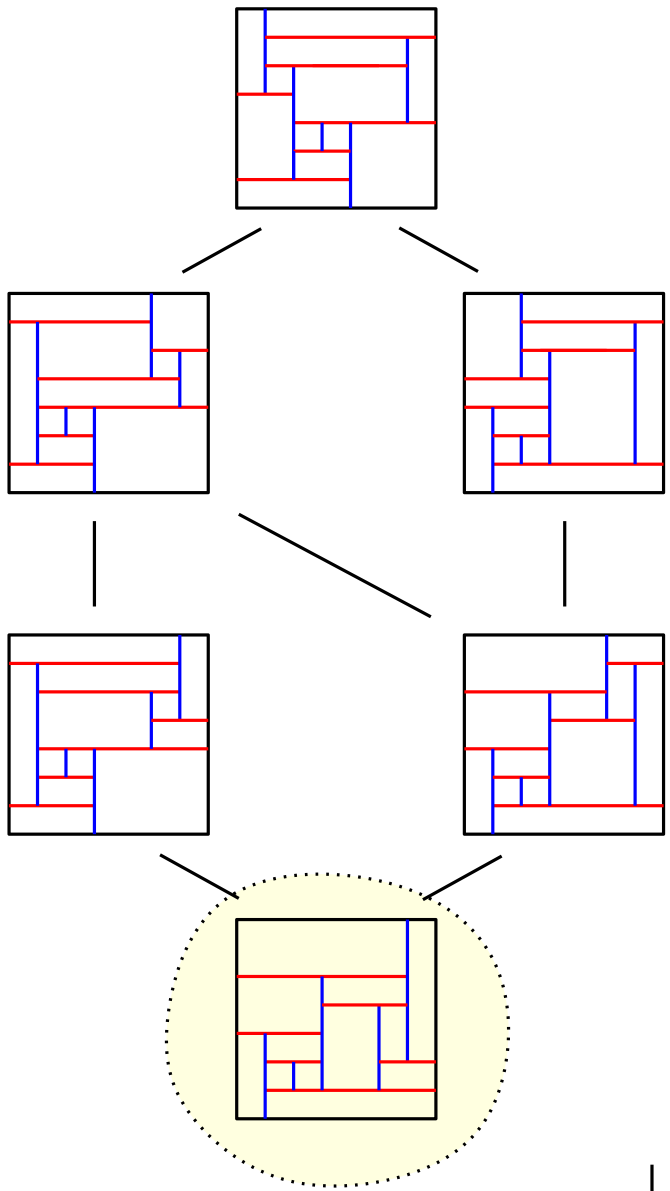

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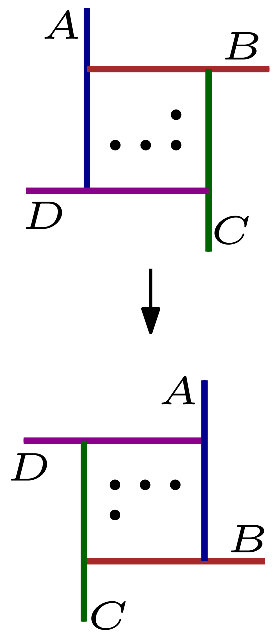
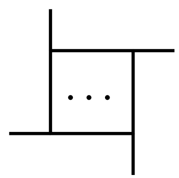
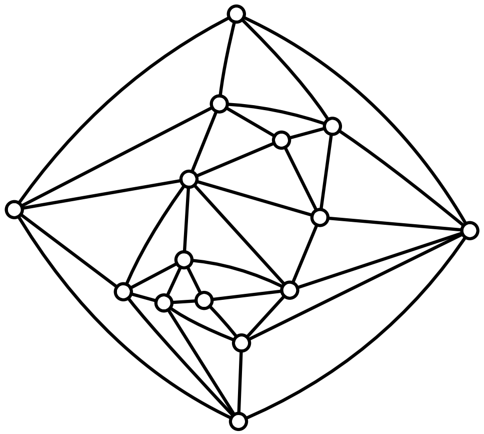
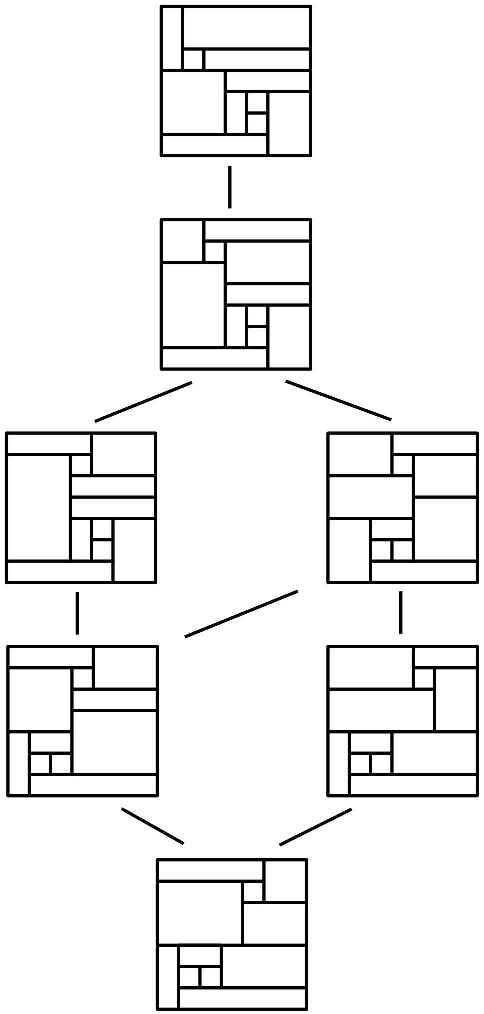


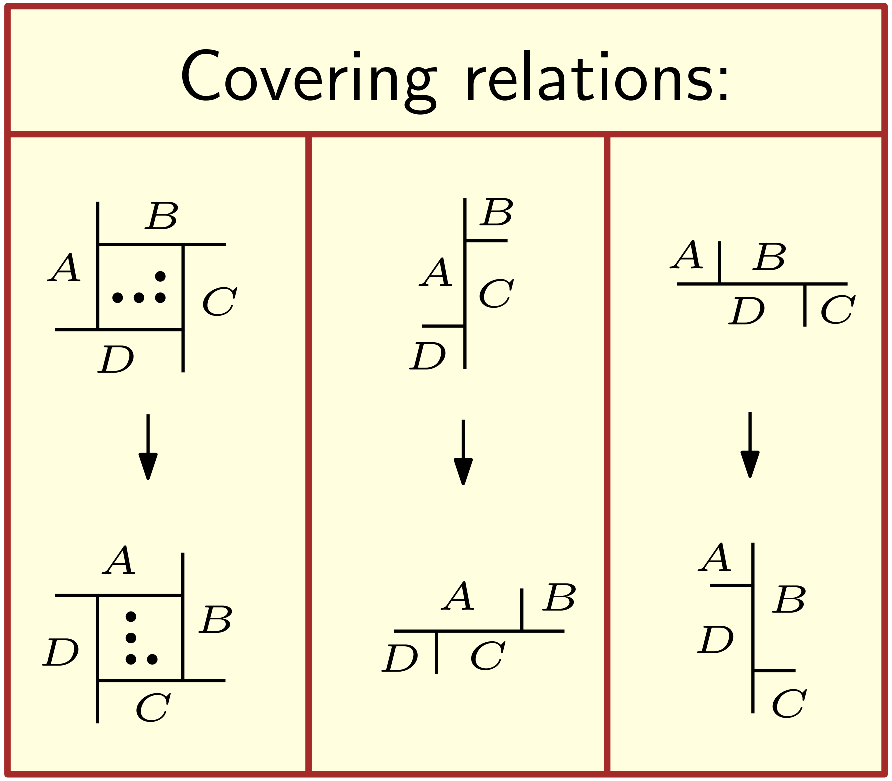
minimal element: no 

Set of strong rectangulations of a triangulation

[F'07]

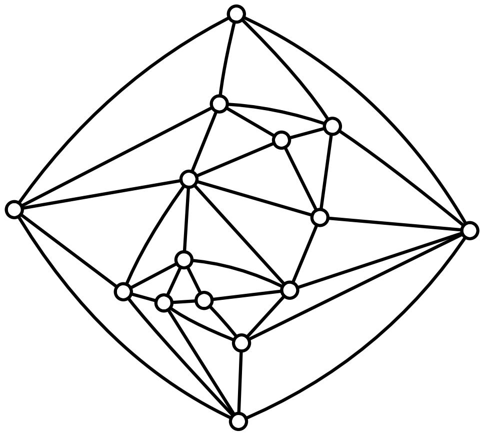


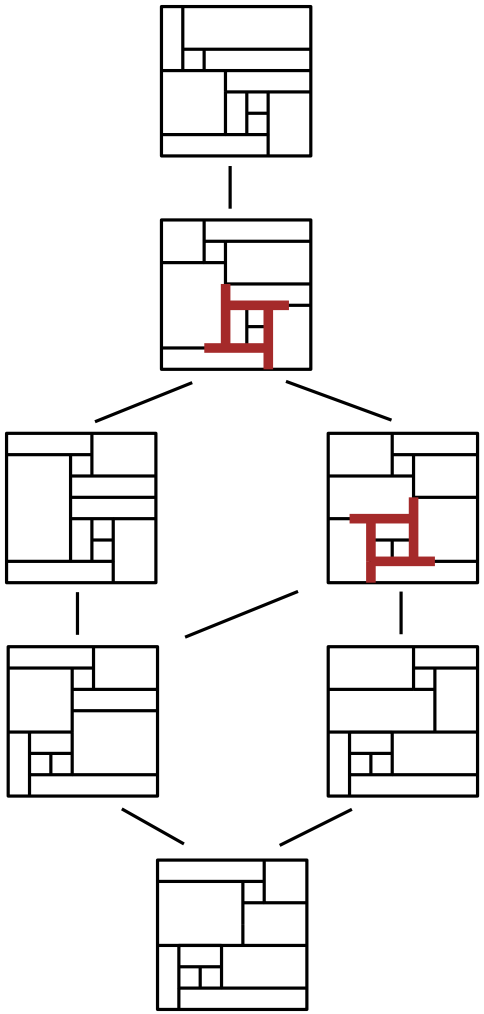


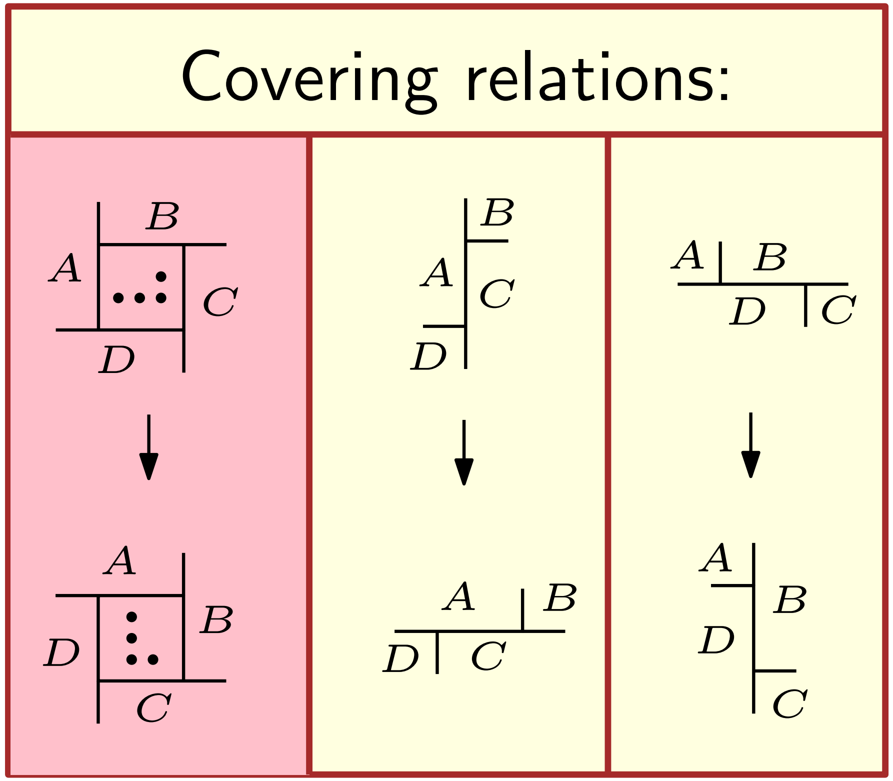


Set of strong rectangulations of a triangulation

[F'07]

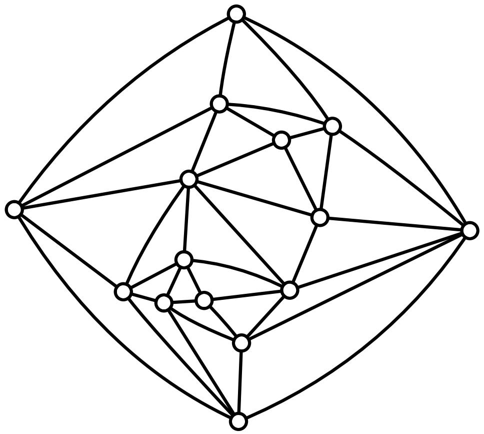


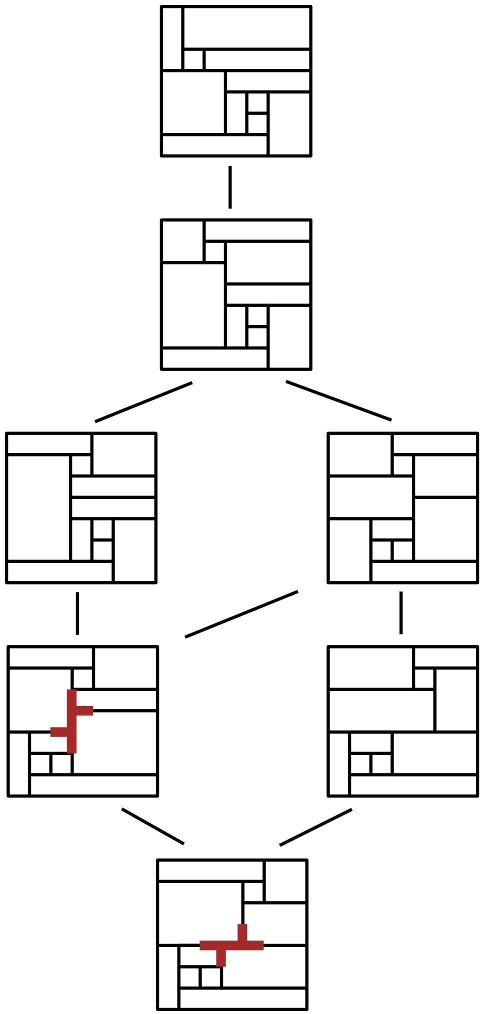


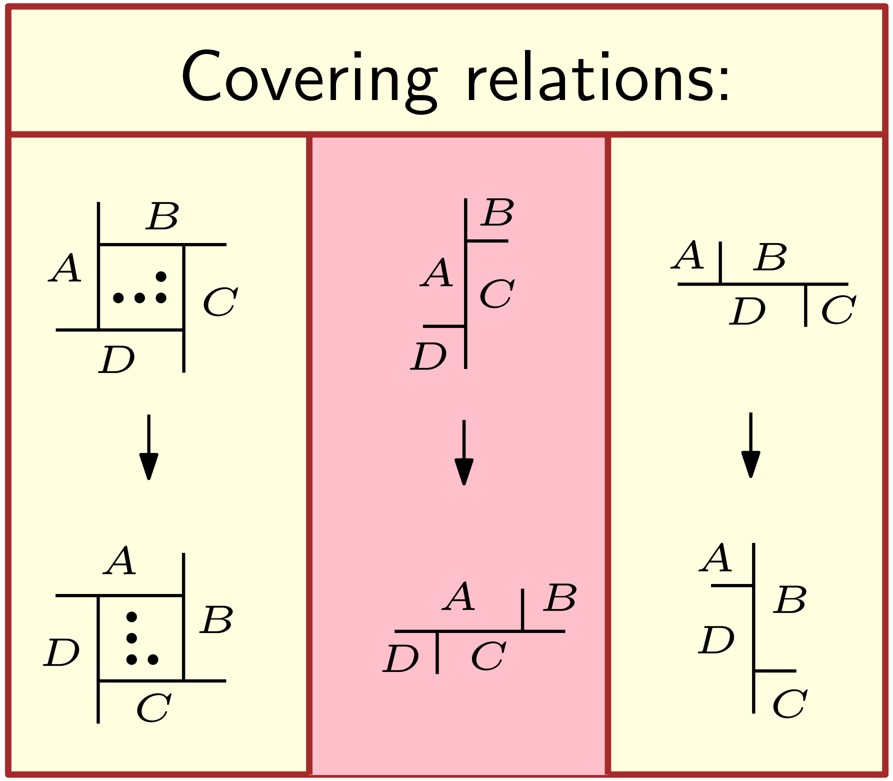


Set of strong rectangulations of a triangulation

[F'07]

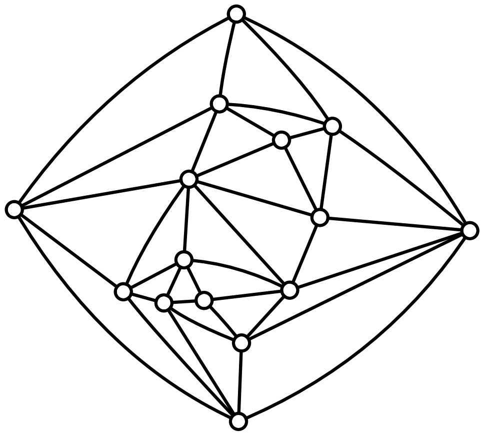


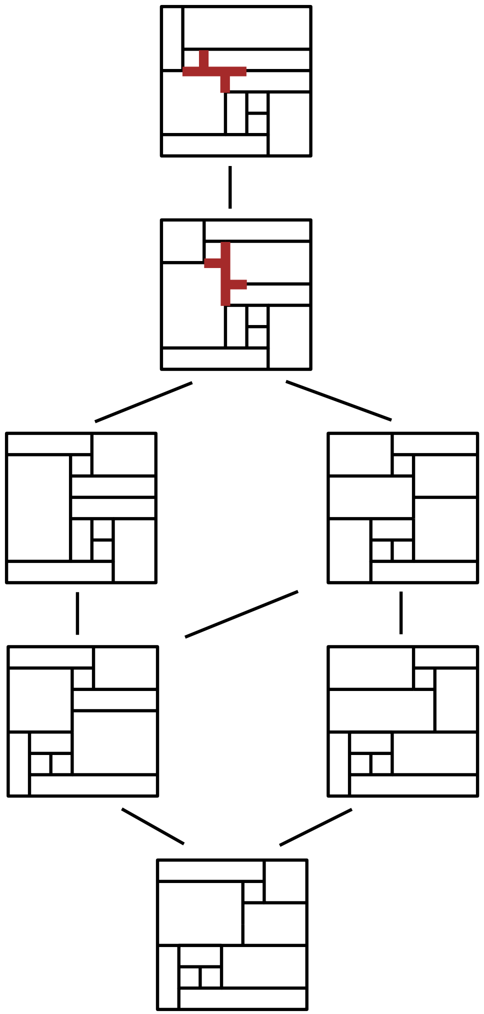


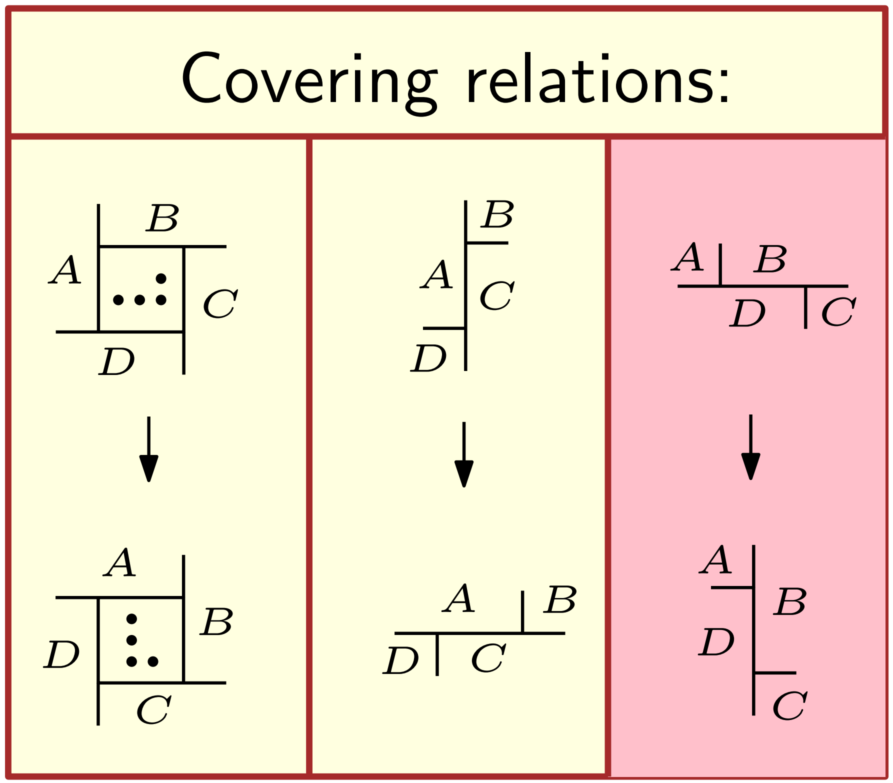


Set of strong rectangulations of a triangulation

[F'07]

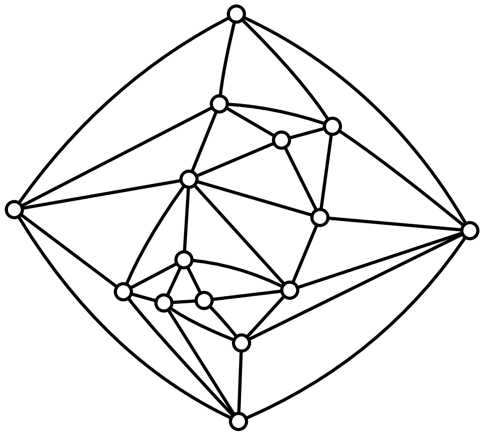


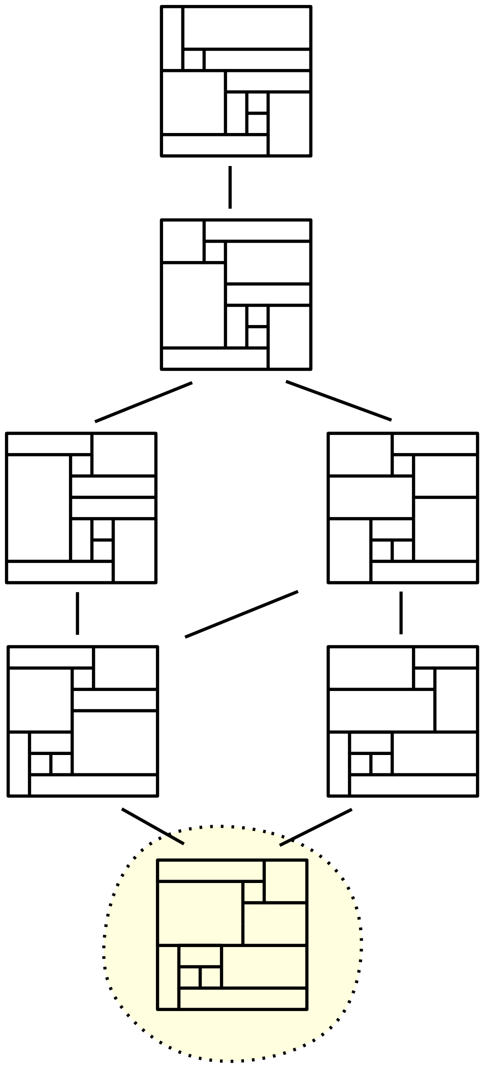




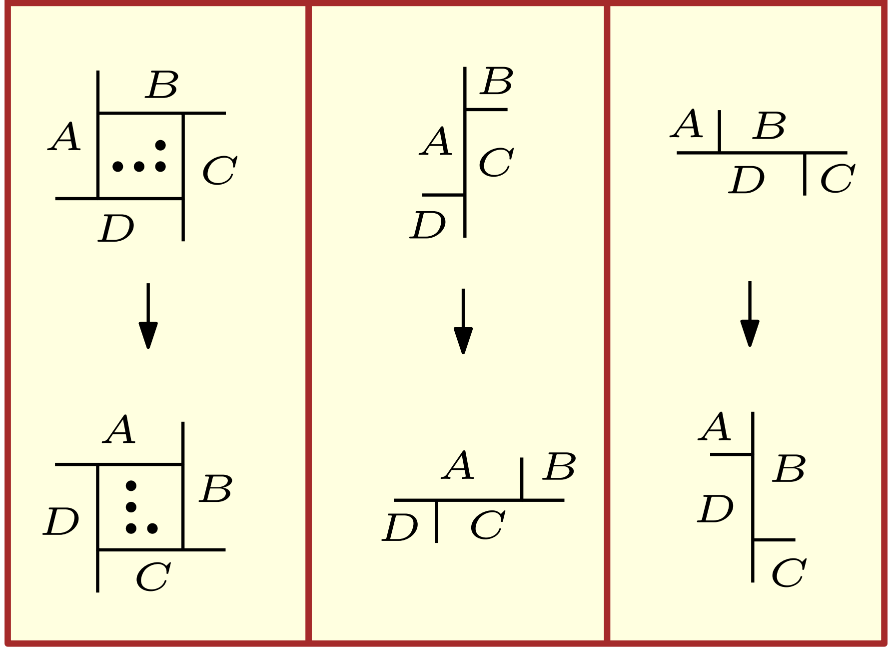
Set of strong rectangulations of a triangulation

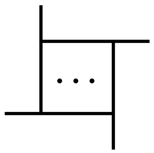
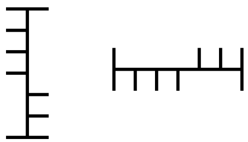
[F'07]





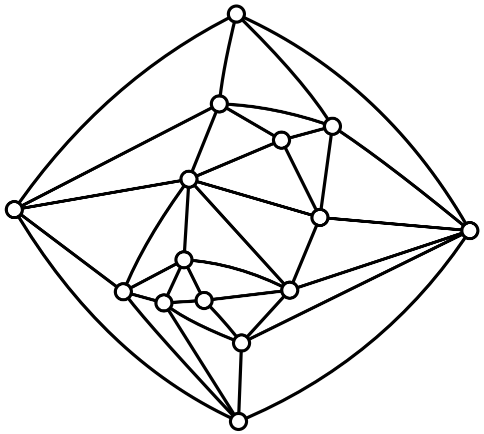
Covering relations:

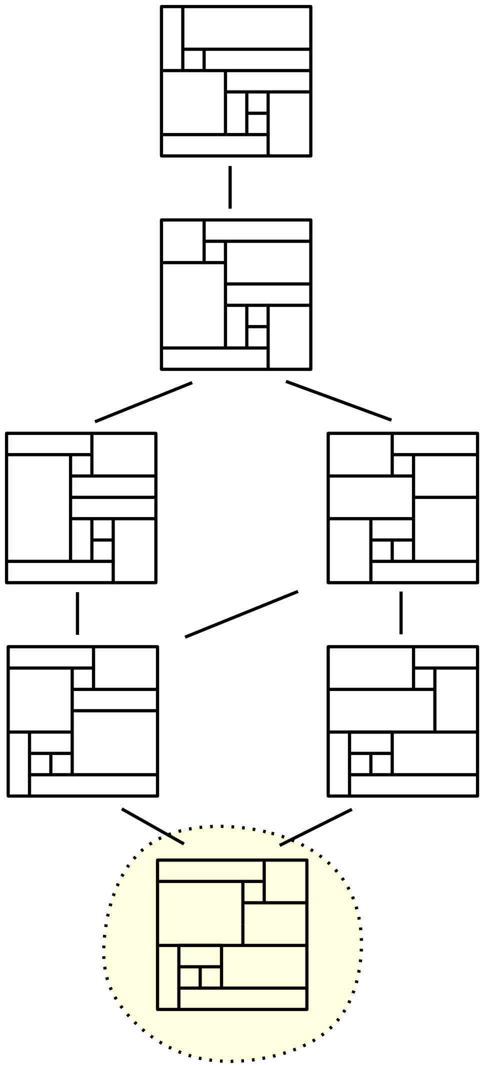


minimal element: no  canonical shuffle: at segments 

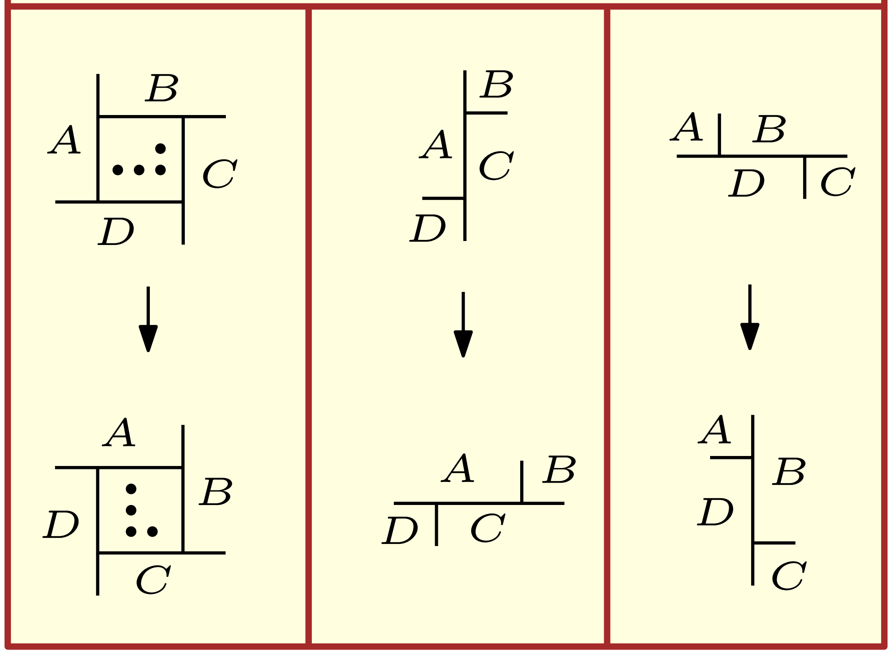
Set of strong rectangulations of a triangulation

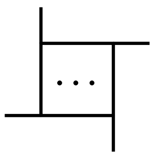
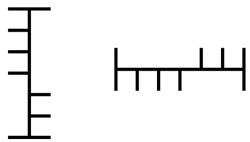
[F'07]





Covering relations:



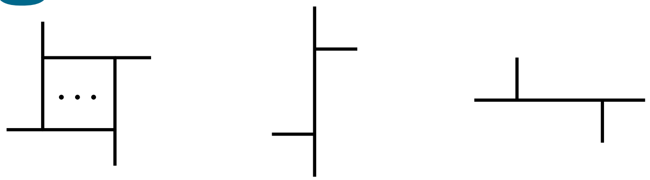
minimal element: no  canonical shuffle: at segments 

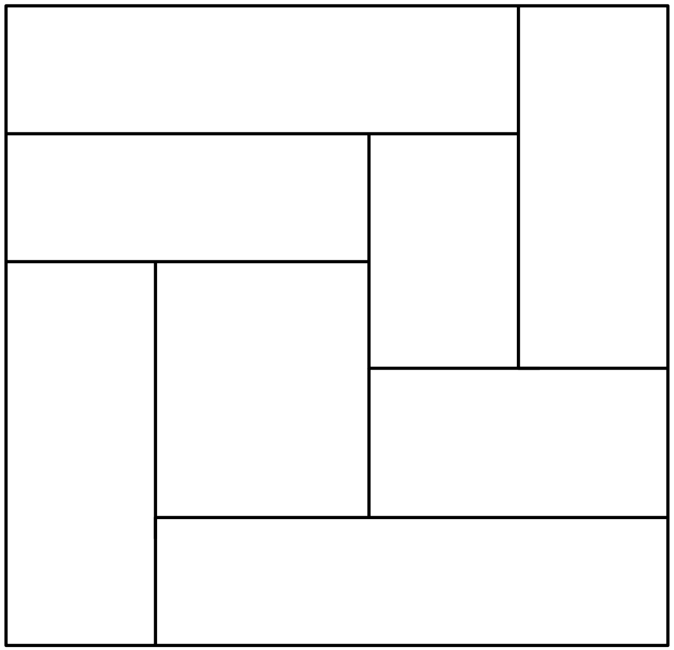
same minimal elements as for 2-connected maps

⇓

bijection between the two map families

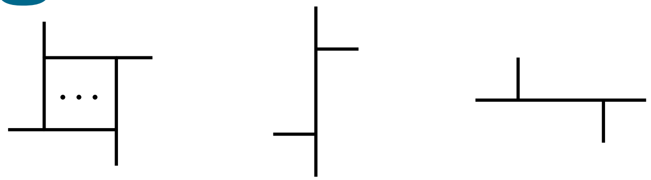
Minimal rectangulations ↔ ternary trees

minimal element: no 



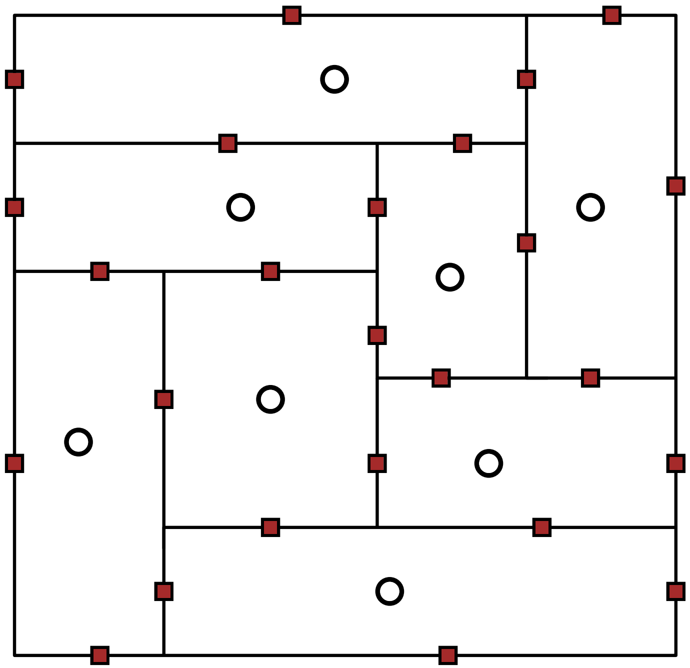
minimal rectangulation
of size $n = 8$

Minimal rectangulations $\leftrightarrow$ ternary trees

minimal element: no 

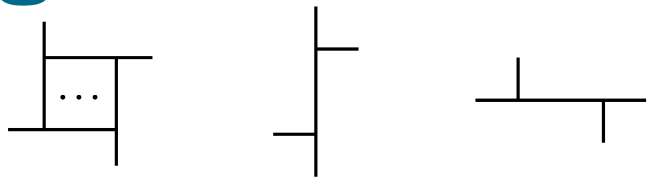
Partial duality mapping: insert

- round vertex in each region
  (n round vertices)
- square vertex in each little segment
  ($3n+1$ square vertices)



minimal rectangulation
of size $n = 8$

Minimal rectangulations ↔ ternary trees

minimal element: no 

Partial duality mapping: insert

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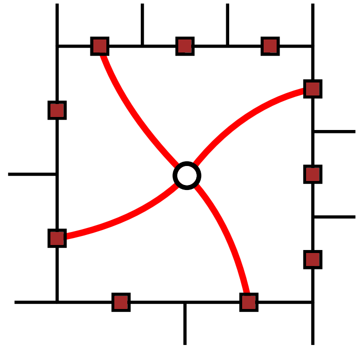
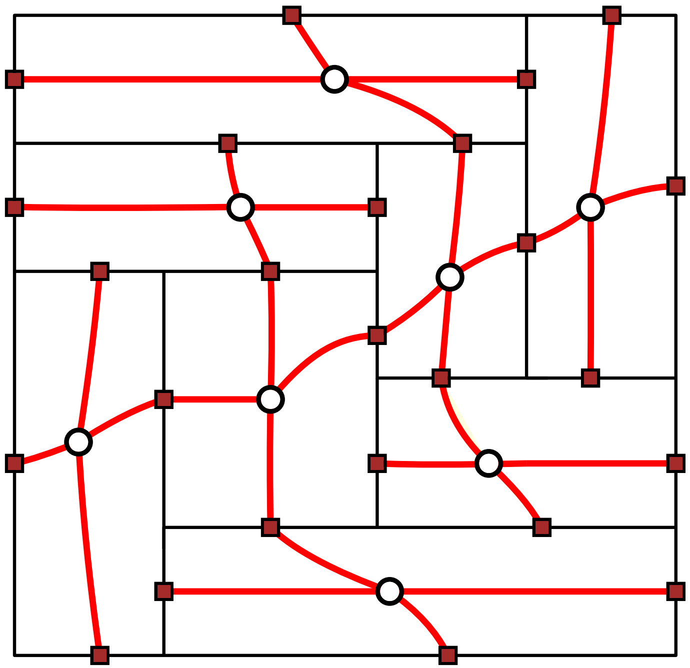
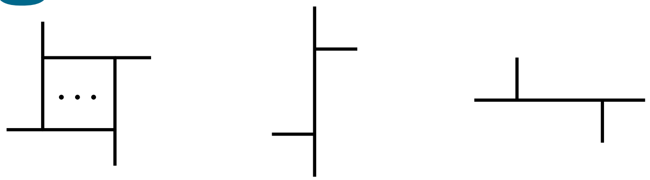
- 4 edges in each region
  ($4n$ edges)





minimal rectangulation of size $n = 8$

Minimal rectangulations $\leftrightarrow$ ternary trees

minimal element: no 

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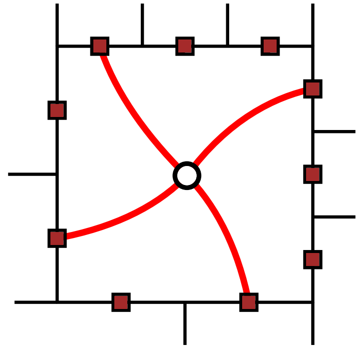

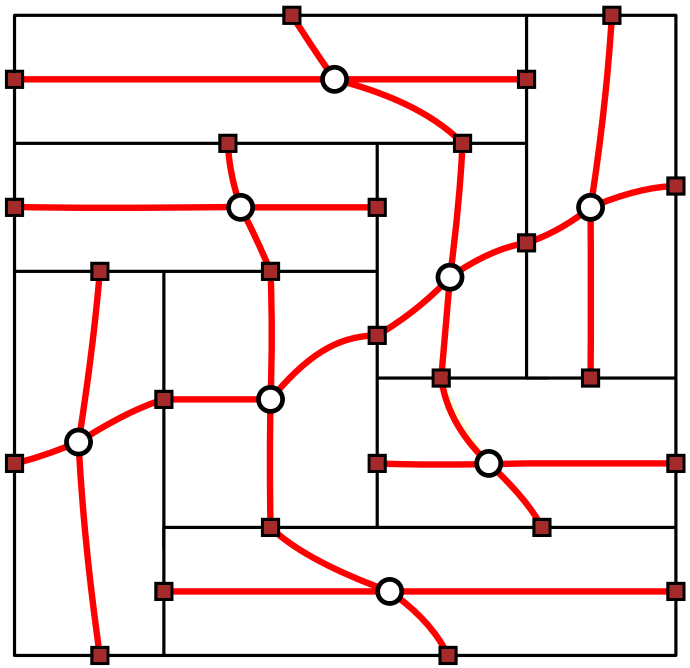

minimal rectangulation
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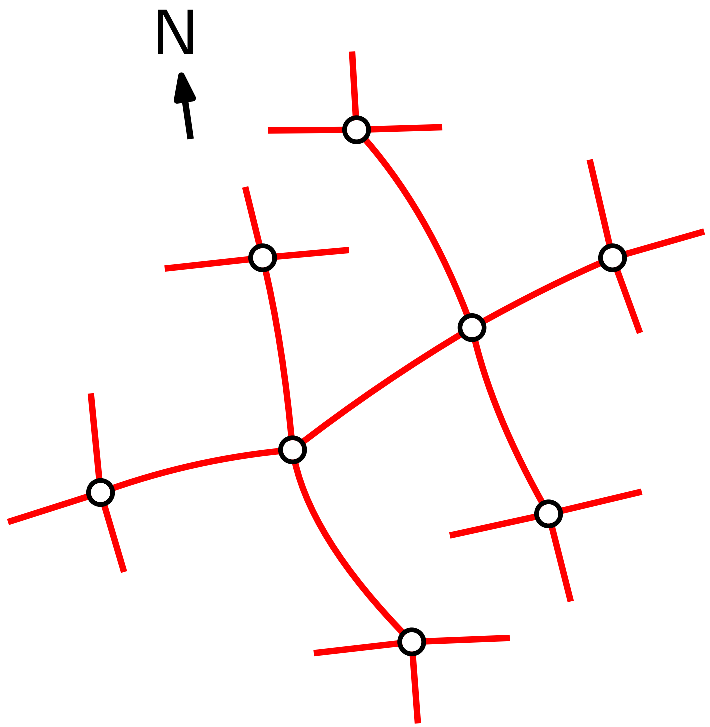
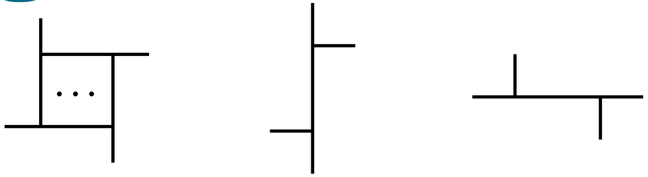


unrooted ternary tree
[Schaeffer'99, F'07]

Minimal rectangulations $\leftrightarrow$ ternary trees

minimal element: no 

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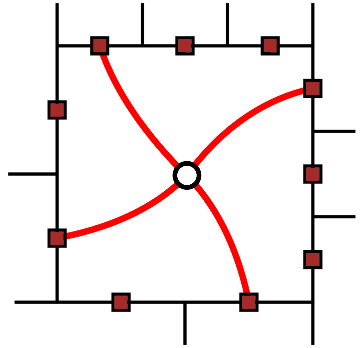

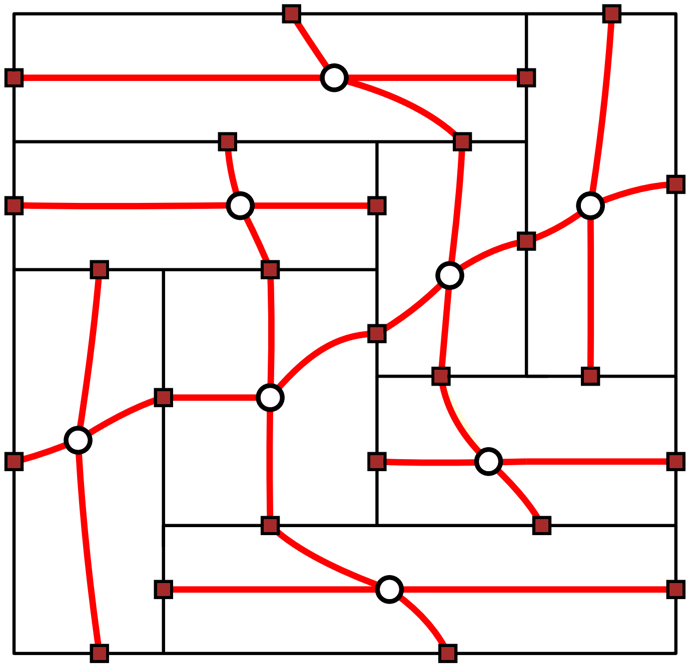

minimal rectangulation
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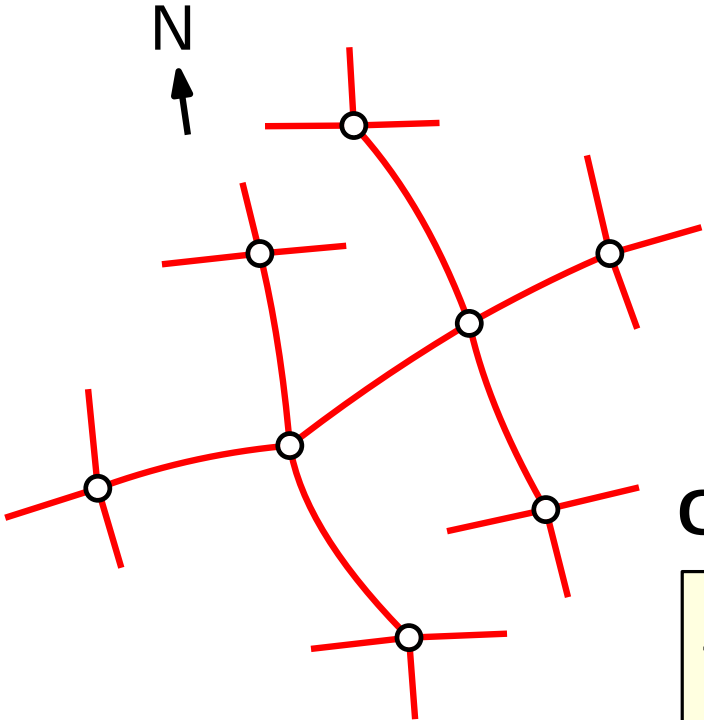


unrooted ternary tree
[Schaeffer'99, F'07]

Counting: [Tutte'62,63]

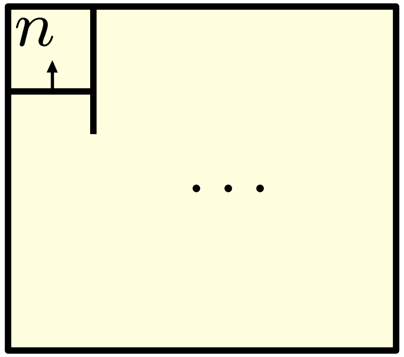
$$\frac{4}{2n+2}\,\frac{1}{2n+1}\binom{3n}{n}$$

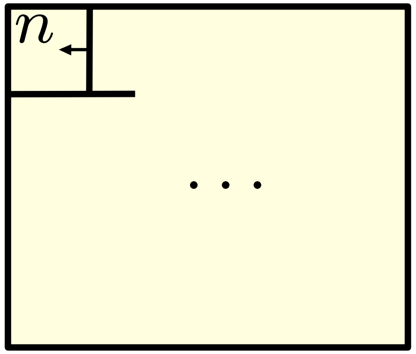
$$\sim c\,(27/4)^n n^{-5/2}$$

Peeling order for weak rectangulations

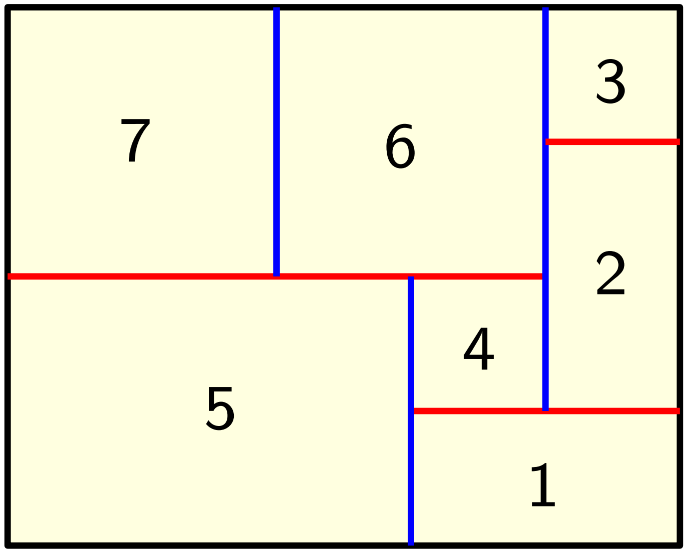
[Hong et al.'00]

Contract top-left region:
two cases



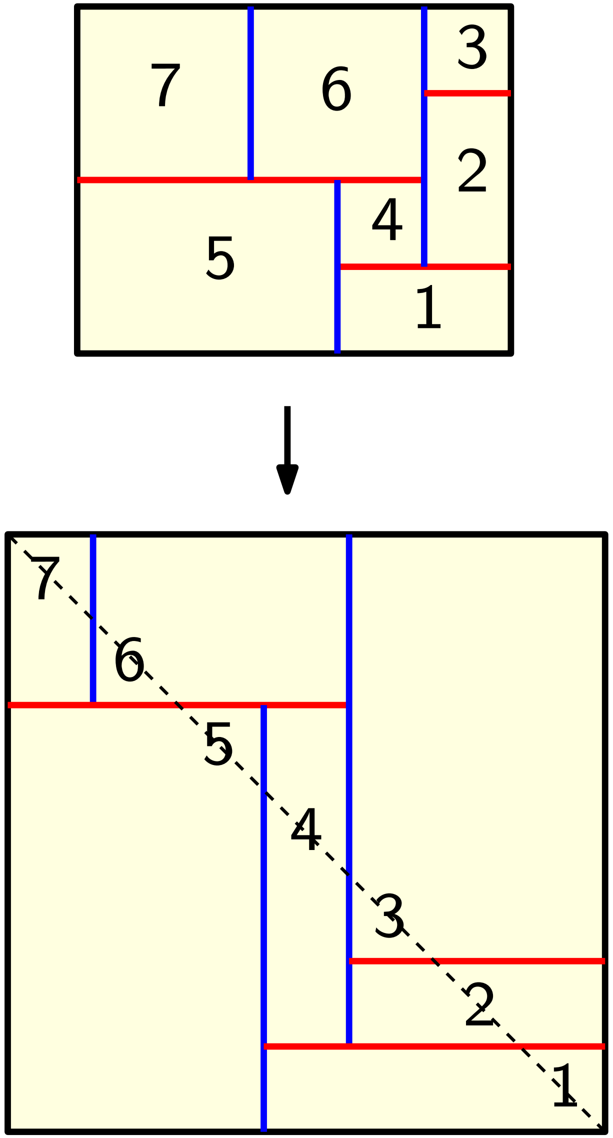


⇒ peeling order on regions



Diagonal representation

[Ackerman, Barequet, Pinter'06]



[Ackerman, Barequet, Pinter'06]

Diagonal representation

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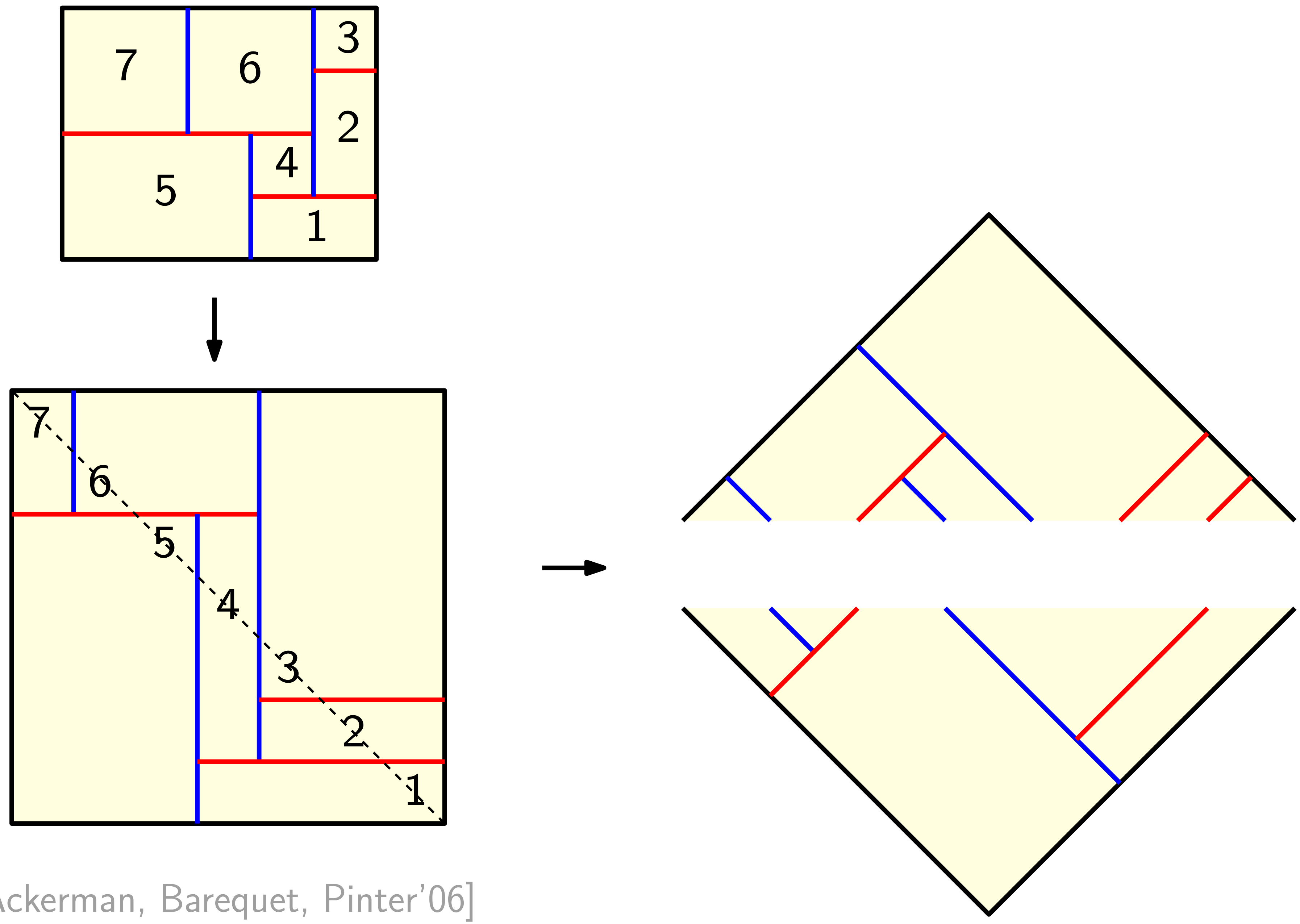

[Ackerman, Barequet, Pinter'06]

Diagonal representation

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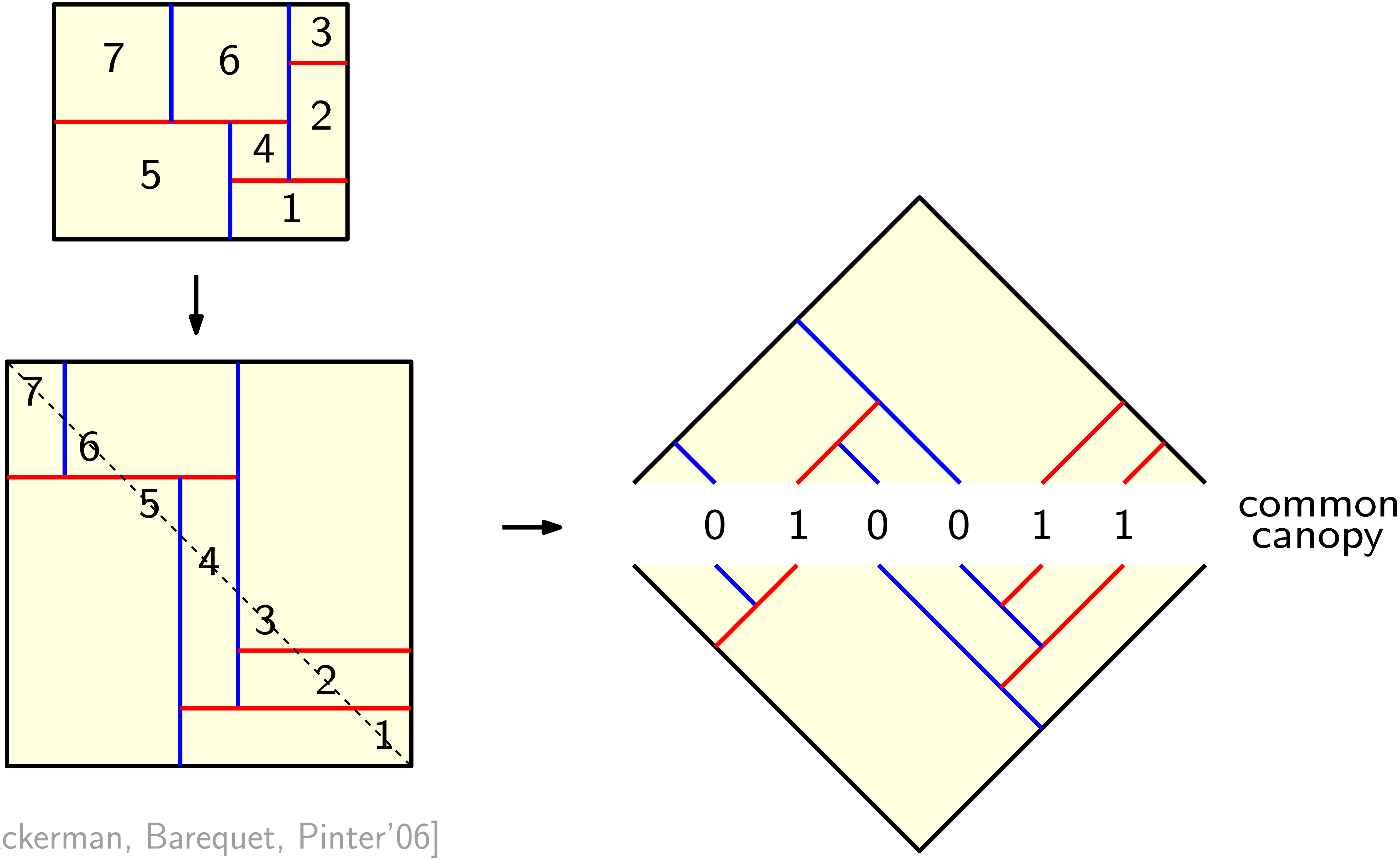
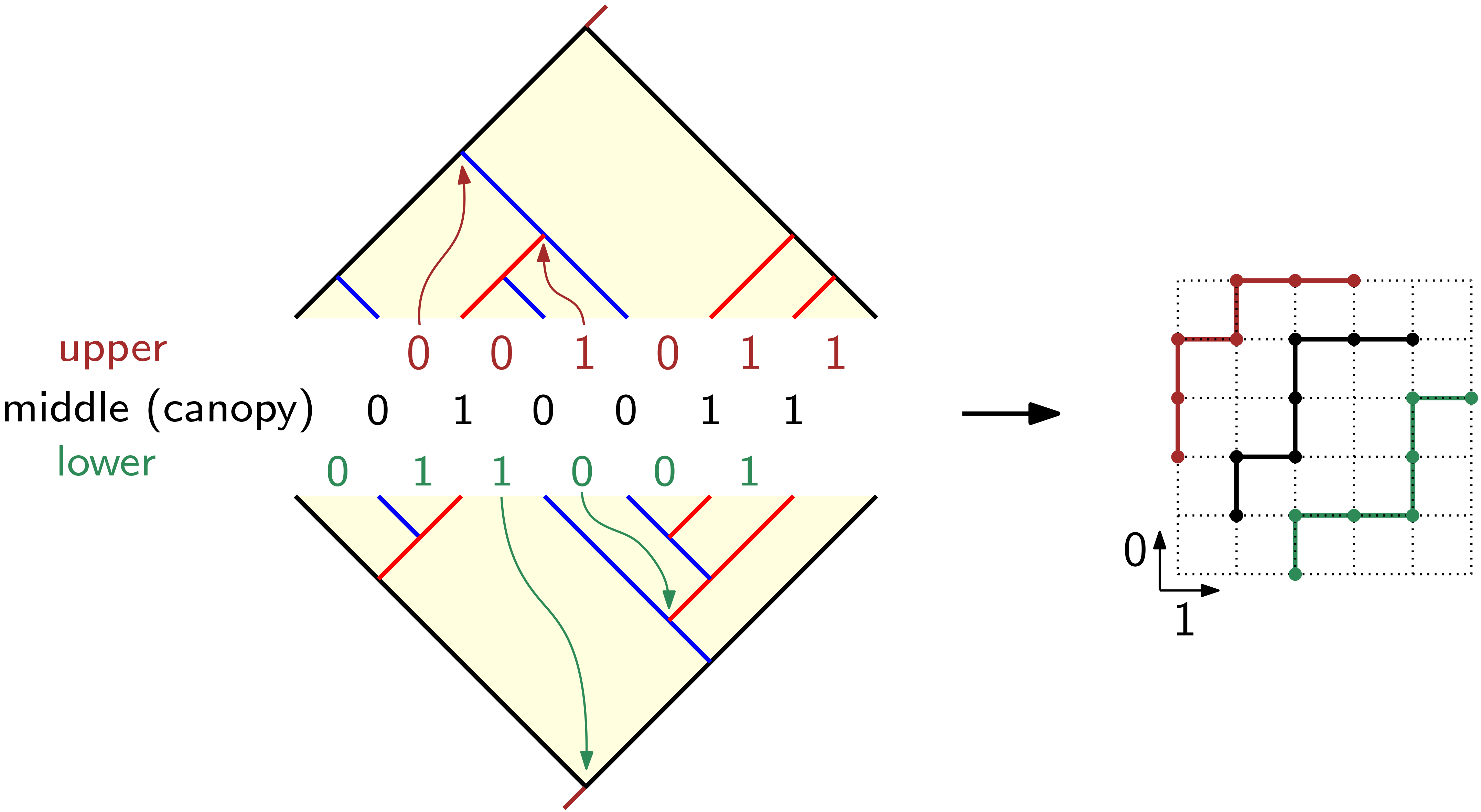


[Ackerman, Barequet, Pinter'06]

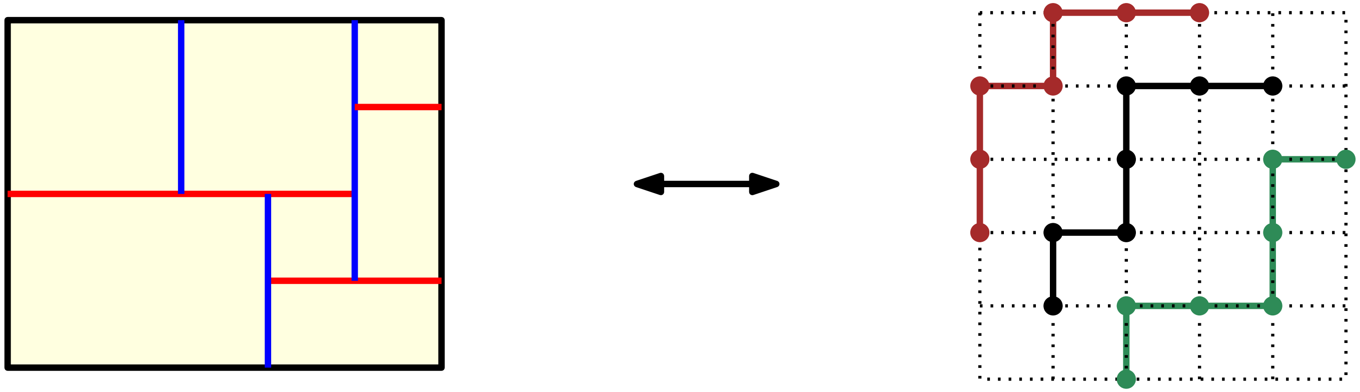
twin pair of binary trees

Encoding by a triple of walks

[Feslner, F, Noy, Order'10]
other encoding in [Dulucq, Guibert'98]



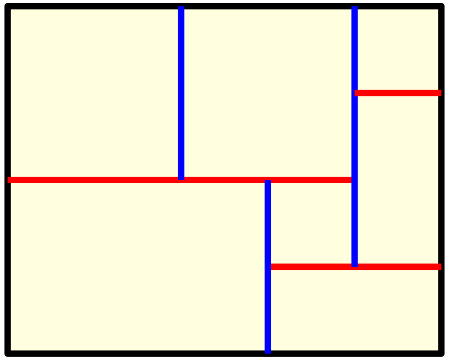
Baxter numbers and Baxter families

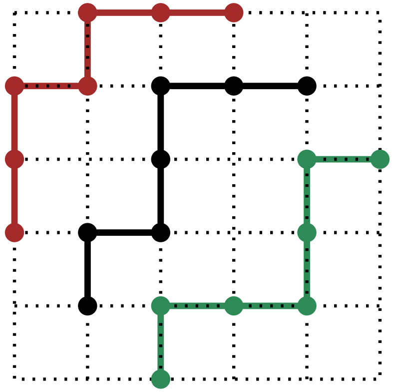


Gessel-Viennot $\Rightarrow$ $$w_n = \frac{2}{n(n+1)^2}\sum_{r=0}^{n-1}\binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2} \sim \frac{2^5}{\pi\sqrt{3}}8^n n^{-4}$$

Baxter numbers

Baxter numbers and Baxter families





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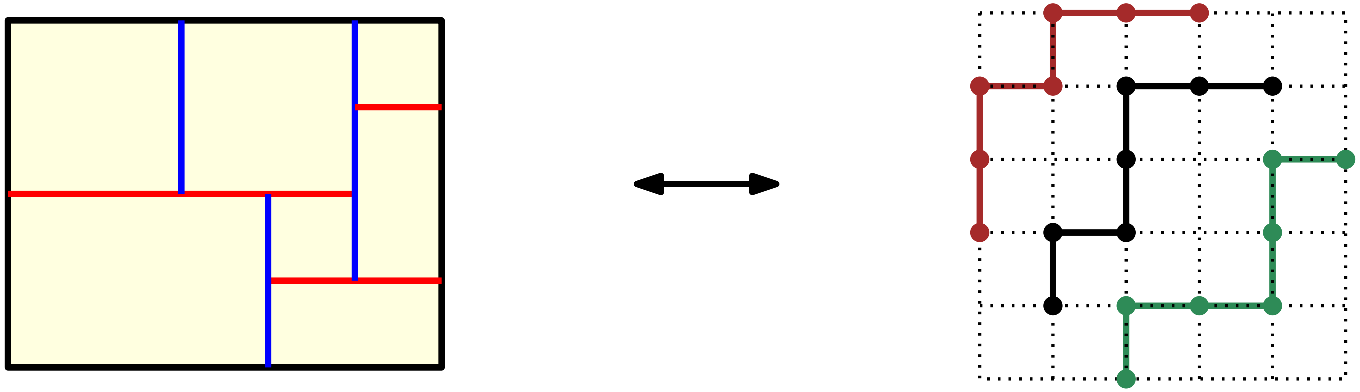
Baxter numbers

D-finite: $$(n+2)(n+3)w_n = (7n^2+7n-2)w_{n-1} + 8(n-1)(n-2)w_{n-2}$$

Baxter numbers and Baxter families



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Baxter numbers

D-finite: $$(n+2)(n+3)w_n = (7n^2+7n-2)w_{n-1} + 8(n-1)(n-2)w_{n-2}$$

Baxter families are families counted by Baxter numbers

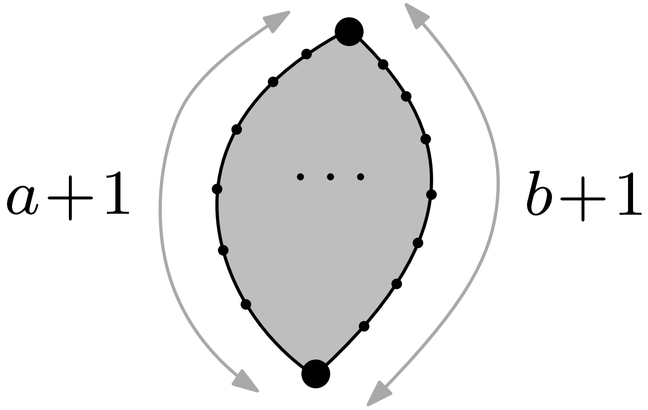
Various bijections relating these families (Baxter generating tree)

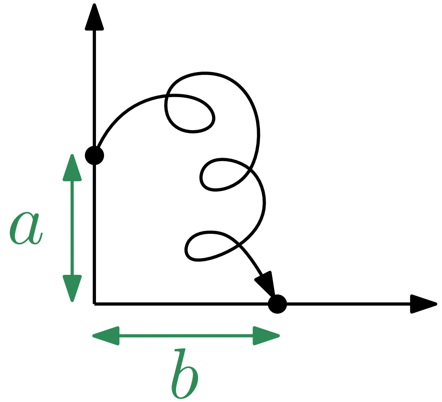
[Chung, Graham, Hoggatt, Kleiman, '78], [Viennot'81], [Dulucq-Guibert'98], [Ackerman-Barequet-Pinter'06]
[F-Poulalhon-Schaeffer'07] [Felsner-F-Noy-Orden'10], [Bonichon-Bousquet-Mélou-F'09], [Albenque-Poulalhon'13], . . .

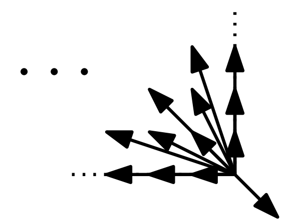
Another walk-encoding: KMSW bijection

[Kenyon, Miller, Sheffield, Wilson'15]

Plane bipolar orientations ⟷ "Tandem walks" in the quadrant







n edges ⟷ length $n-1$

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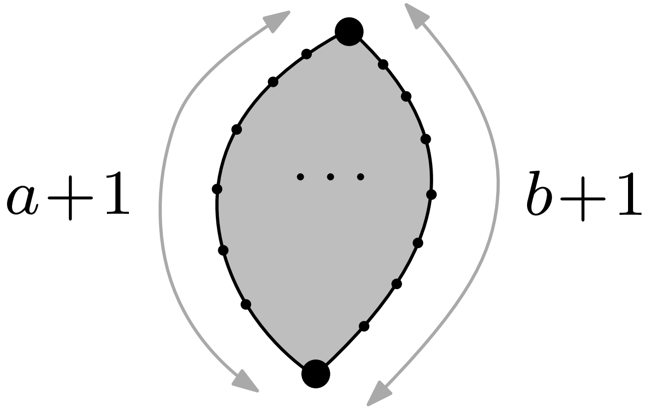
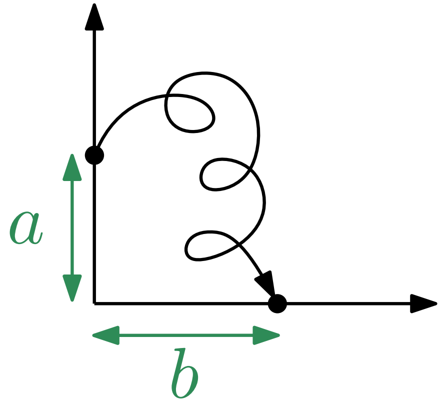
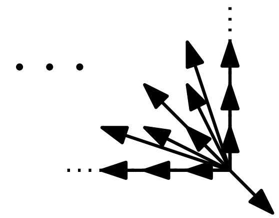


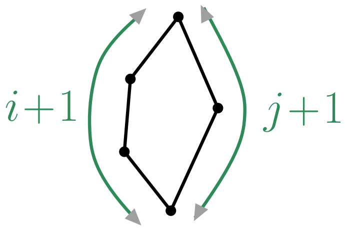


step-set



SE $\cup \{(-i,j), i,j \geq 0\}$

n edges ⟷ length $n-1$

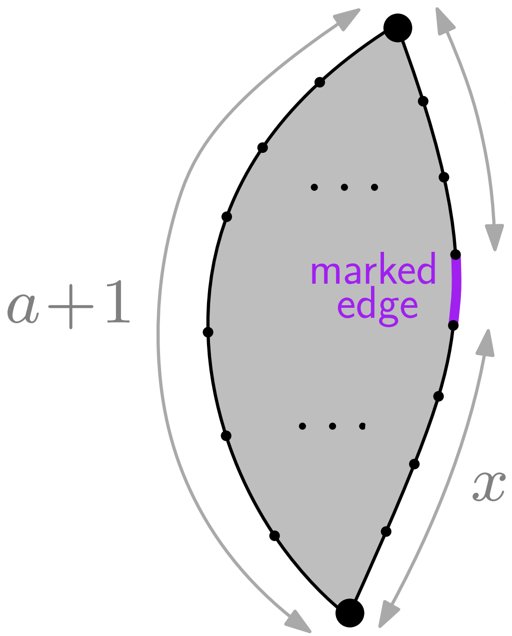
face 

⟷ face-step $(-i,j)$

non-pole vertex ⟷ SE step

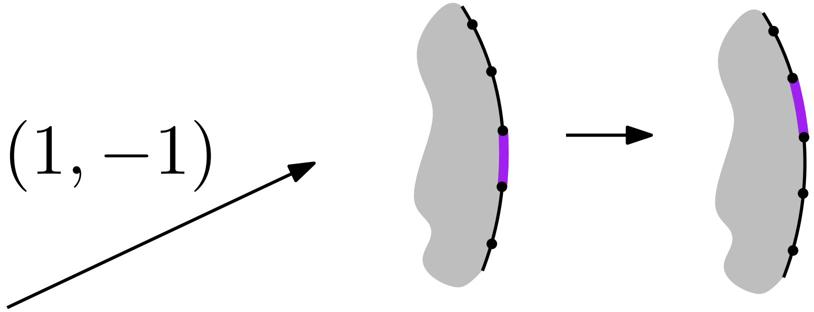
Another walk-encoding: KMSW bijection

Orientation is built step by step from the walk,

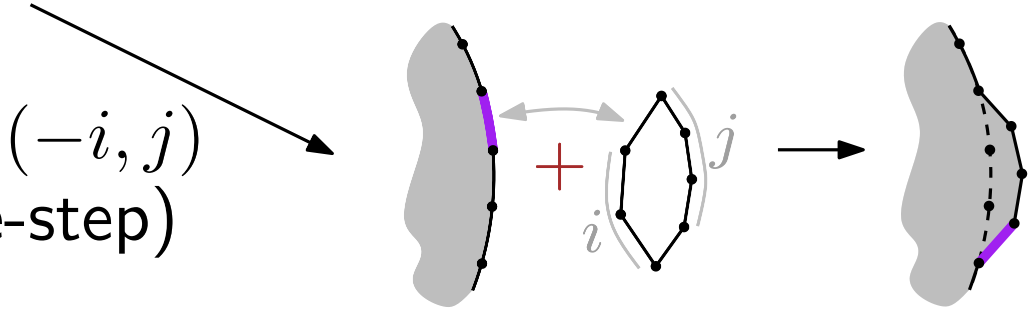


orientation
currently built

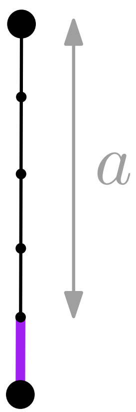
add $(1,-1)$



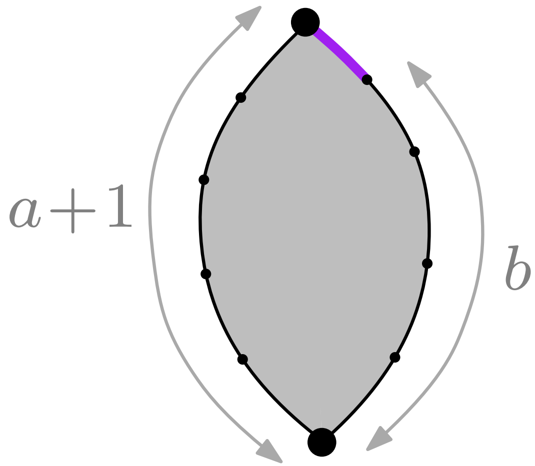
add $(-i,j)$
(face-step)



Starts with

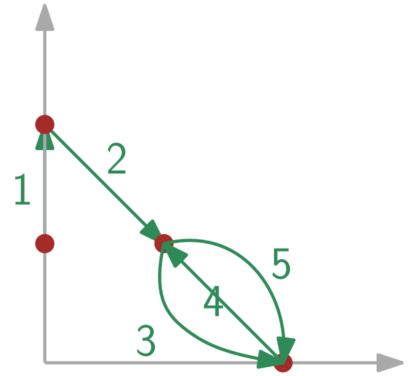


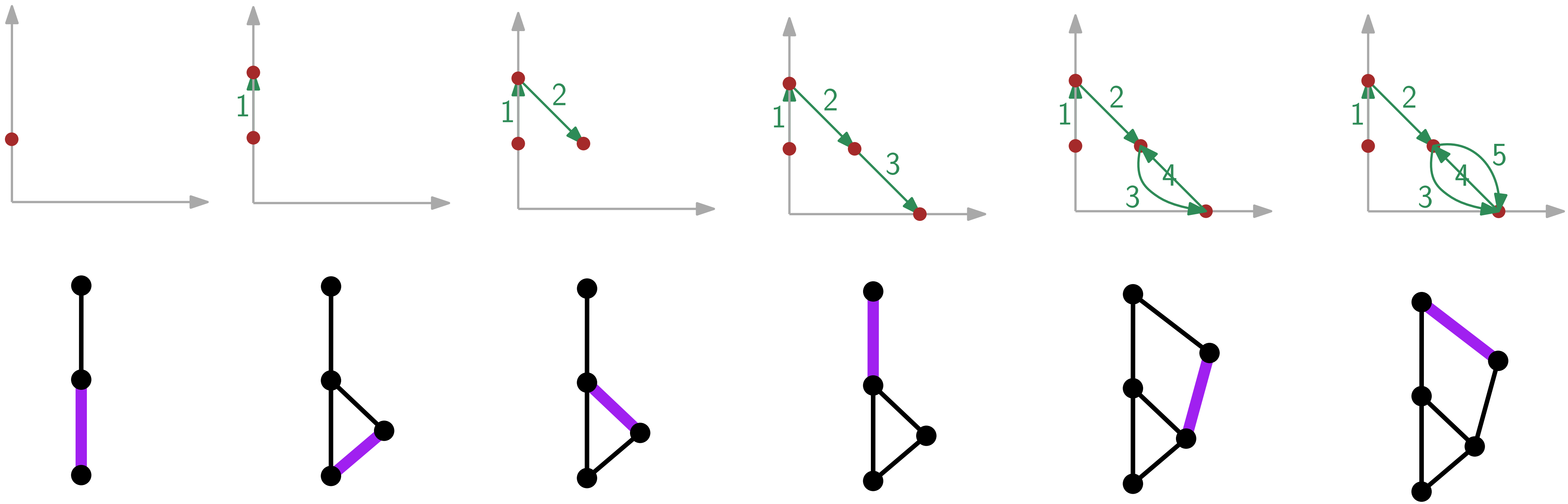
Ends with



Another walk-encoding: KMSW bijection

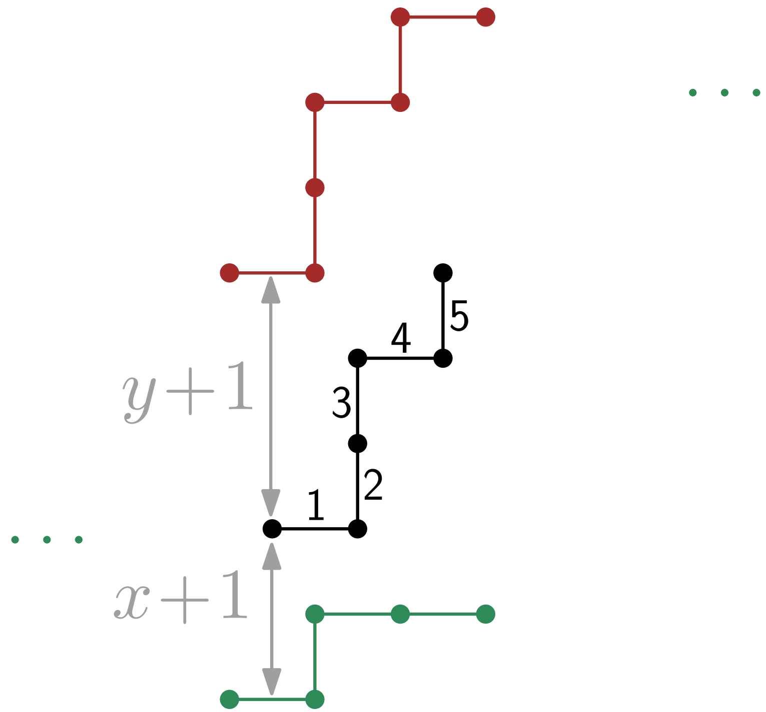
Example: build orientation associated to



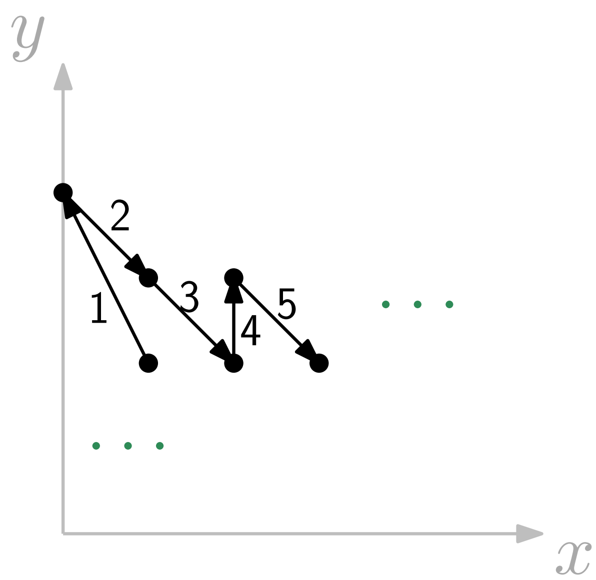


Link with non-intersecting triples of walks

[Bousquet-Mélou,F,Raschel'20]

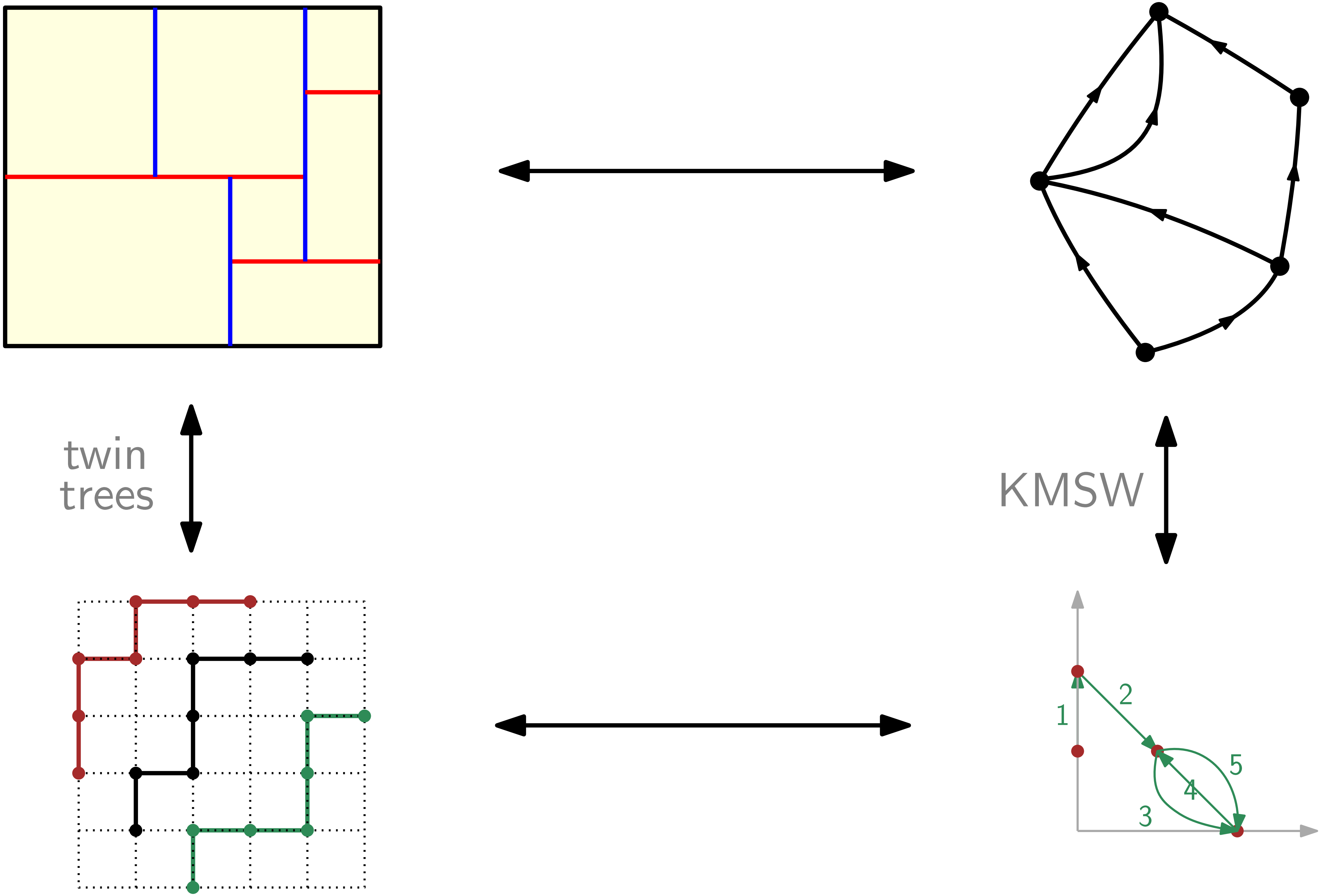


non-intersecting triple



tandem walk

Bijective links



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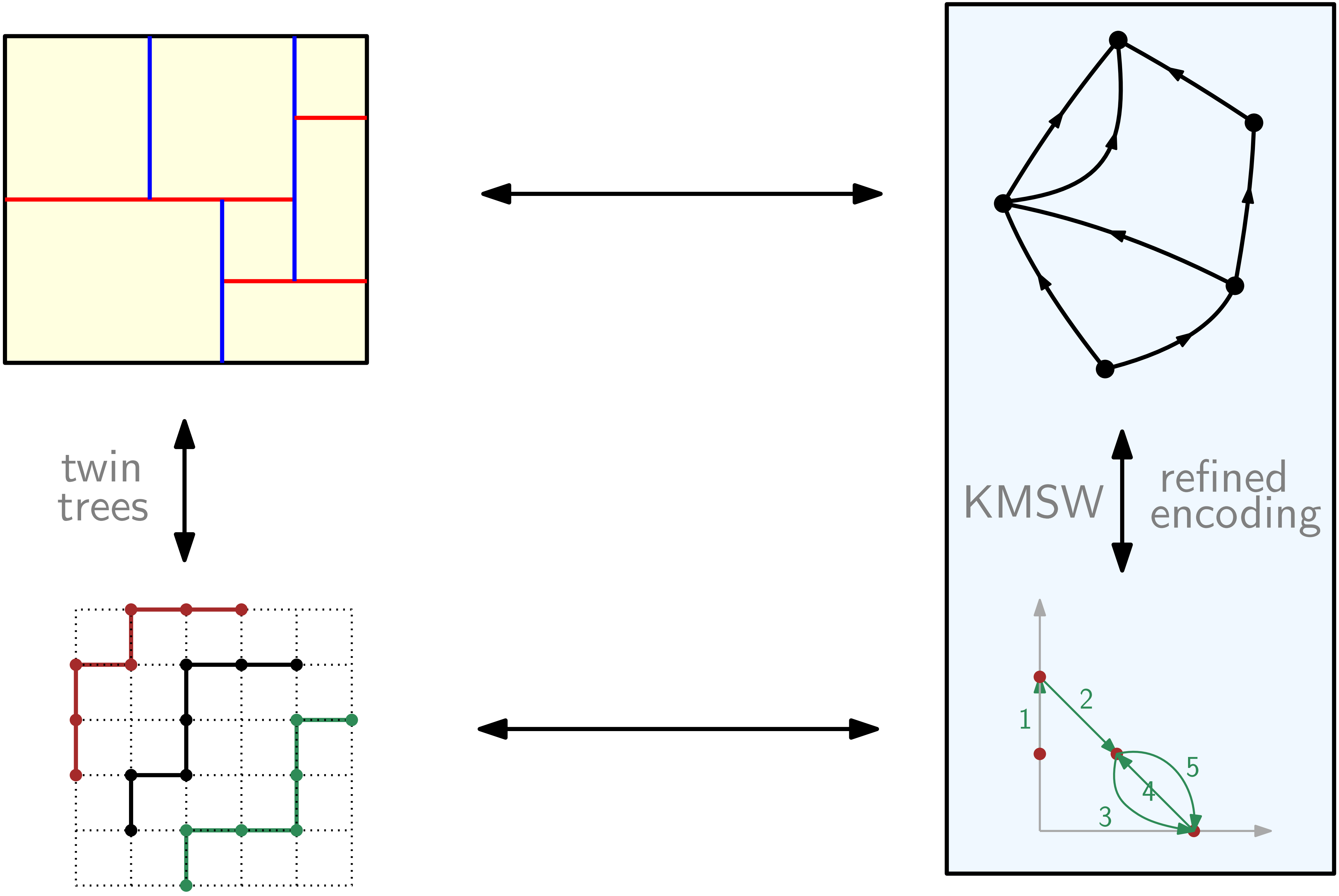
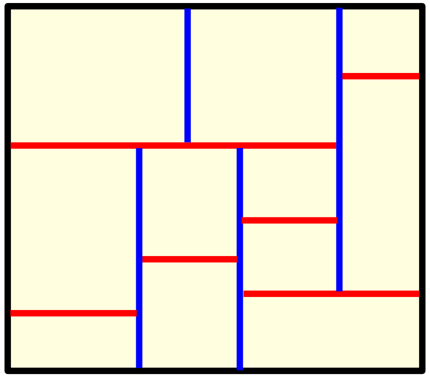


Application to strong rectangulations



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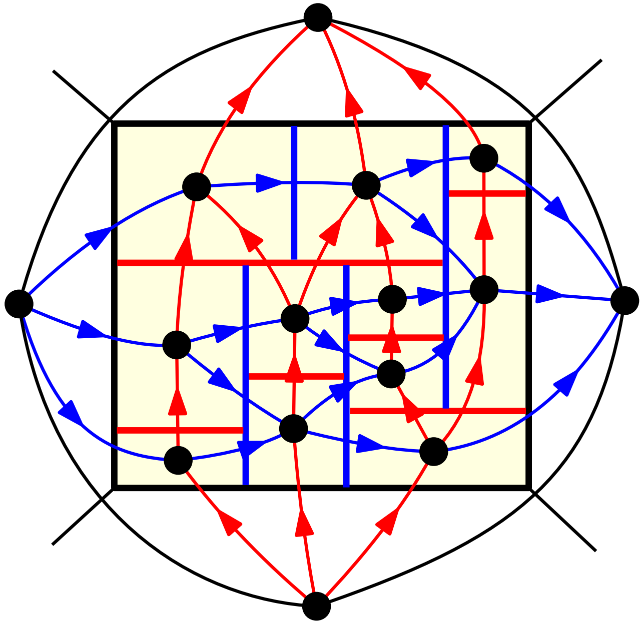
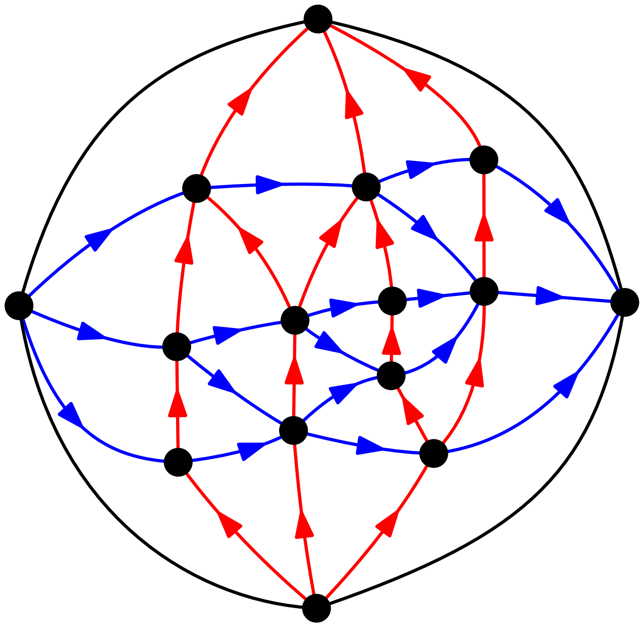


Transversal structure
$n + 4$ vertices

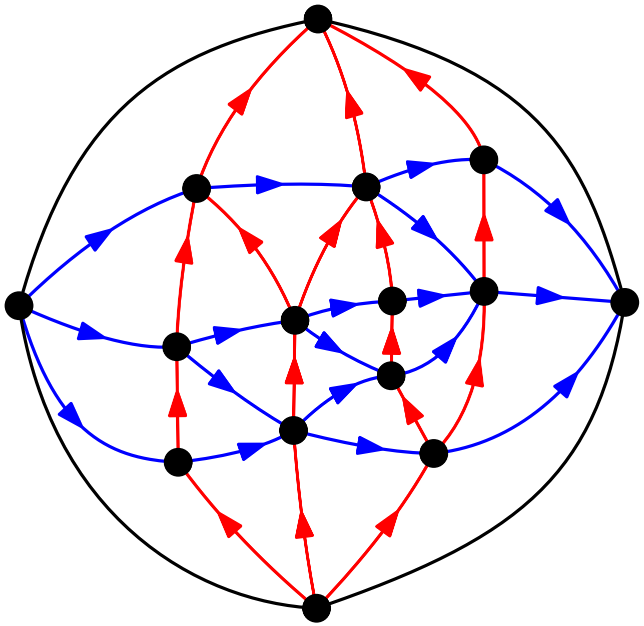
Application to strong rectangulations



Transversal structure
$n + 4$ vertices

Application to strong rectangulations

[F-Narmanli-Schaeffer'21]



Transversal structure
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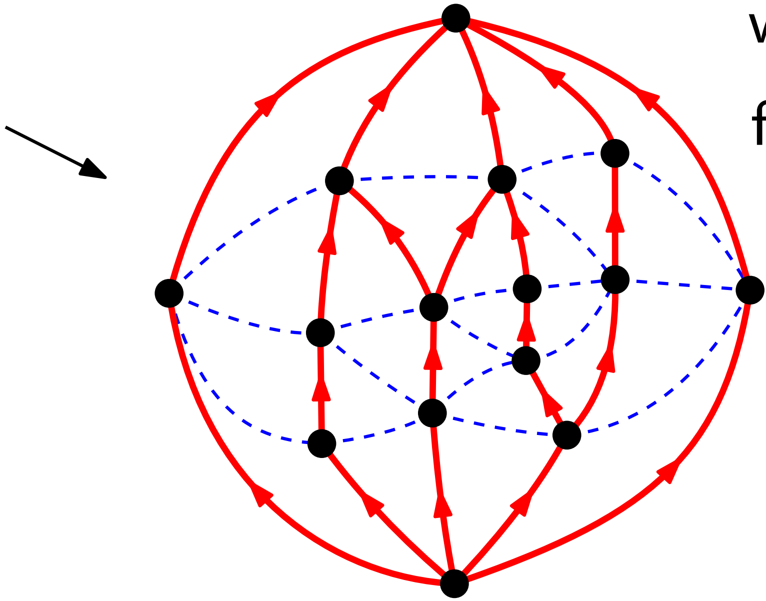
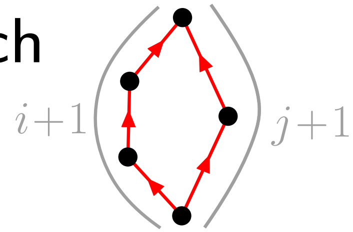


red bipolar poset
$+$ transversal edges

weight $\binom{i+j-2}{i-1}$

for each



Application to strong rectangulations

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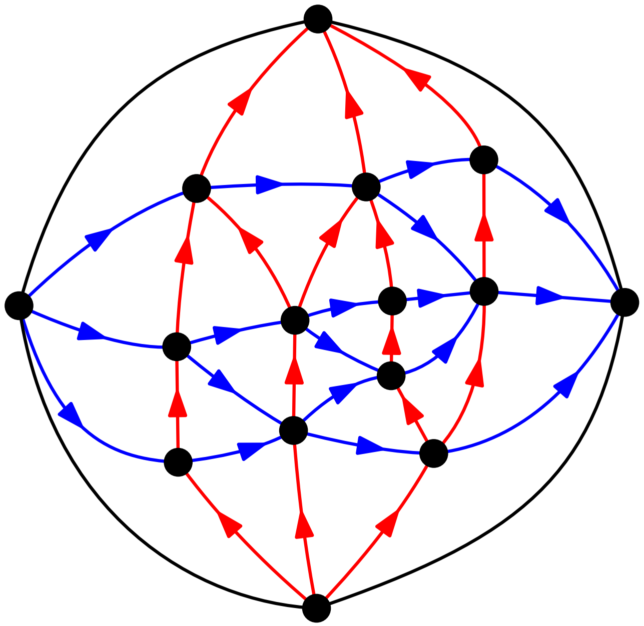

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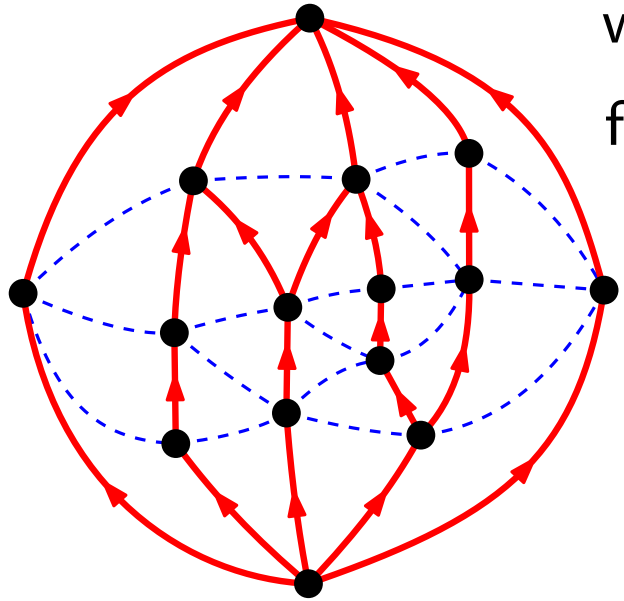

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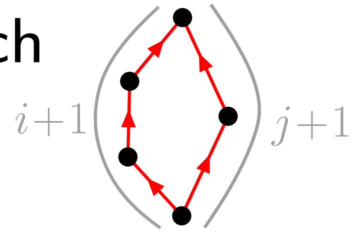
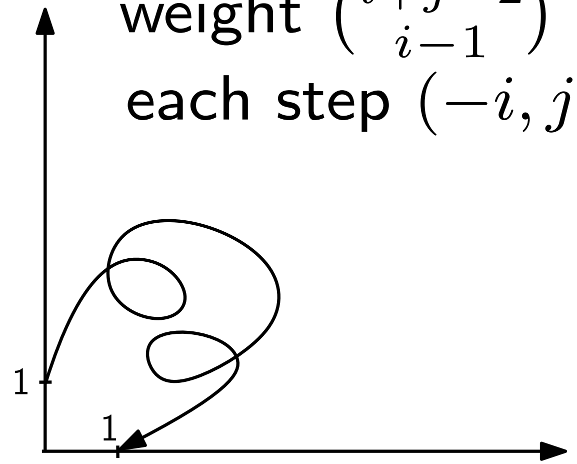


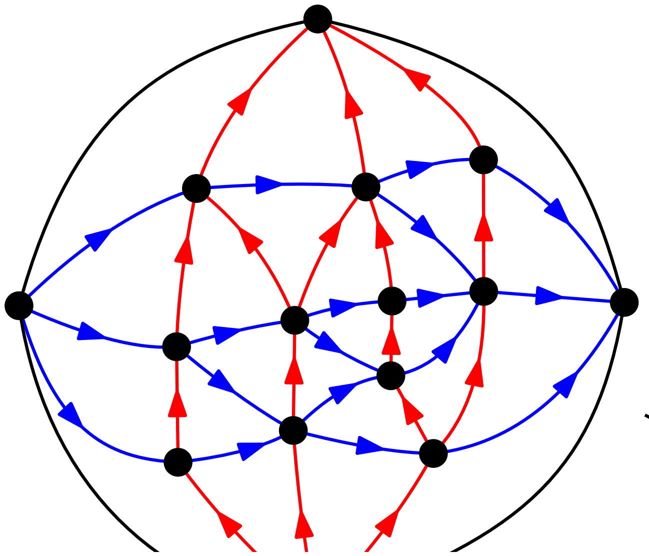
weight $\binom{i+j-2}{i-1}$ for each step $(-i,j)$

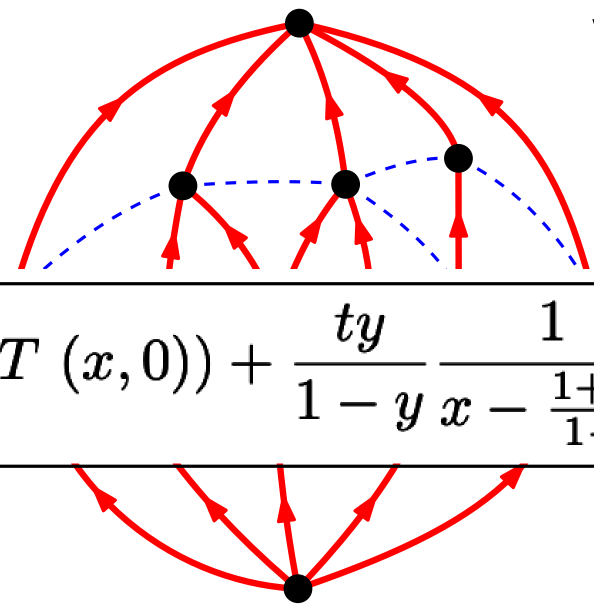


weighted tandem walk
with n SE steps

Application to strong rectangulations

[F-Narmanli-Schaeffer'21]





weight $\binom{i+j-2}{i-1}$

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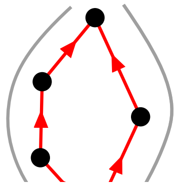
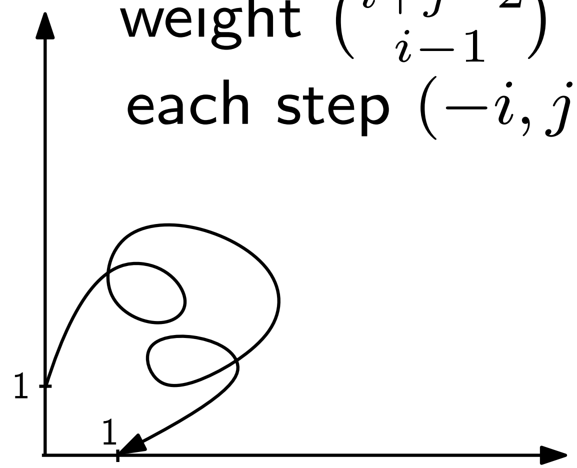


$$T(x,y) = y + tu\frac{x}{y}\left(T(x,y) - T(x,0)\right) + \frac{ty}{1-y}\,\frac{1}{x - \frac{1+vy}{1-y}}\left(T(x,y) - T\left(\frac{1+vy}{1-y}, y\right)\right)$$

$n+4$ vertices

red bipolar poset
+ transversal edges

weight $\binom{i+j-2}{i-1}$ for each step $(-i, j)$



weighted tandem walk
with n SE steps

Asymptotic enumeration

[F-Narmanli-Schaeffer'21]

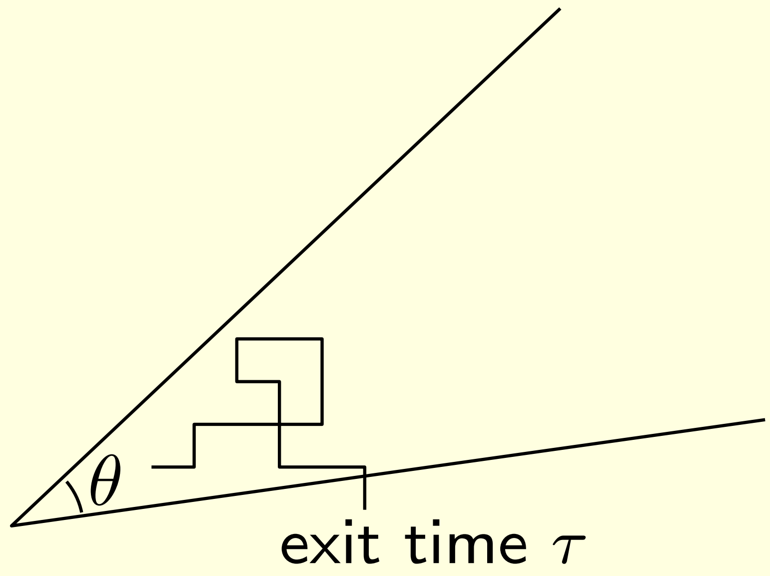
relies on [Denisov-Wachtel'11, Bostan-Raschel-Salvy'14]

Each of the counting sequences w_n, s_n has asymptotics of the form

$$c\,\gamma^n\,n^{-\alpha}$$

$\alpha = 1 + \frac{\pi}{\theta}$

	weak	strong
γ	8	$27/2$
$\cos(\theta)$	$1/2$	$7/8$
α	4	$\approx 7.21 \notin \mathbb{Q}$



$$\mathbb{P}(\tau > n) \sim c\,n^{-\frac{\pi}{2\theta}}$$

$$\mathbb{P}(\tau > n \text{ \& excursion}) \sim c'\,n^{-1-\frac{\pi}{\theta}}$$

Asymptotic enumeration

[F-Narmanli-Schaeffer'21]

relies on [Denisov-Wachtel'11, Bostan-Raschel-Salvy'14]

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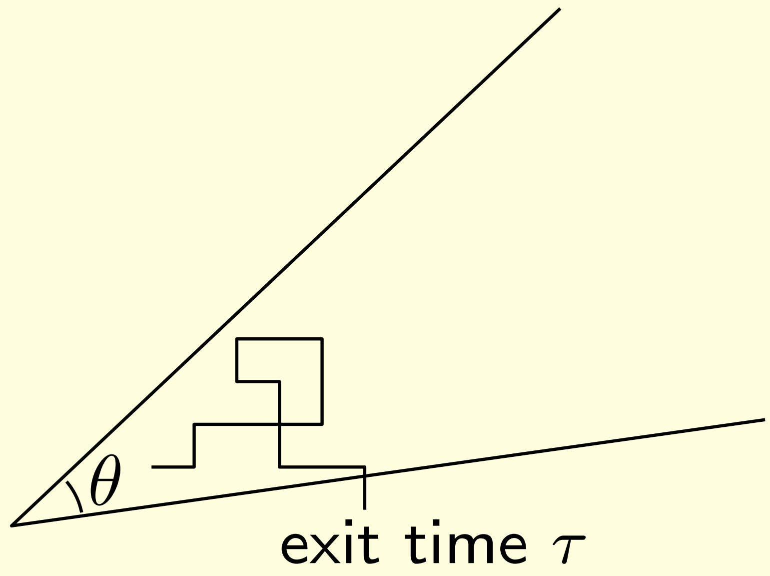
γ	8	$27/2$
$\cos(\theta)$	$1/2$	$7/8$
α	4	$\approx 7.21 \notin \mathbb{Q}$

$\notin \mathbb{Q}$ → not D-finite



$$\mathbb{P}(\tau > n) \sim c\,n^{-\frac{\pi}{2\theta}}$$

$$\mathbb{P}(\tau > n \;\&\; \text{excursion}) \sim c'\,n^{-1-\frac{\pi}{\theta}}$$

Illustration of the method on tandem walks

Let $\mathcal{S}$ be a step-set for $\mathbb{Z}^2$ $\quad S(x,y) := \sum_{(i,j)\in\mathcal{S}} x^i y^j$

Let $a_n(i,j) = \#\mathcal{S}$-walks of length n ending at (i,j)

Then $\boxed{\sum_{i,j} a_n(i,j) x^i y^j = S(x,y)^n}$ $\quad \forall x, y \geq 0$

Hence $\quad a_n(0,0) \leq \gamma^n \quad$ with $\gamma = \min_{x,y>0} S(x,y)$

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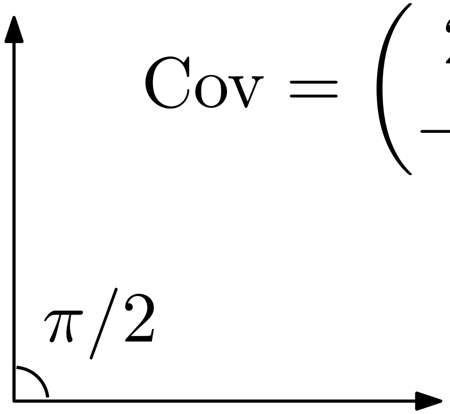
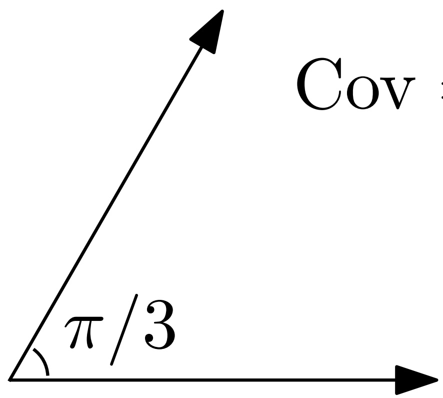


$\mathrm{Cov} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

sheer $\longrightarrow$



$\mathrm{Cov} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

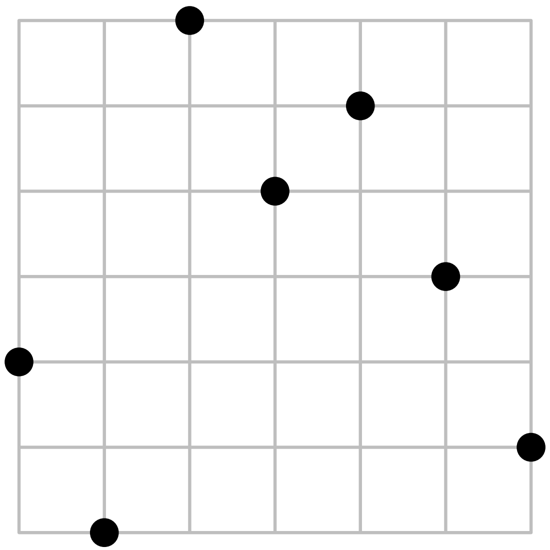
Rectangulations and permutations

Permutations

Example of permutation in size $n = 7$

word $\quad 3 \ 1 \ 7 \ 5 \ 6 \ 4 \ 2$

point diagram



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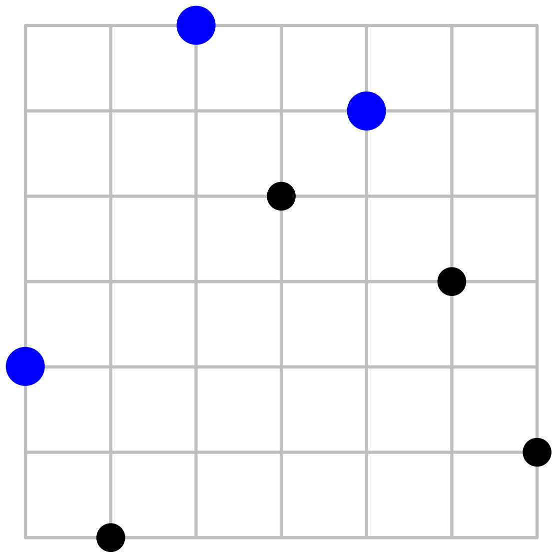
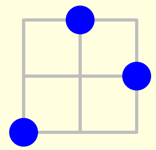
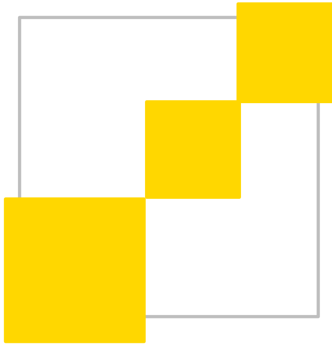
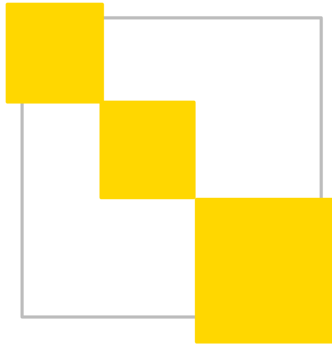


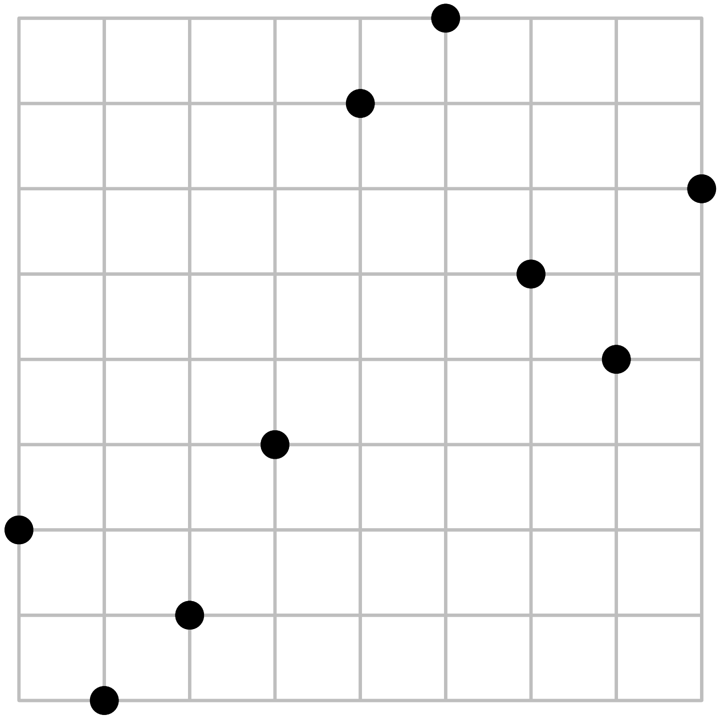
Pattern occurence

contains the pattern 132



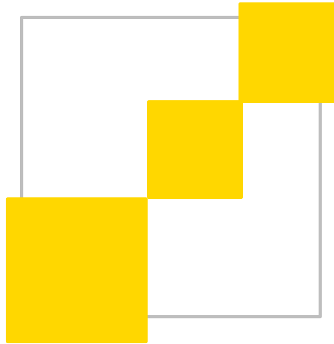
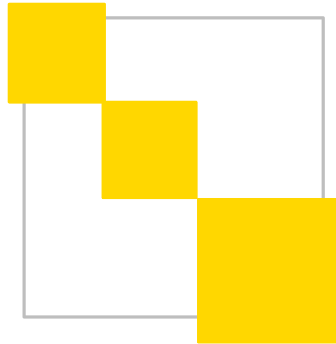
Separable permutations

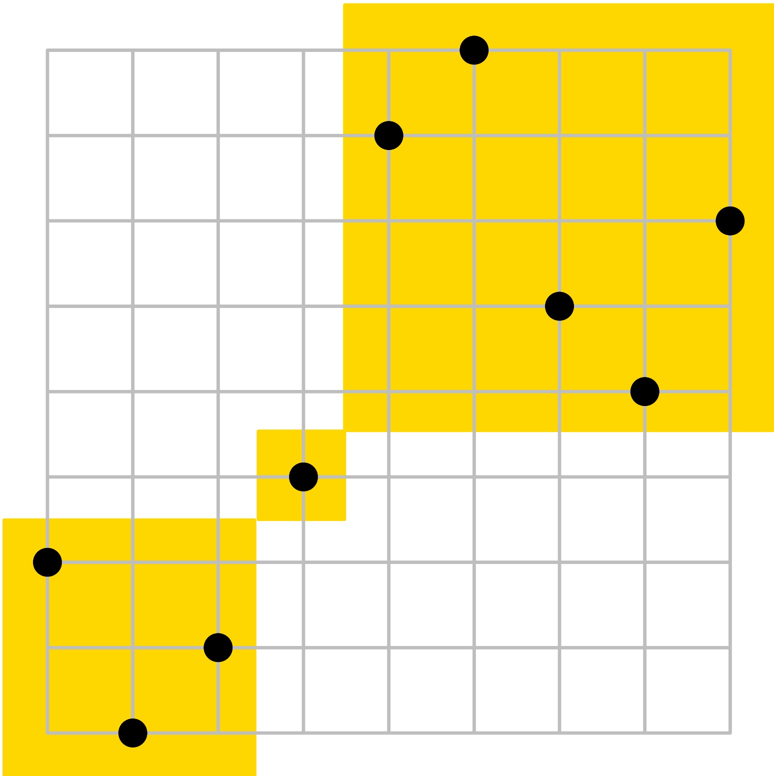
can be recursively decomposed by  or 



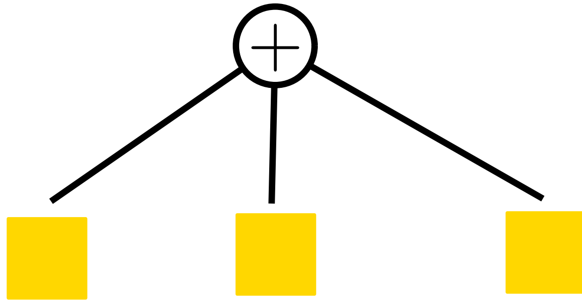
3 1 2 4 8 9 6 5 7

Separable permutations

can be recursively decomposed by  or 



3 1 2 4 8 9 6 5 7



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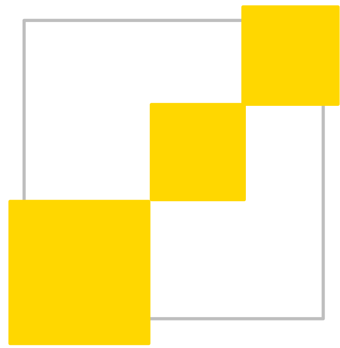
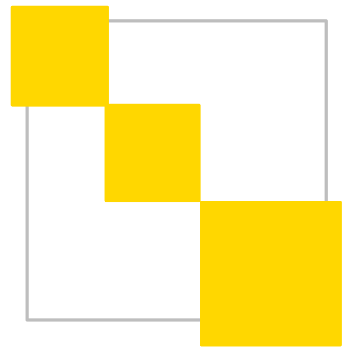
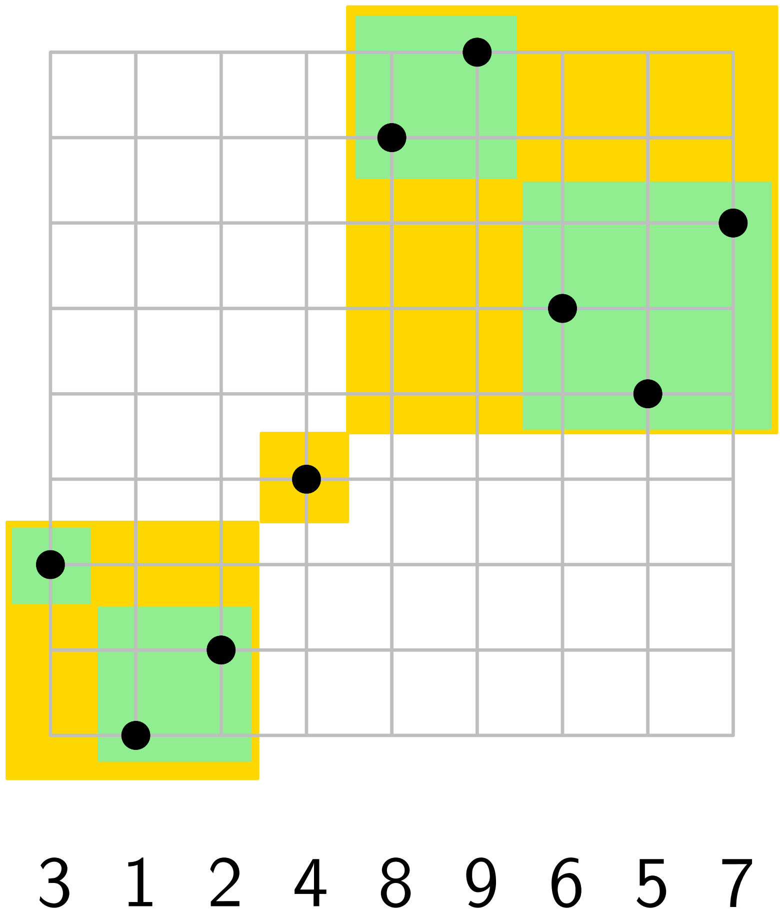
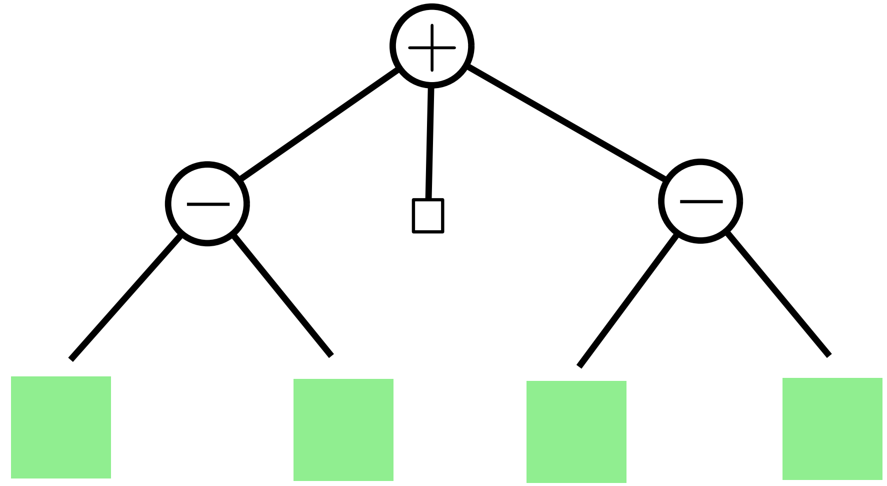
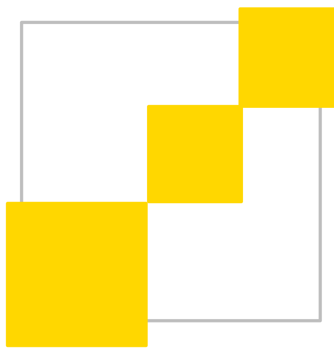
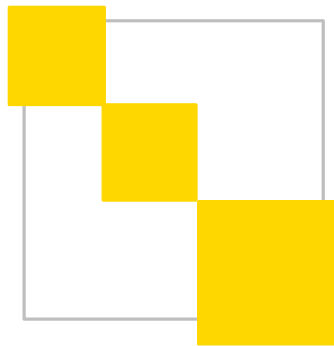
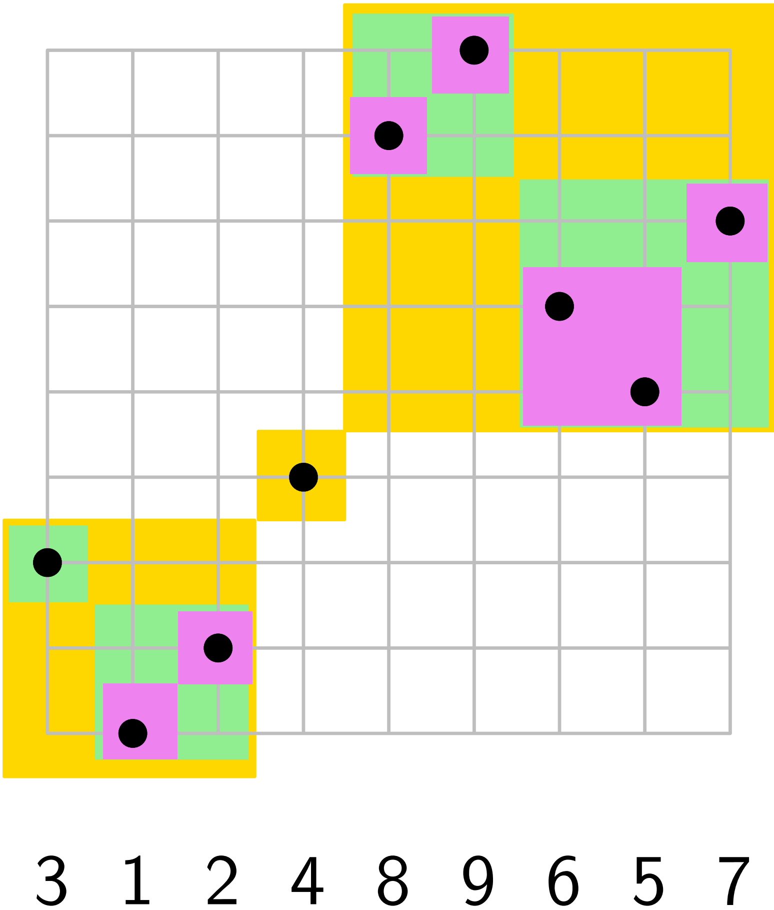
can be recursively decomposed by  or 

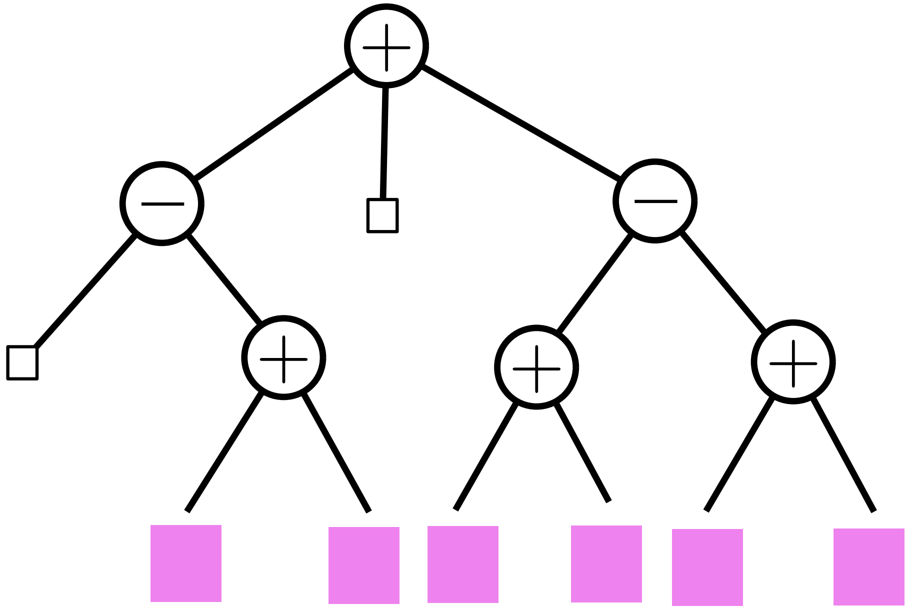




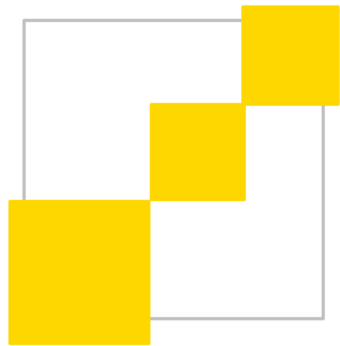
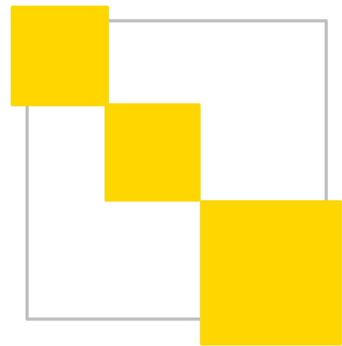
Separable permutations

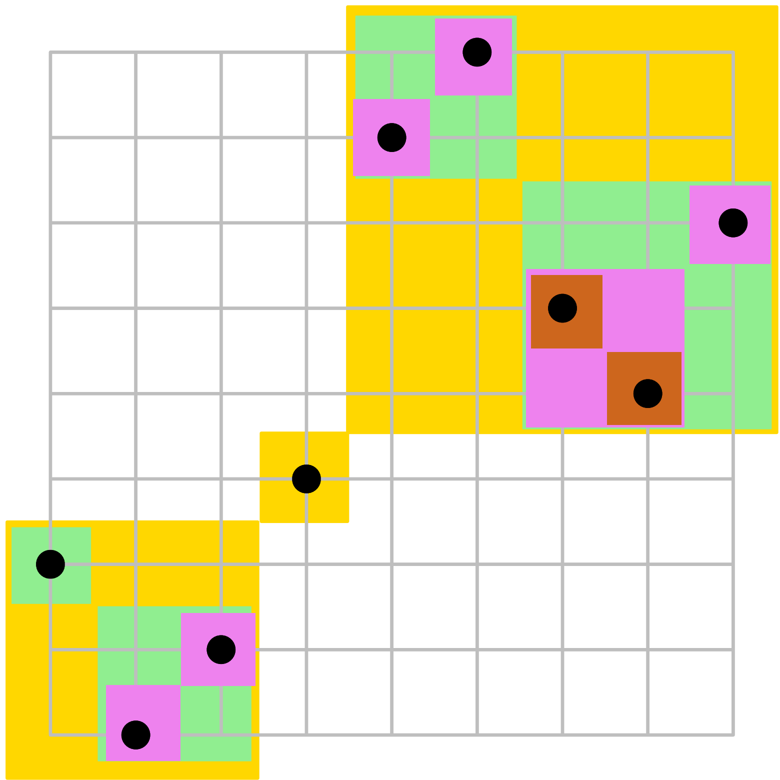
can be recursively decomposed by  or 



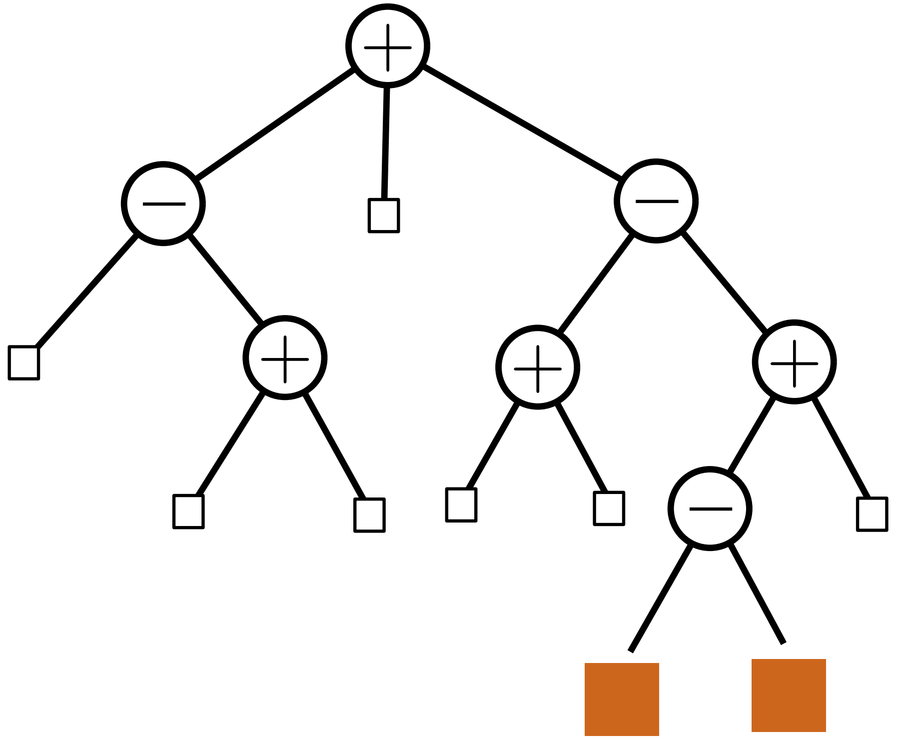


Separable permutations

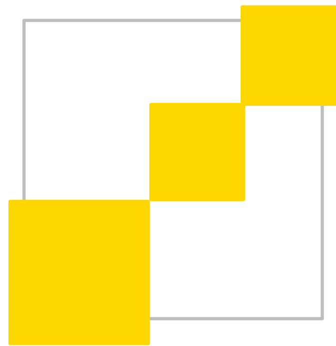
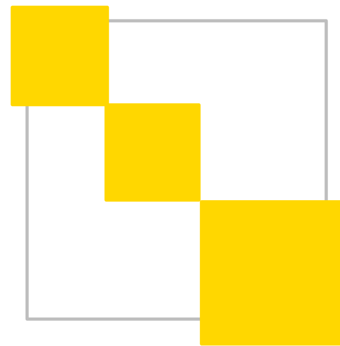
can be recursively decomposed by  or 

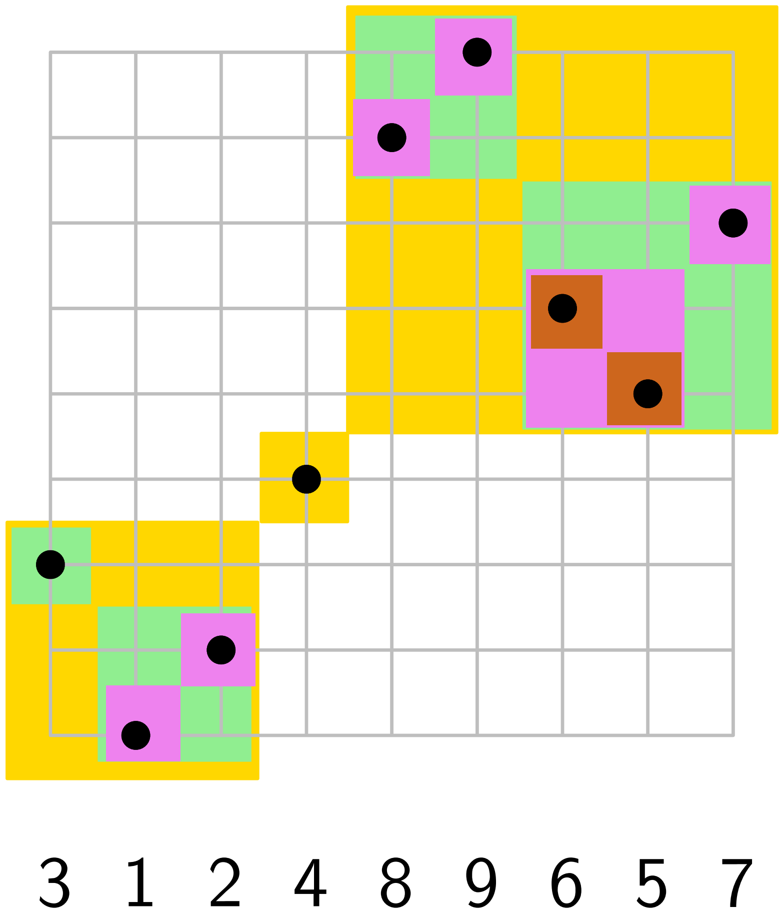


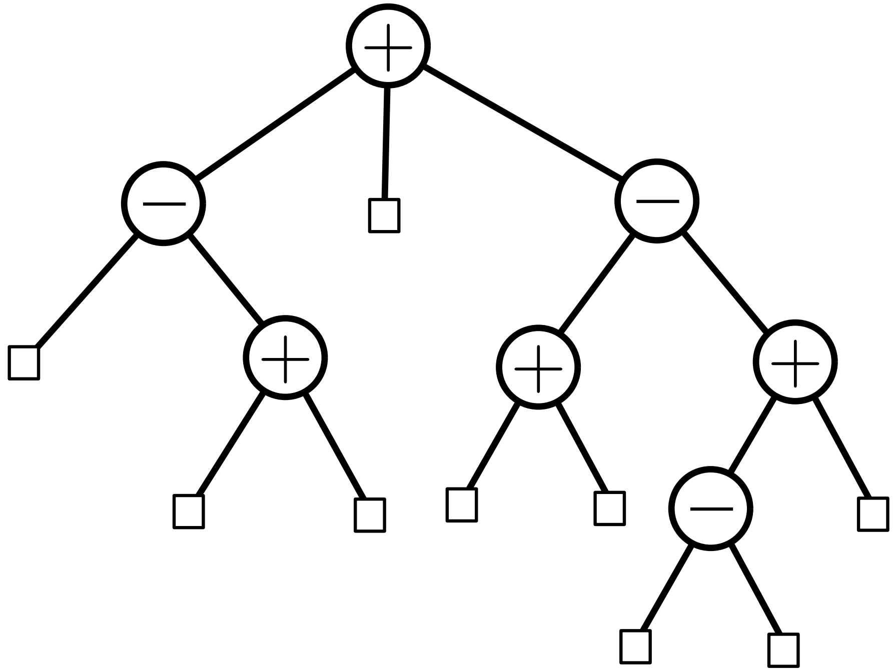
3 1 2 4 8 9 6 5 7



Separable permutations

can be recursively decomposed by  or 





Separable permutations

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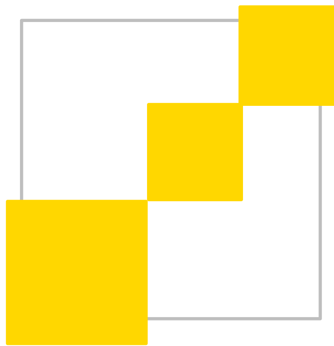
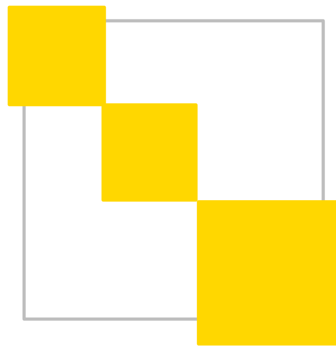
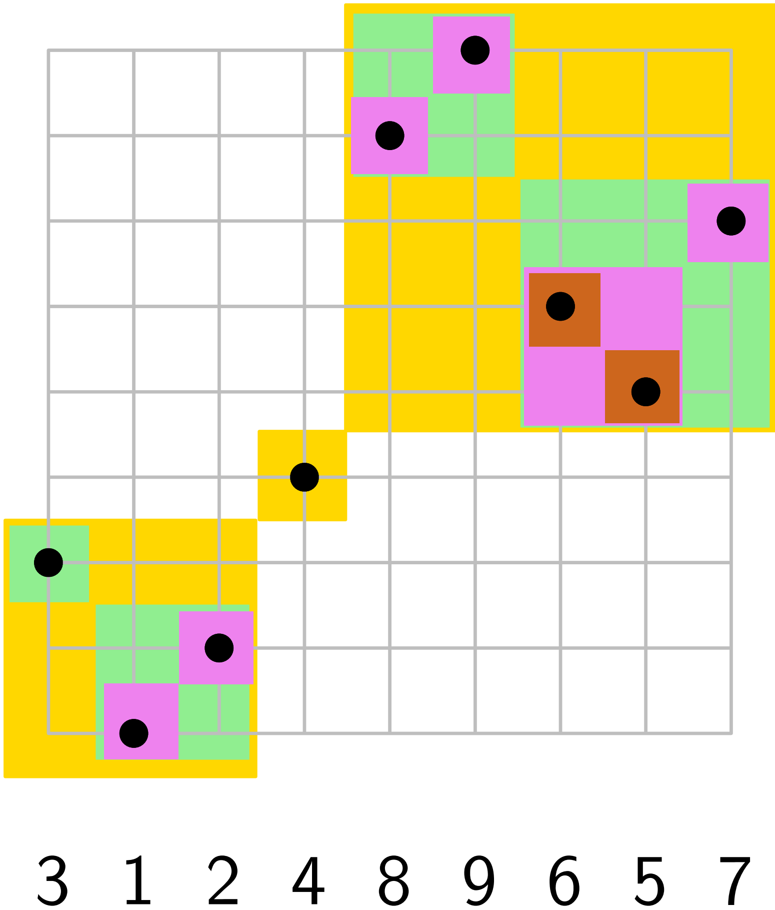
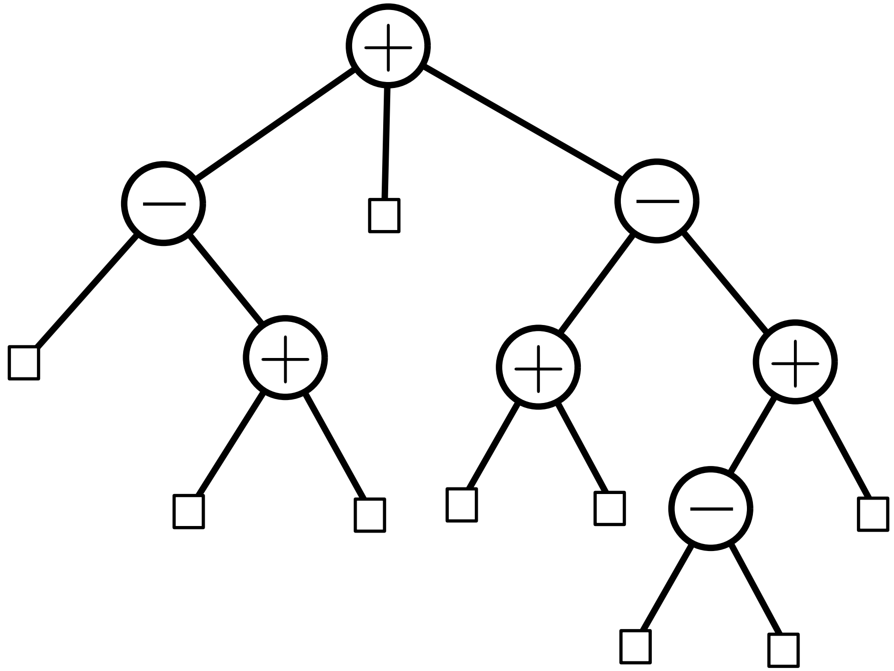


or

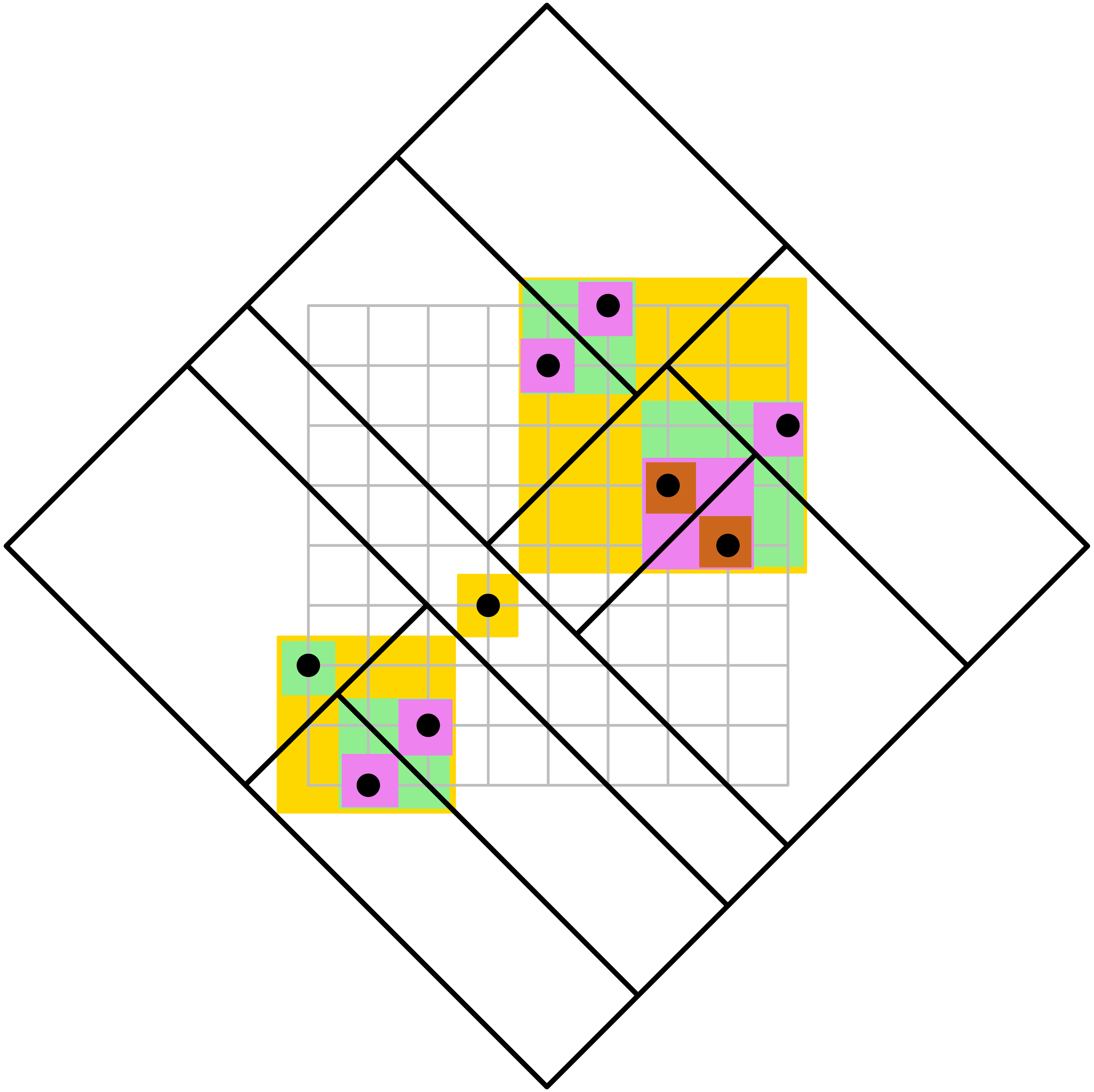






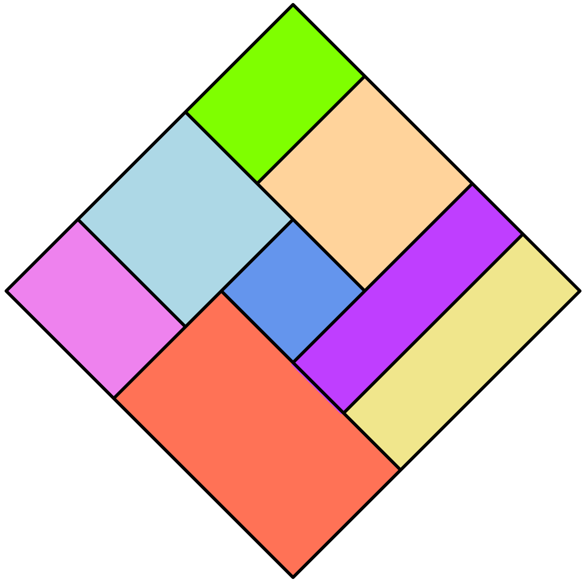
Pattern characterization: separable $\Leftrightarrow$ avoids 3142 and 2413

Bijection to guillotine rectangulations



From a weak rectangulation to a permutation

[Ackerman et al.'06]



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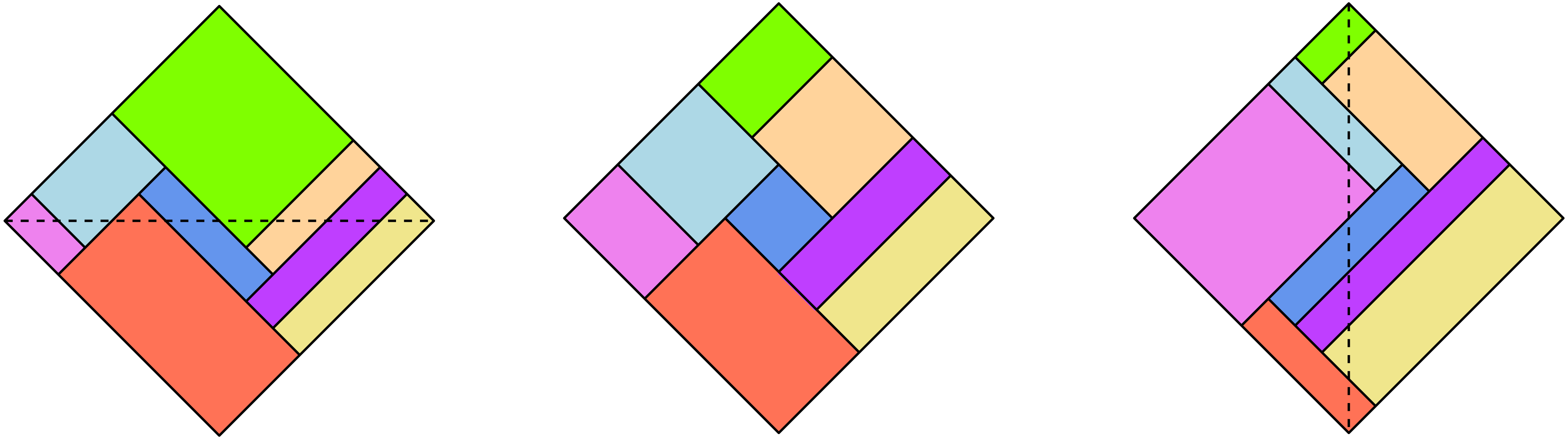
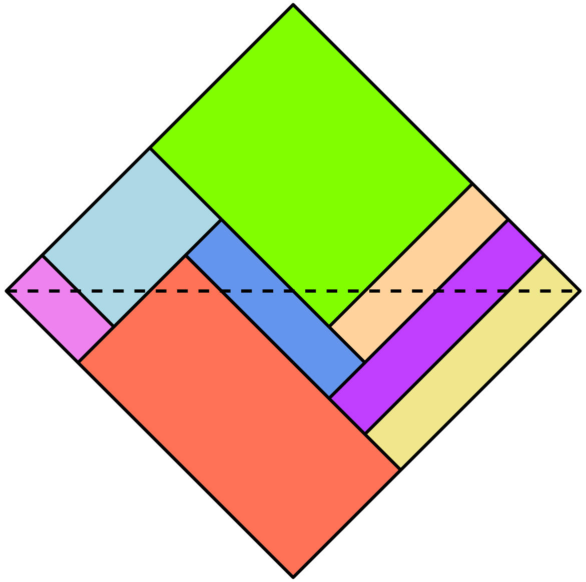
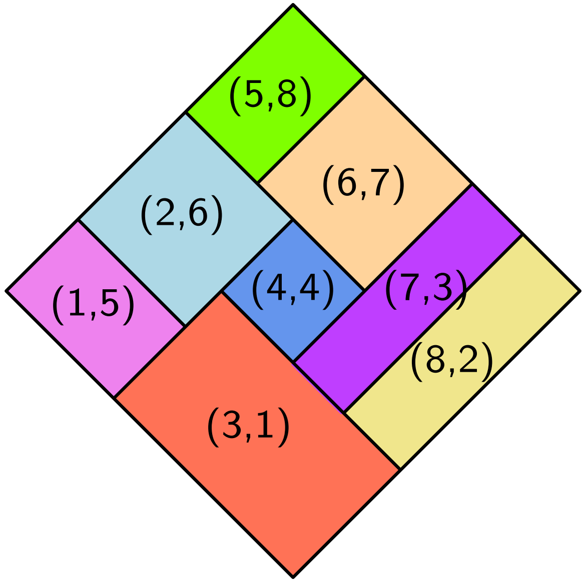


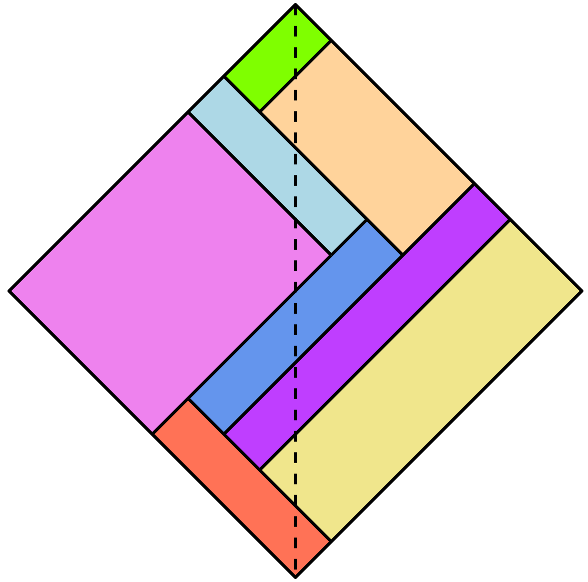
From a weak rectangulation to a permutation

[Ackerman et al.'06]

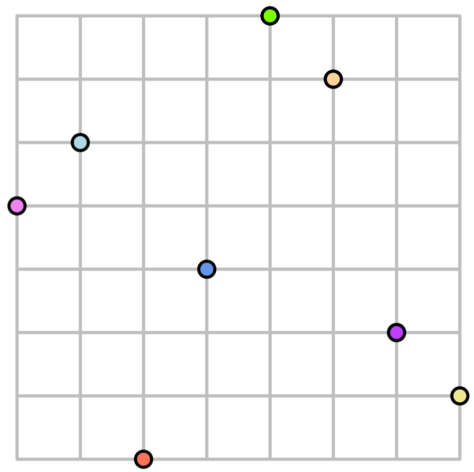


x-coordinates



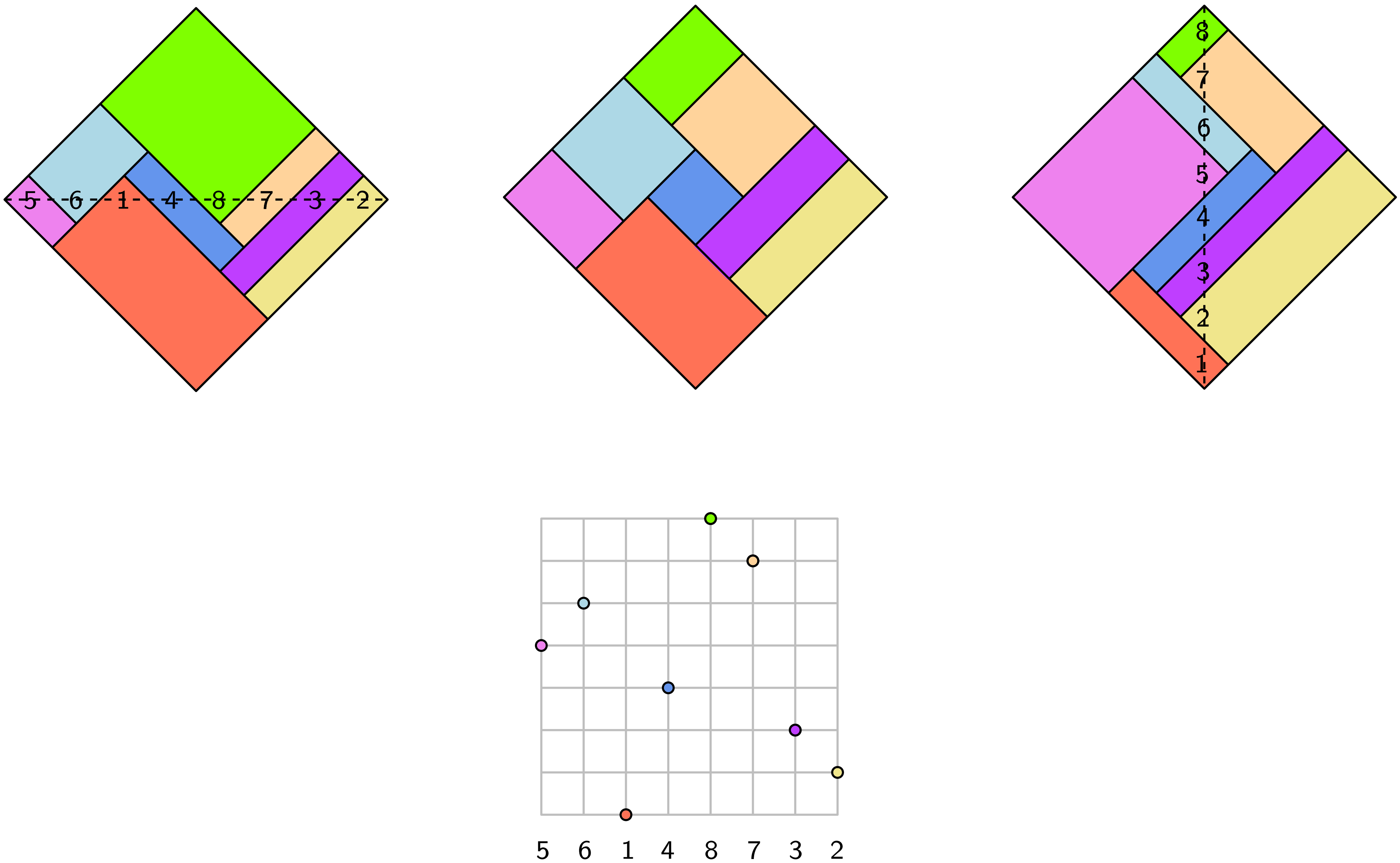


y-coordinates



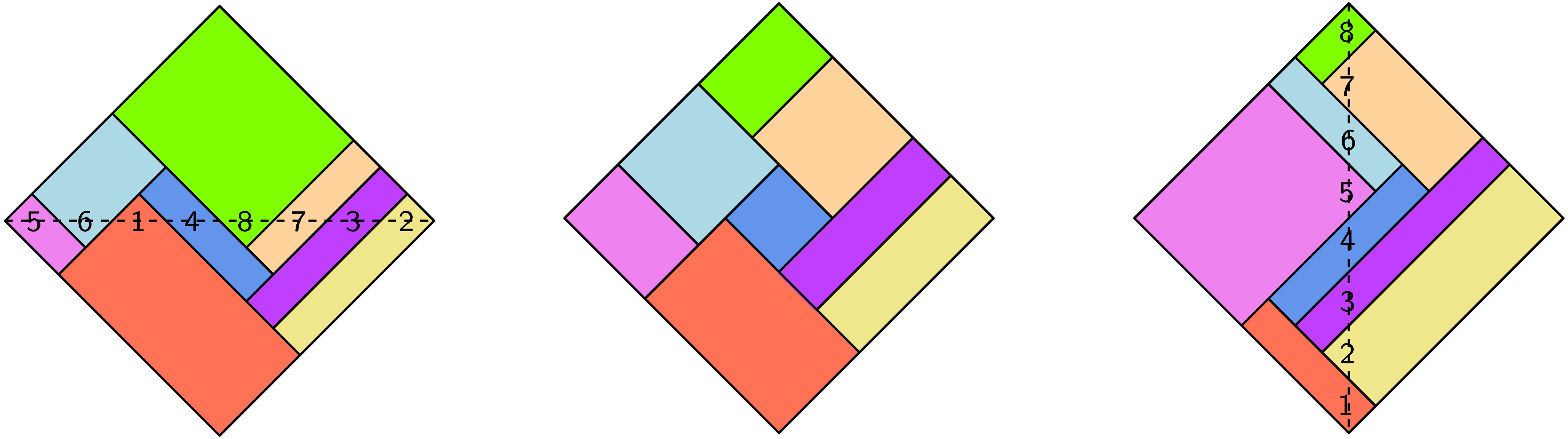
From a weak rectangulation to a permutation

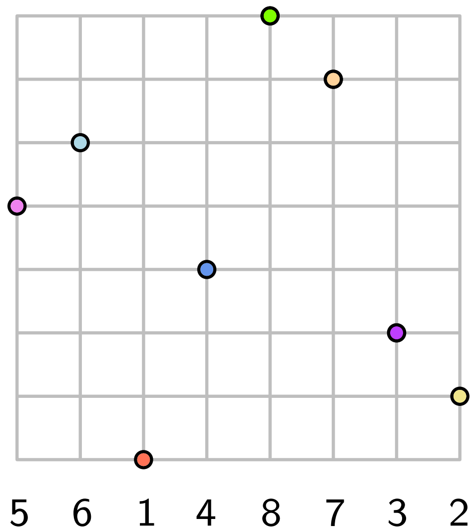
[Ackerman et al.'06]



From a weak rectangulation to a permutation

[Ackerman et al.'06]



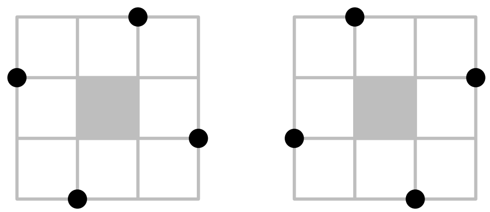


Bijection to permutations
(called Baxter permutations)

avoid $3\underbrace{14}2$ and $2\underbrace{41}3$

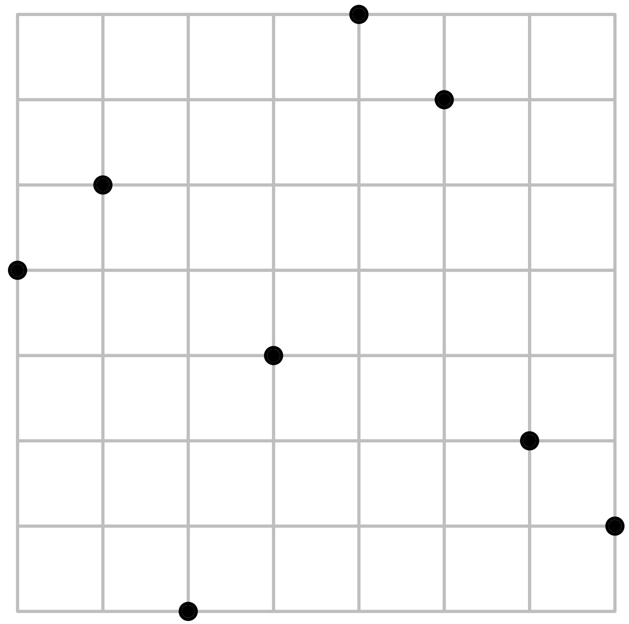
$\Updownarrow$

avoid



Inverse mapping

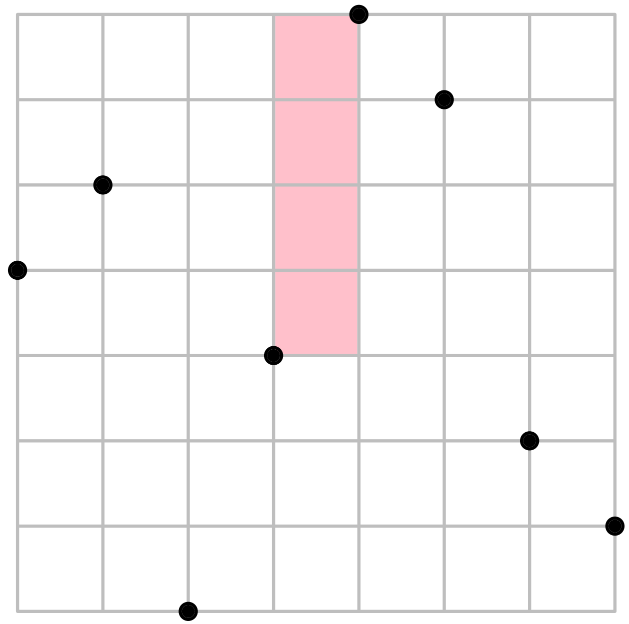
[Bonichon, Bousquet-Mélou, F'08]



Inverse mapping

[Bonichon, Bousquet-Mélou, F'08]

cross-property for ascents



Inverse mapping

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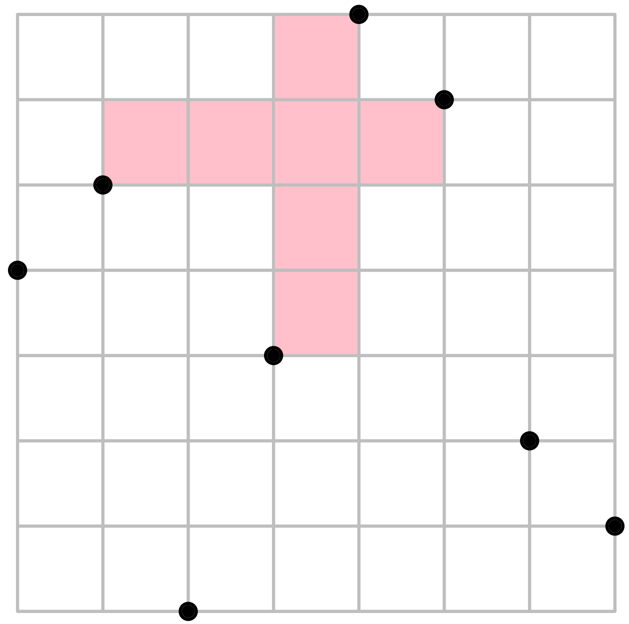
cross-property for ascents



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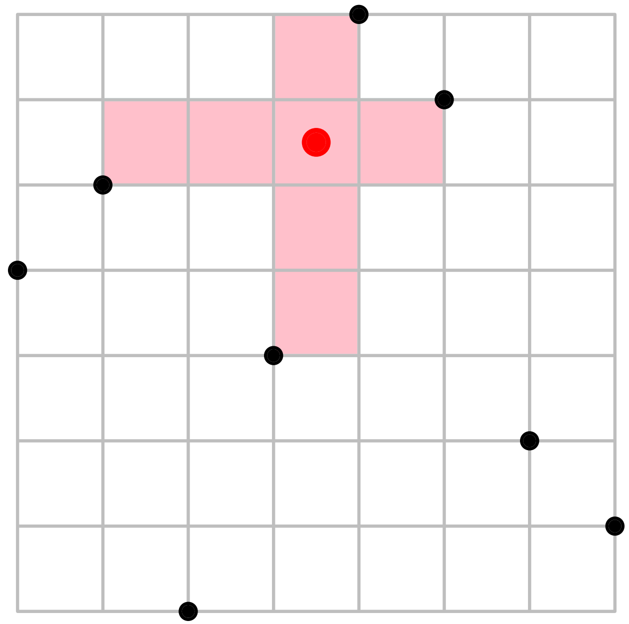
cross-property for ascents



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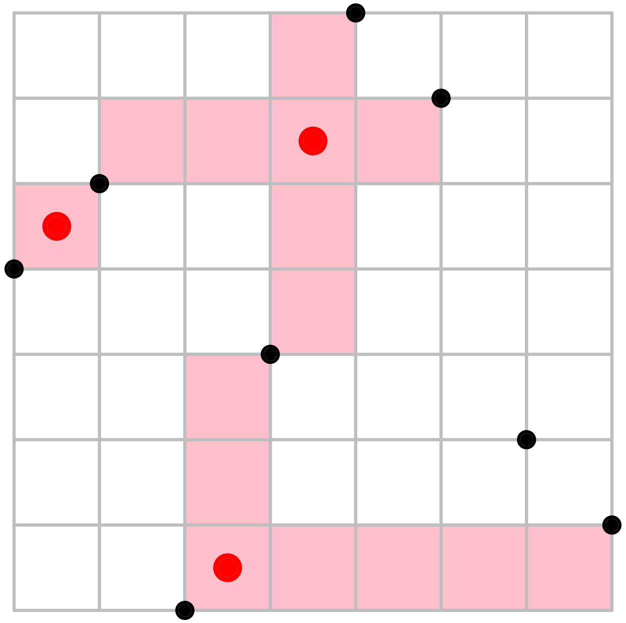
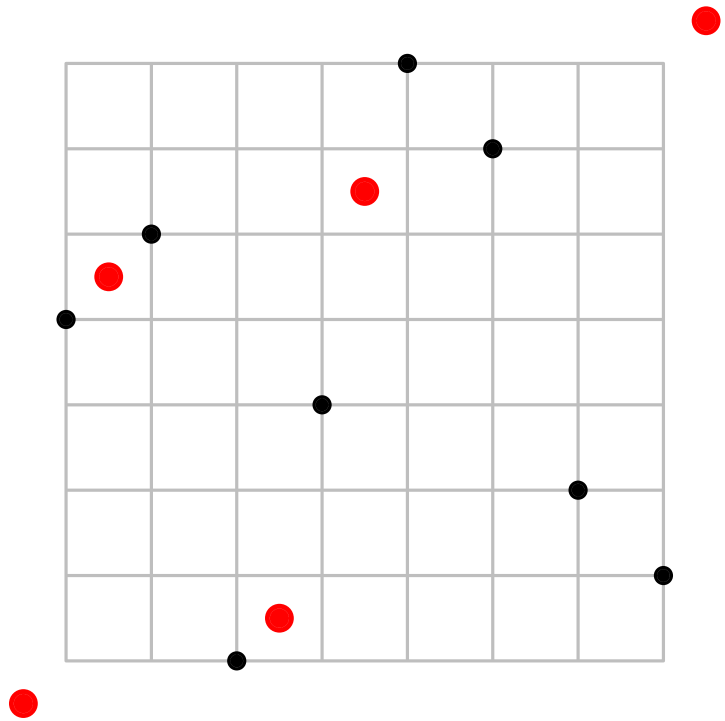


Inverse mapping

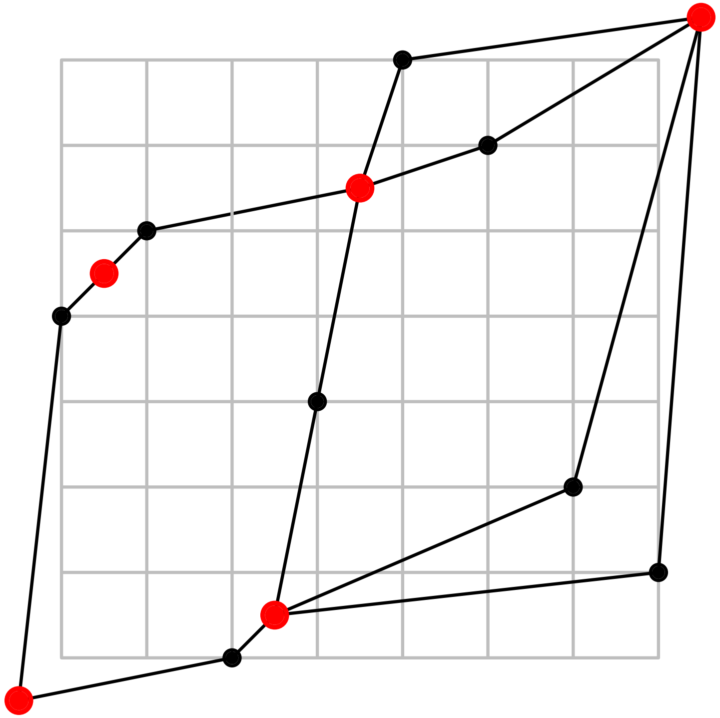
[Bonichon, Bousquet-Mélou, F'08]

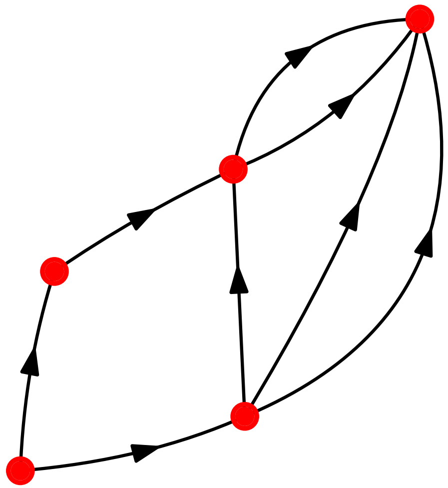


Inverse mapping

[Bonichon, Bousquet-Mélou, F'08]

dominance drawing

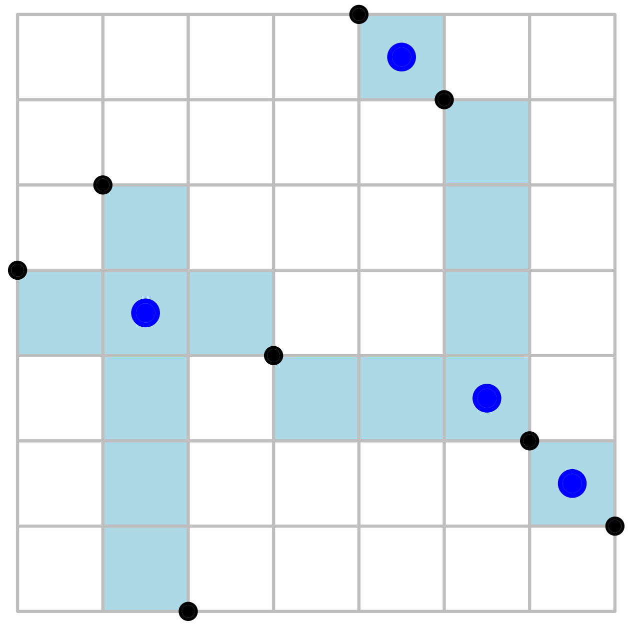




Inverse mapping

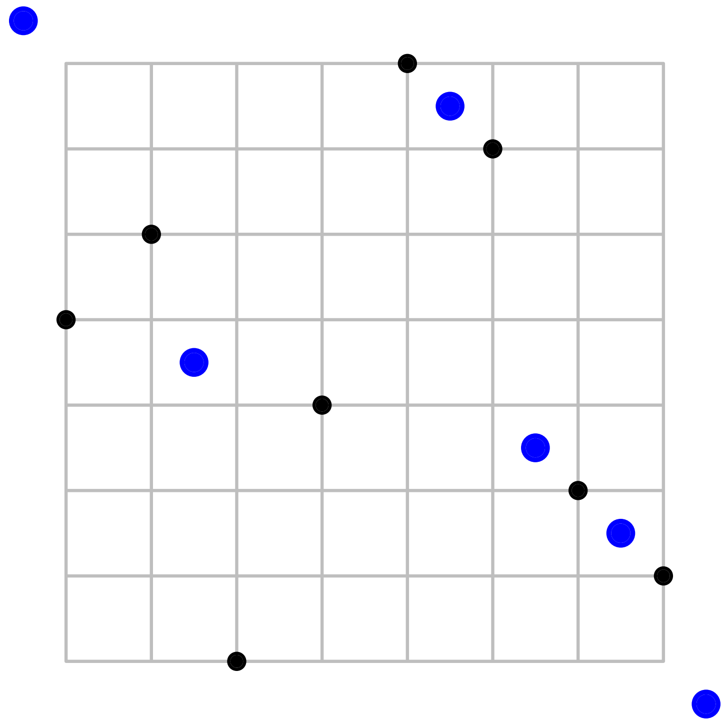
[Bonichon, Bousquet-Mélou, F'08]

cross property at descents



Inverse mapping

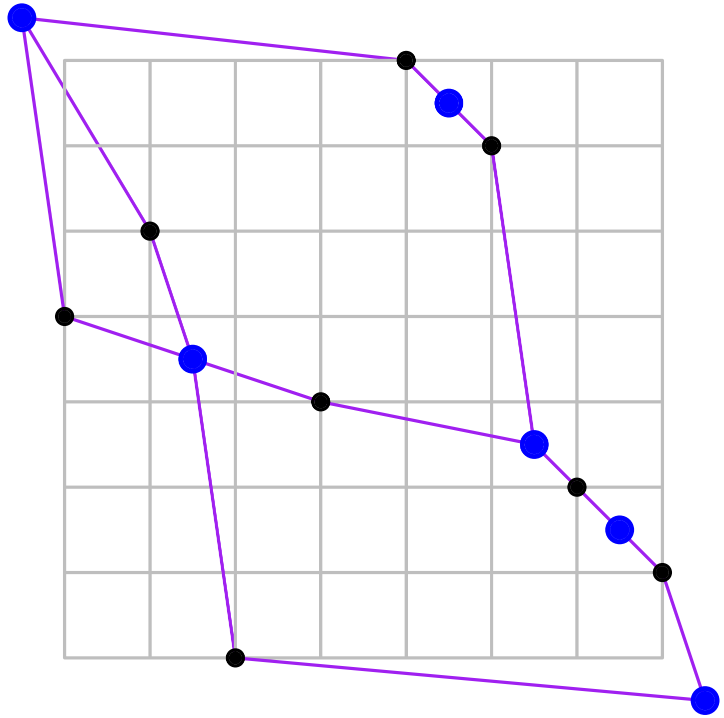
[Bonichon, Bousquet-Mélou, F'08]



Inverse mapping

[Bonichon, Bousquet-Mélou, F'08]

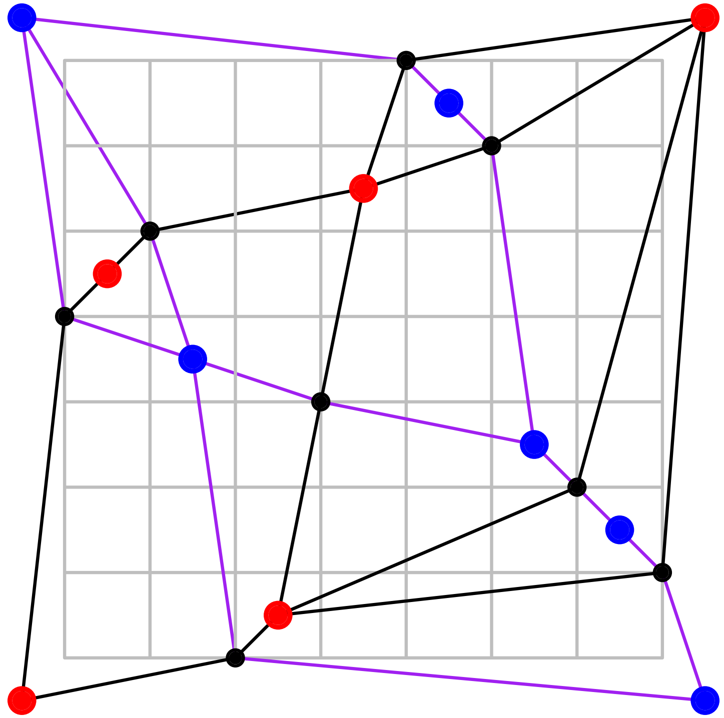
dominance drawing

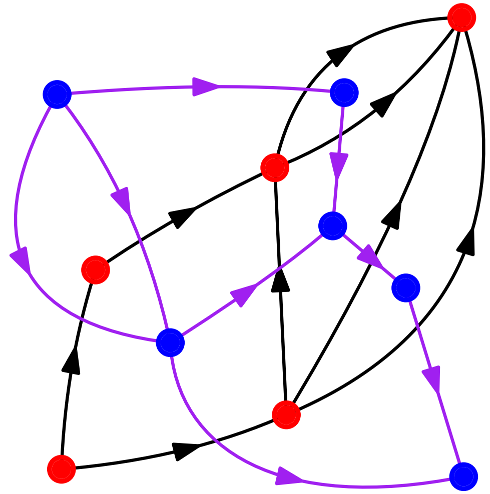


Inverse mapping

[Bonichon, Bousquet-Mélou, F'08]

dominance drawing gives dual bipolar orientation

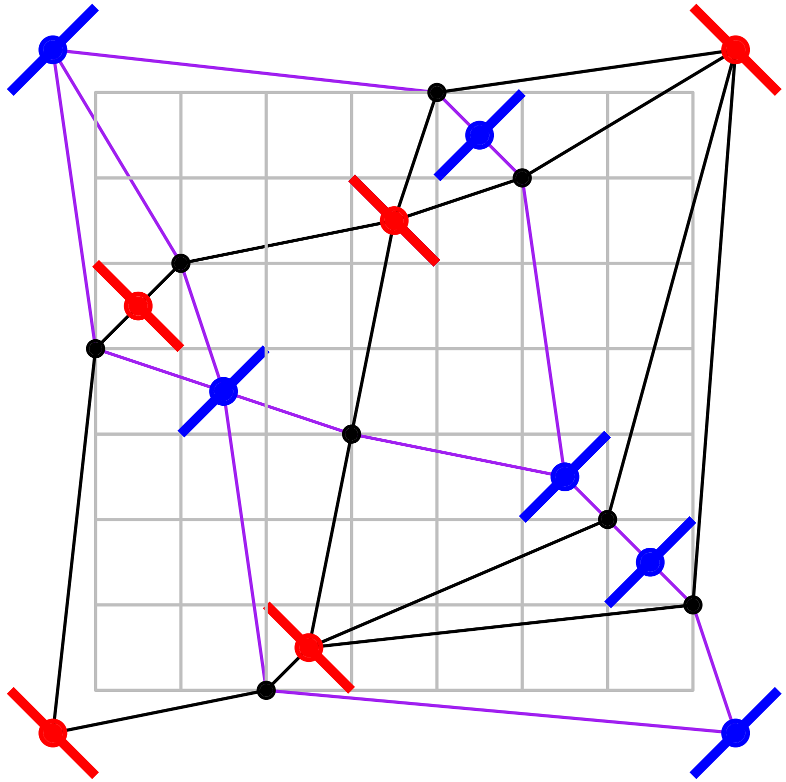


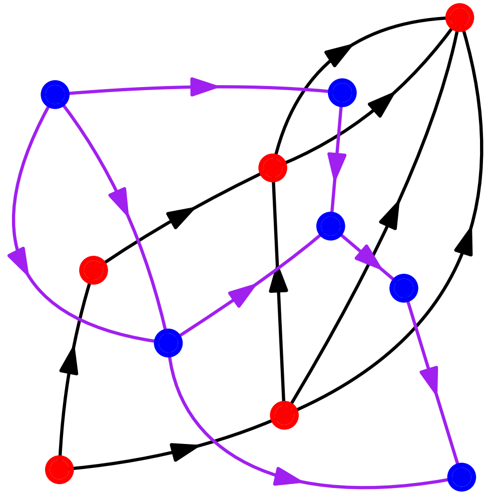


Inverse mapping

[Bonichon, Bousquet-Mélou, F'08]

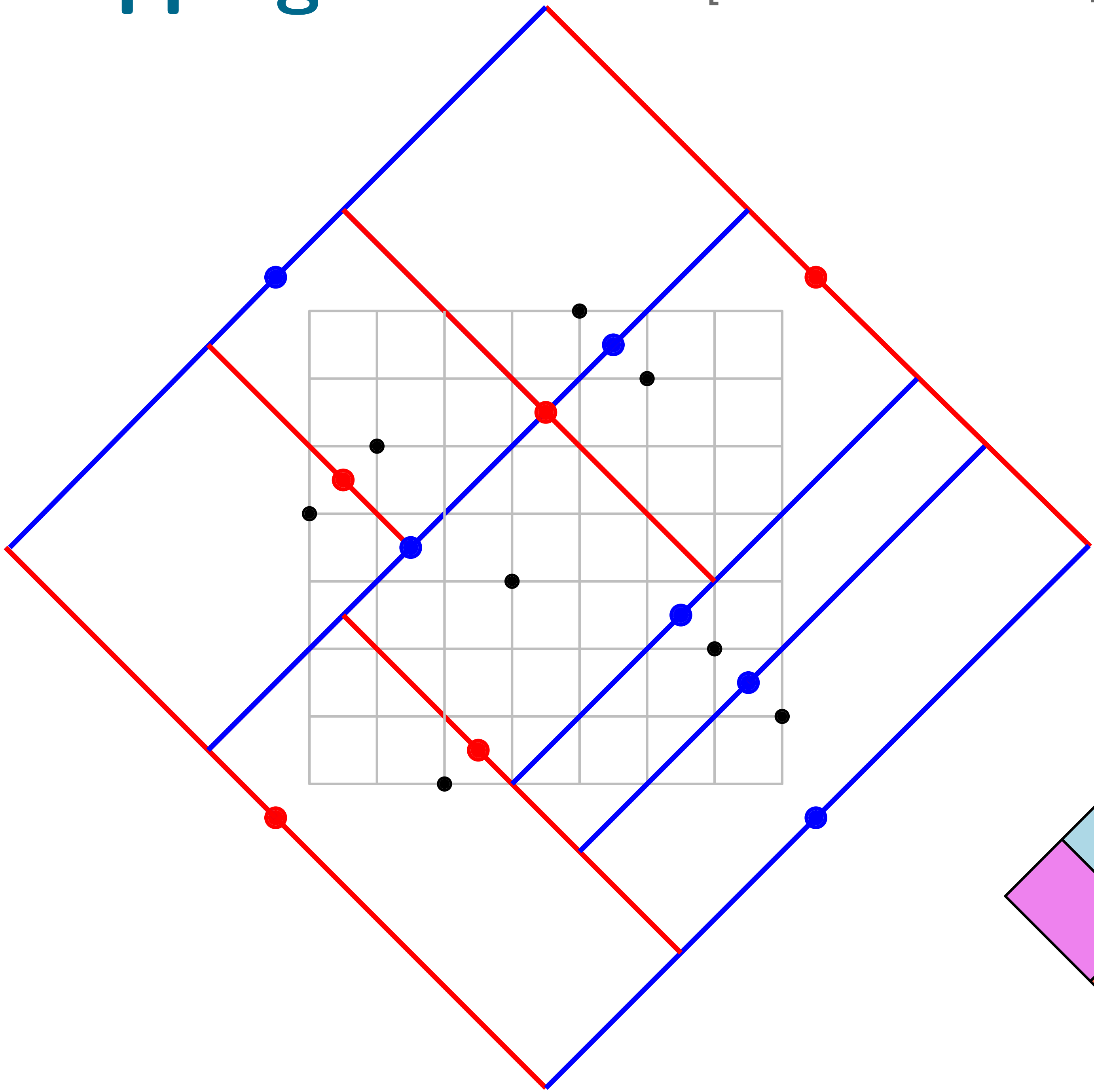
dominance drawing gives dual bipolar orientation

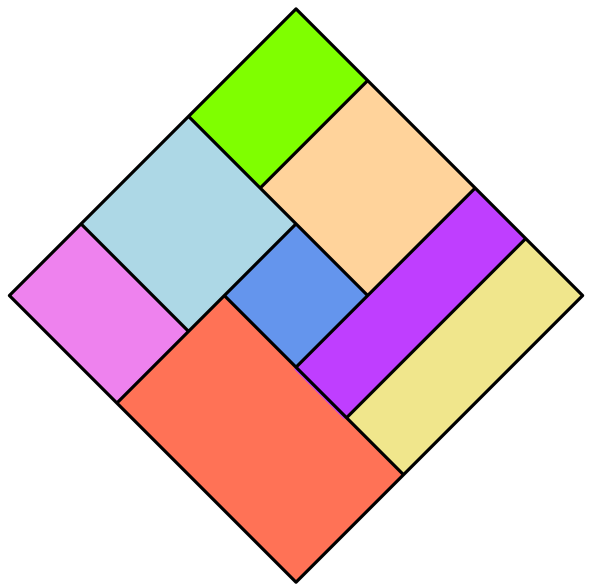




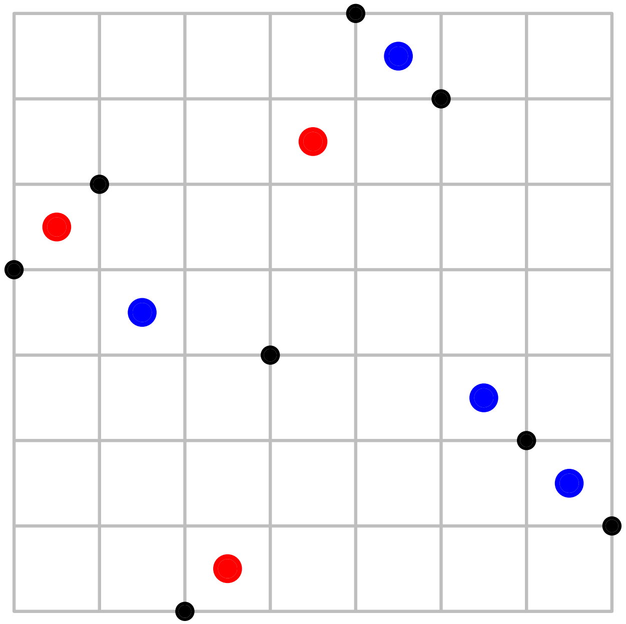
Inverse mapping

[Bonichon, Bousquet-Mélou, F'08]





Inverse mapping



Permutation on black points (size n) avoids $3\underbrace{14}2$ and $2\underbrace{41}3$ (Baxter)

Permutation on blue/red points (size n-1) avoids $2\underbrace{14}3$ and $3\underbrace{41}2$ (S-permutations)

[Asinowski, Barequet, Bousquet-Mélou, Mansour, Pinter'10]

Baxter-like families

[Bouvel, Guerrini, Rinaldi'19]

	$2\underbrace{14}3$	$3\underbrace{14}2$	$2\underbrace{41}3$	$3\underbrace{41}2$
Baxter		✓	✓	
twisted Baxter			✓	✓
S-permutations	✓			✓
plane	✓			
semi-Baxter			✓	
strong-Baxter		✓	✓	✓
twisted S-perm.	✓		✓	✓
fully Baxter	✓	✓	✓	✓

Baxter-like families

[Bouvel, Guerrini, Rinaldi'19]

		2143	3142	2413	3412	growth rate
①	Baxter		✓	✓		8
②	twisted Baxter			✓	✓	
③	S-permutations	✓			✓	$4 + 2\sqrt{2}$ ≈ 6.82
	plane	✓				
	semi-Baxter			✓		
	strong-Baxter		✓	✓	✓	
	twisted S-perm.	✓		✓	✓	
	fully Baxter	✓	✓	✓	✓	

a: Baxter ↔ twisted Baxter

D-finite/not D-finite

① [Chung et al.78], [Mallows'79], [Viennot,81], [Dulucq-Guibert'98], [Bousquet-Mélou'02],. . .

② [Reading'05,][West,06]

bijection for a in [Law, Reading,10] and [Giraudo'10]

③ [Asinowski, Barequet, Bousquet-Mélou, Mansour, Pinter'10]

Baxter-like families

[Bouvel, Guerrini, Rinaldi'19]

		2143	3142	2413	3412	growth rate
①	Baxter		✓	✓		8
②	twisted Baxter			✓	✓	8
③	S-permutations	✓			✓	$4+2\sqrt{2}$ ≈ 6.82
④	plane	✓				$\frac{1}{2}(11+5\sqrt{5})$ ≈ 11.09
⑤	semi-Baxter			✓		$\frac{1}{2}(11+5\sqrt{5})$ ≈ 11.09
	strong-Baxter		✓	✓	✓	
	twisted S-perm.	✓		✓	✓	
	fully Baxter	✓	✓	✓	✓	

a: ① ↔ ②; b: ④ ↔ ⑤

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① [Chung et al.78], [Mallows'79], [Viennot,81], [Dulucq-Guibert'98], [Bousquet-Mélou'02],. . .

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③ [Asinowski, Barequet, Bousquet-Mélou, Mansour, Pinter'10]

④ [Bousquet-Mélou, Butler'07]

④ ⑤ [Bouvel, Guerrini, Rechnitzer, Rinaldi'18] bijections for b also in [Law, Reading,10], [Kasraoui'13]

Baxter-like families

[Bouvel, Guerrini, Rinaldi'19]

		2143	3142	2413	3412	growth rate	
①	Baxter		✓	✓		8	
②	twisted Baxter			✓	✓		
③	S-permutations	✓			✓	$4+2\sqrt{2}$	≈ 6.82
④	plane	✓				$\frac{1}{2}(11+5\sqrt{5})$	≈ 11.09
⑤	semi-Baxter			✓			
⑥	strong-Baxter		✓	✓	✓	Root of $x^3-5x^2-10x-11$	≈ 6.72
⑦	twisted S-perm.	✓		✓	✓	?	
	fully Baxter	✓	✓	✓	✓		≈ 5.56

a: ① ↔ ②; b: ④ ↔ ⑤

D-finite/not D-finite

① [Chung et al.78], [Mallows'79], [Viennot,81], [Dulucq-Guibert'98], [Bousquet-Mélou'02],...

② [Reading'05,][West,06] bijection for a in [Law, Reading,10] and [Giraudo'10]

③ [Asinowski, Barequet, Bousquet-Mélou, Mansour, Pinter'10]

④ [Bousquet-Mélou, Butler'07]

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③ ⑦ [Bouvel, Guerrini, Rinaldi'19]

Baxter-like families

[Bouvel, Guerrini, Rinaldi'19]

		$2\underline{14}3$	$3\underline{14}2$	$2\underline{41}3$	$3\underline{41}2$	growth rate	
①	Baxter		✓	✓		8	
②	twisted Baxter			✓	✓	8	
③	S-permutations	✓			✓	$4+2\sqrt{2}$	≈ 6.82
④	plane	✓				$\frac{1}{2}(11+5\sqrt{5})$	≈ 11.09
⑤	semi-Baxter			✓		$\frac{1}{2}(11+5\sqrt{5})$	≈ 11.09
⑥	strong-Baxter		✓	✓	✓	Root of $x^3-5x^2-10x-11$	≈ 6.72
⑦	twisted S-perm.	✓		✓	✓	?	
⑧	fully Baxter	✓	✓	✓	✓	$\frac{1}{2}(7+\sqrt{17})$	≈ 5.56

a: ① ↔ ②

b: ④ ↔ ⑤

D-finite/not D-finite

① [Chung et al.78], [Mallows'79], [Viennot,81], [Dulucq-Guibert'98], [Bousquet-Mélou'02],. . .

② [Reading'05,][West,06] bijection for a in [Law, Reading,10] and [Giraudo'10]

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④ [Bousquet-Mélou, Butler'07]

④ ⑤ ⑥ [Bouvel, Guerrini, Rechnitzer, Rinaldi'18] bijections for b also in [Law, Reading,10], [Kasraoui'13]

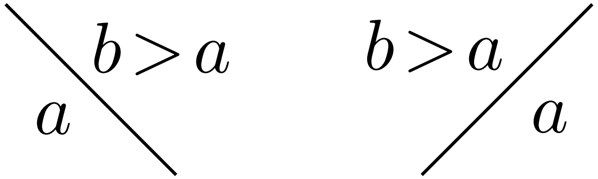
③ ⑦ ⑧ [Bouvel, Guerrini, Rinaldi'19]

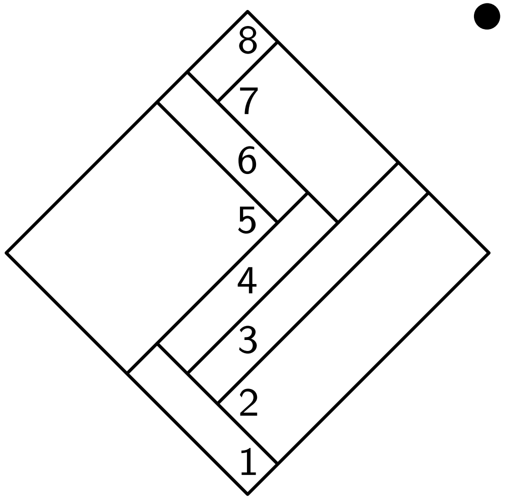
⑧ [Asinowski, Cardinal, Felsner, F'25]

Inverse mapping (2nd way)

[Dulucq-Guibert'96]

Rk: labeling is increasing from bottom to top





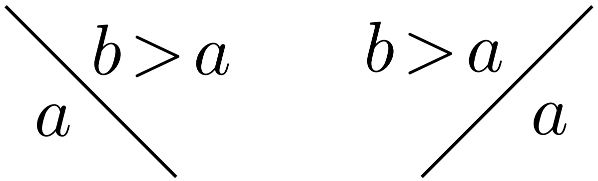
• in diagonal representative
(unique increasing labeling)

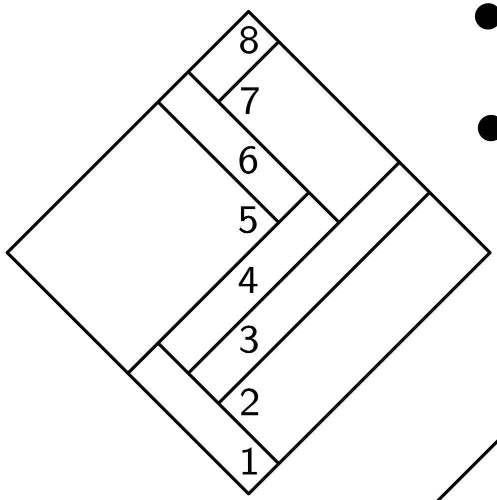
Inverse mapping (2nd way)

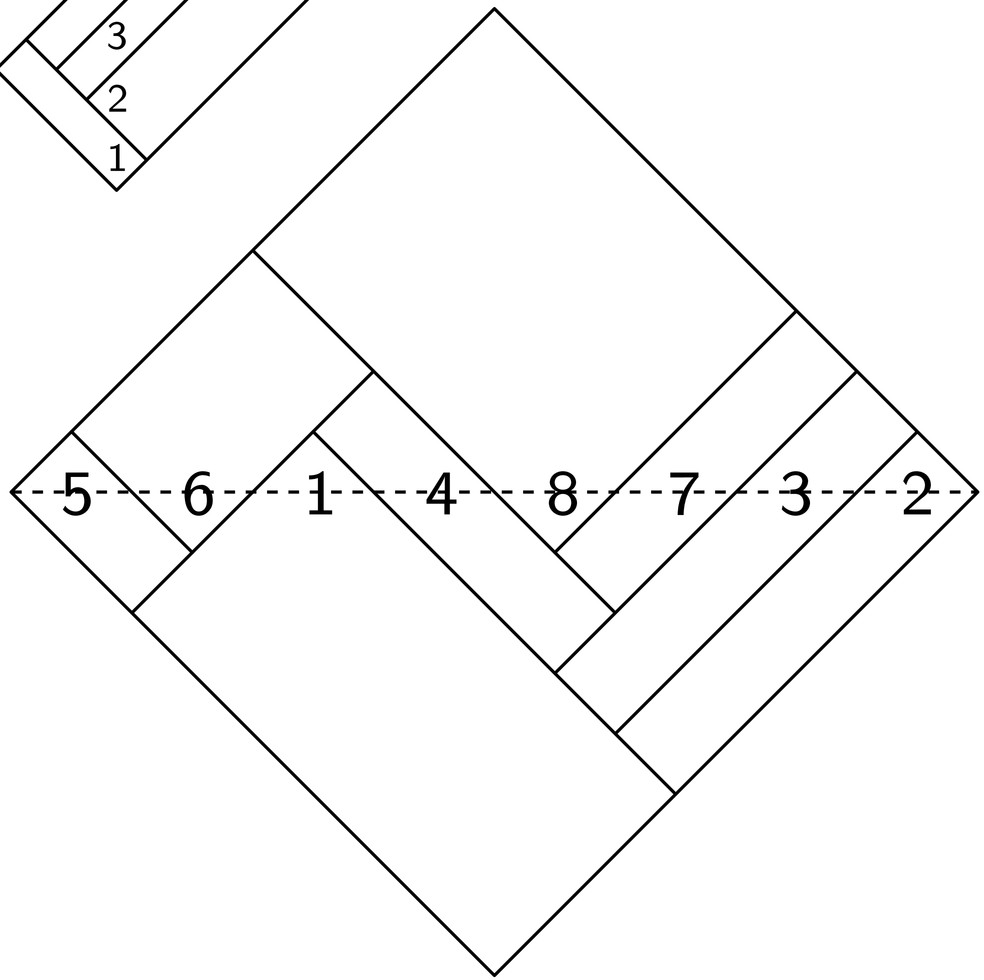
[Dulucq-Guibert'96]

Rk: labeling is increasing from bottom to top

- in diagonal representative
  (unique increasing labeling)
- and also in anti-diagonal representative
  (not unique, less demanding)



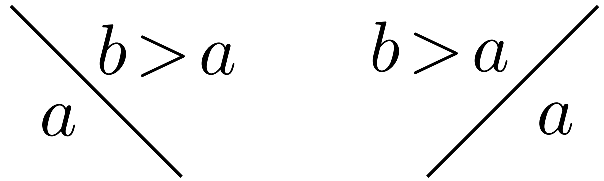




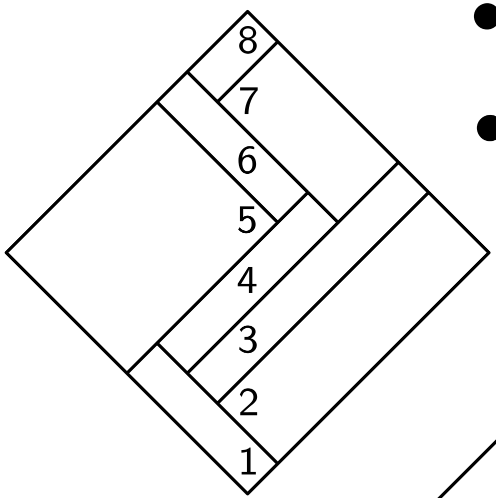
Inverse mapping (2nd way)

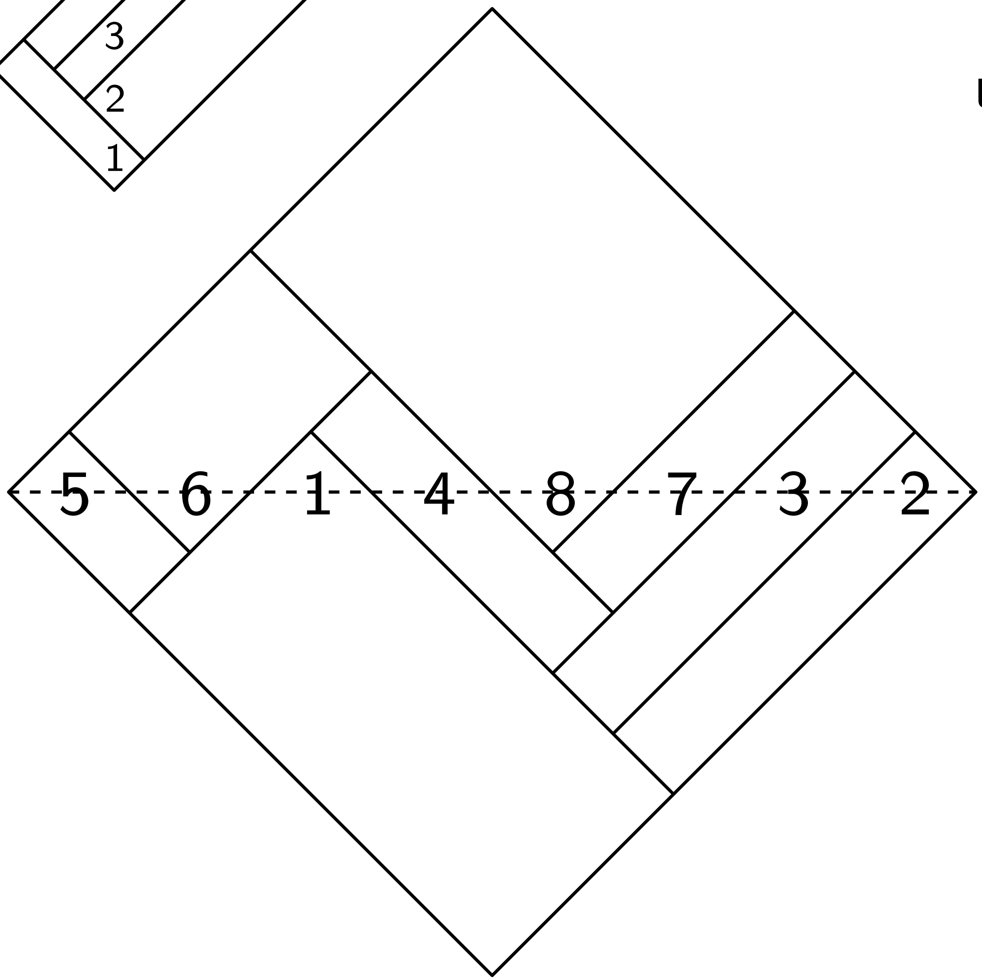
[Dulucq-Guibert'96]

Rk: labeling is increasing from bottom to top

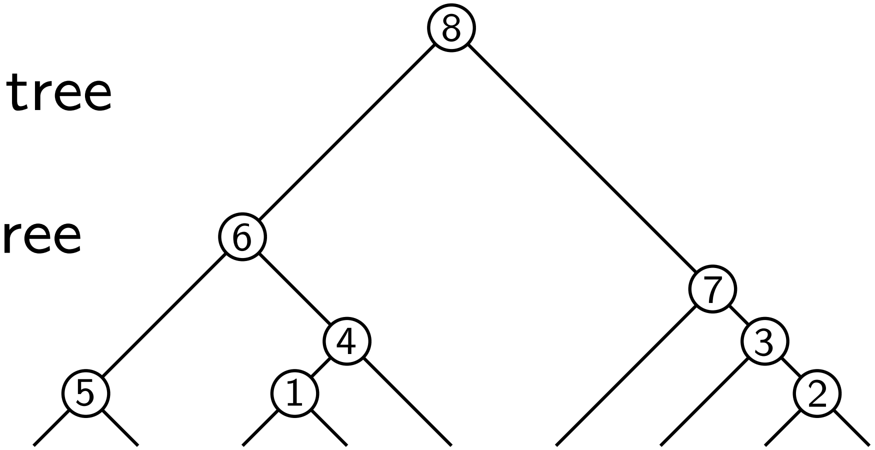


- in diagonal representative
  (unique increasing labeling)
- and also in anti-diagonal representative
  (not unique, less demanding)





upper tree
||
max tree



Inverse mapping (2nd way)

[Dulucq-Guibert'96]

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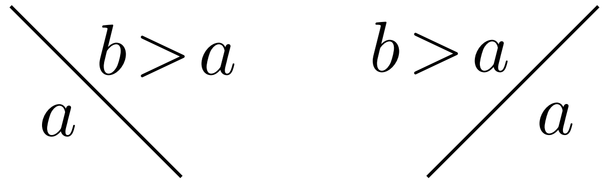

- in diagonal representative
  (unique increasing labeling)
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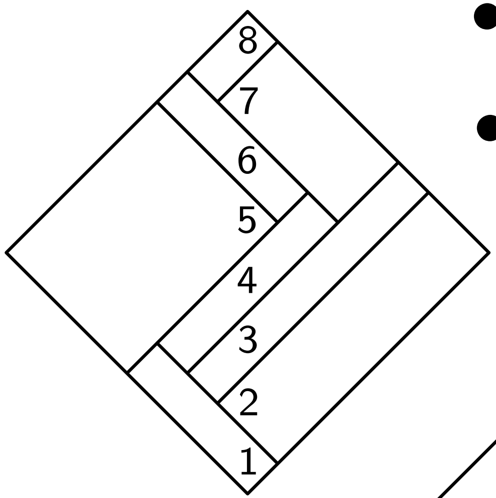
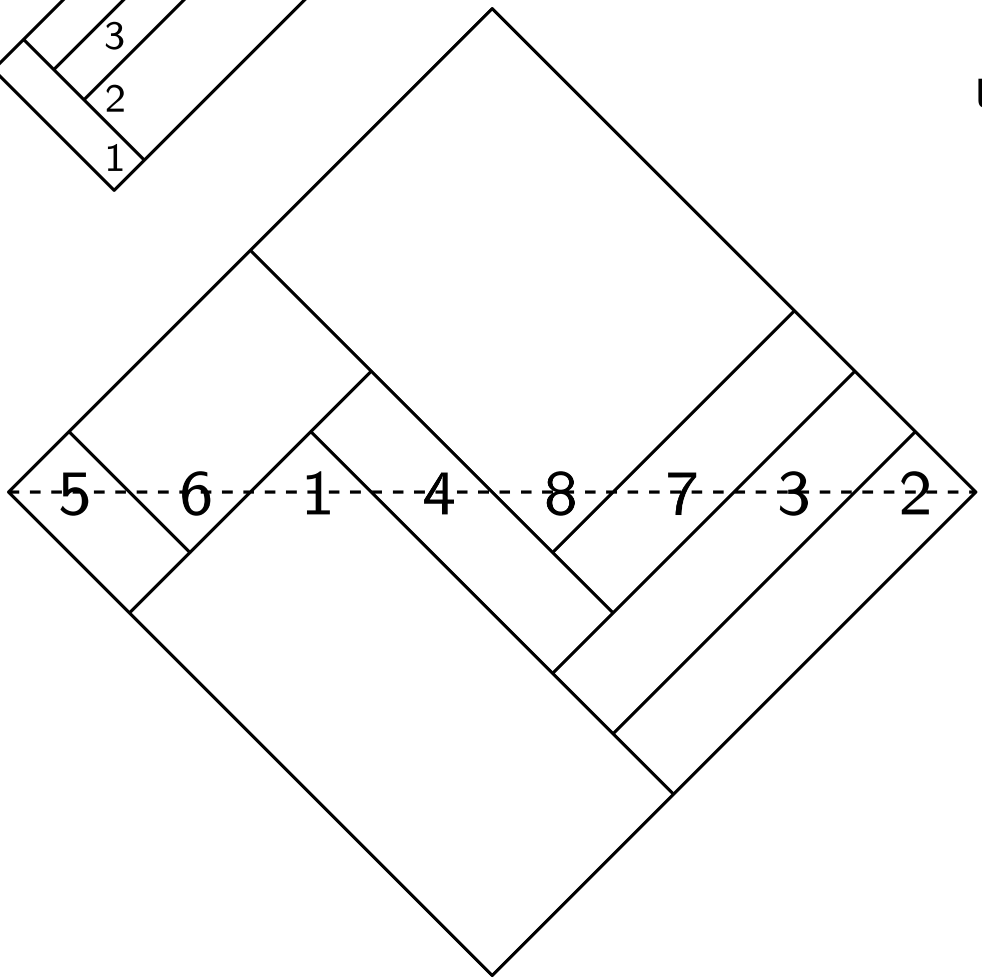
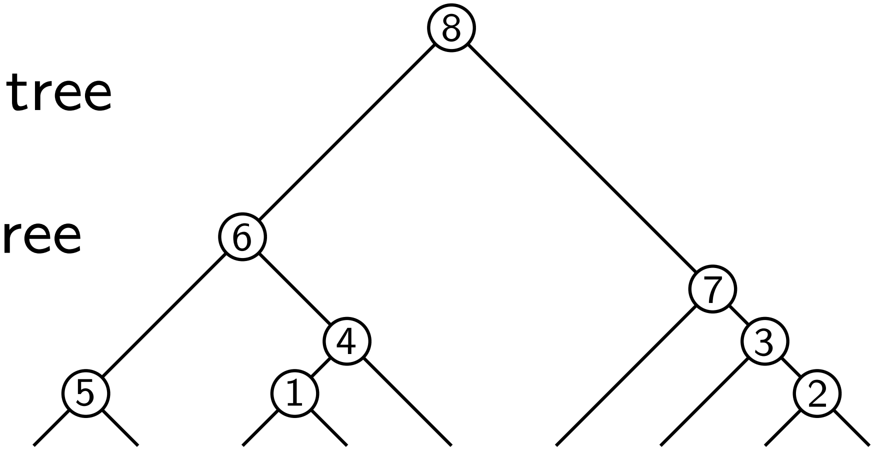
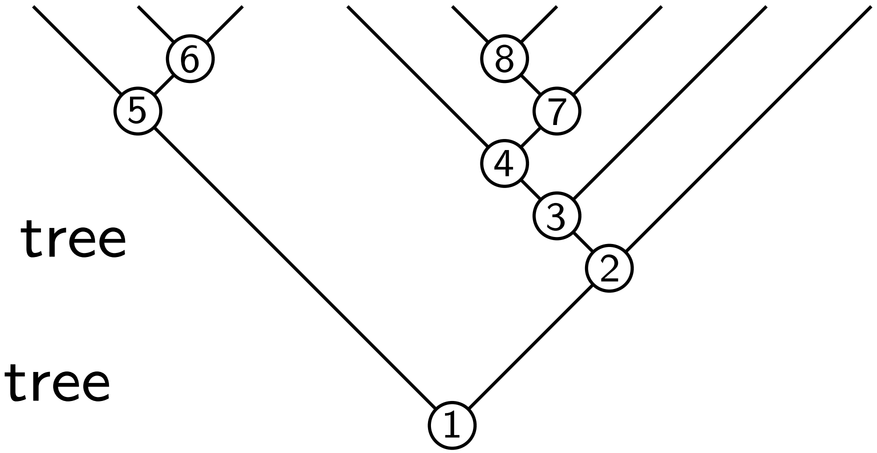




upper tree
||
max tree

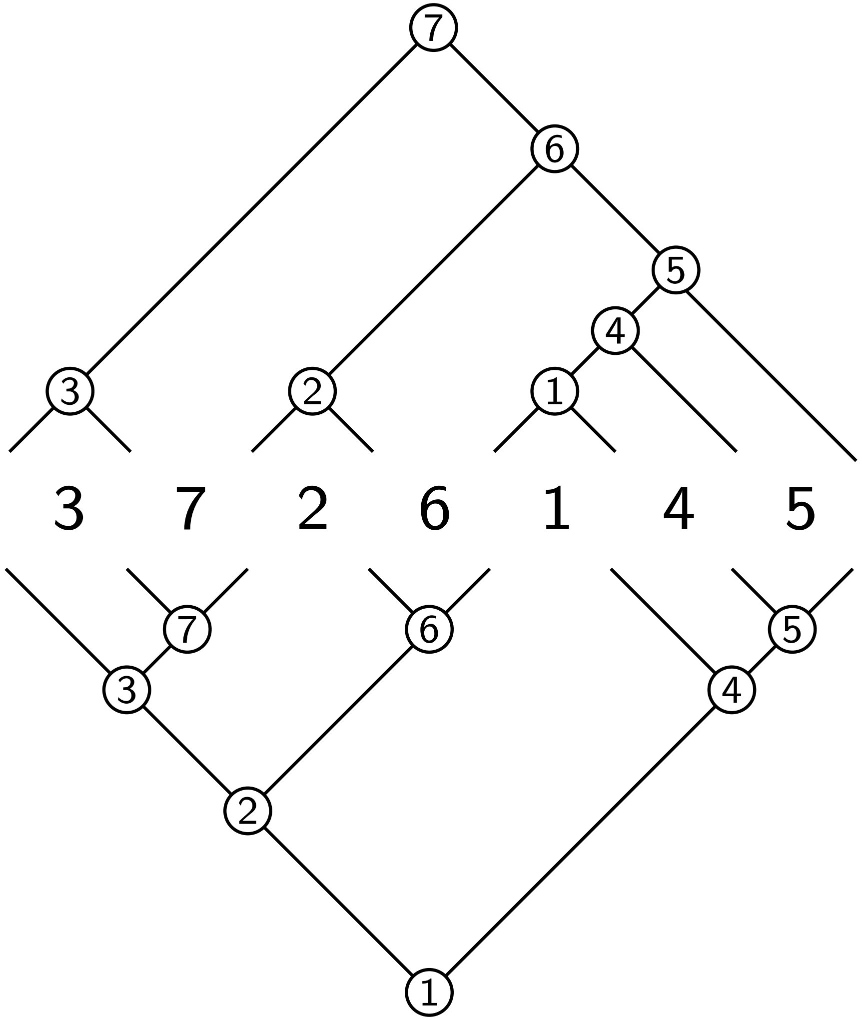


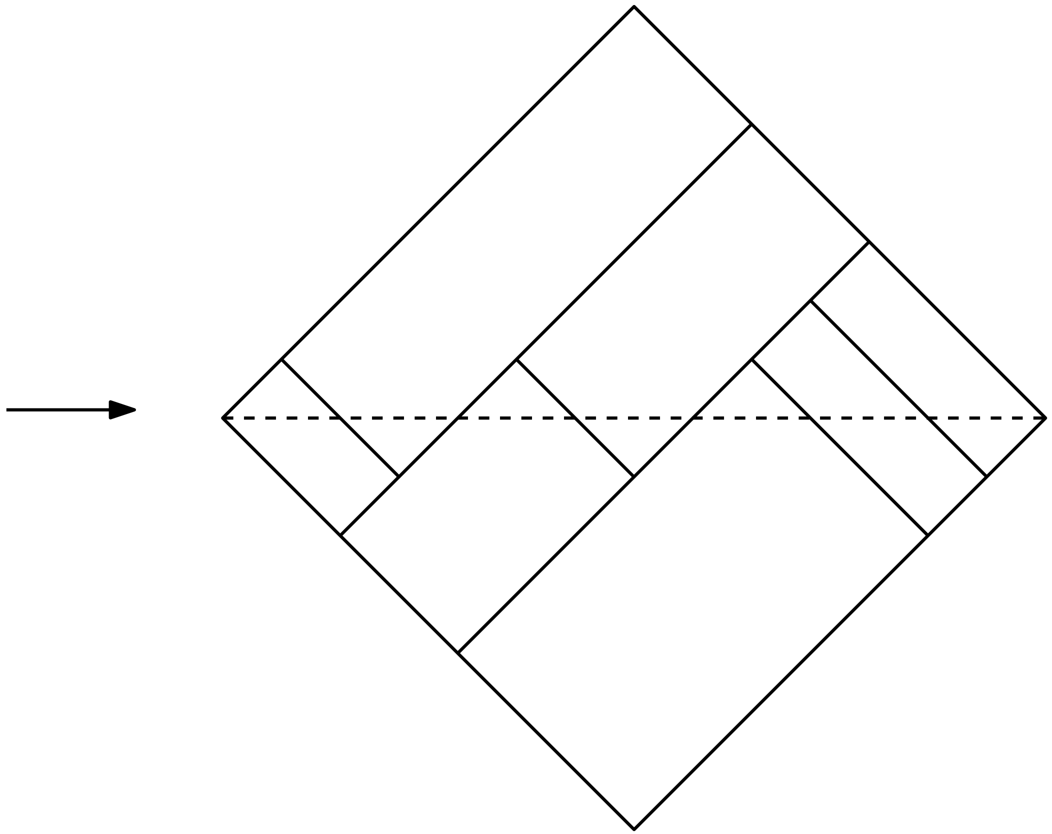


lower tree
||
min tree

Extension to all permutations

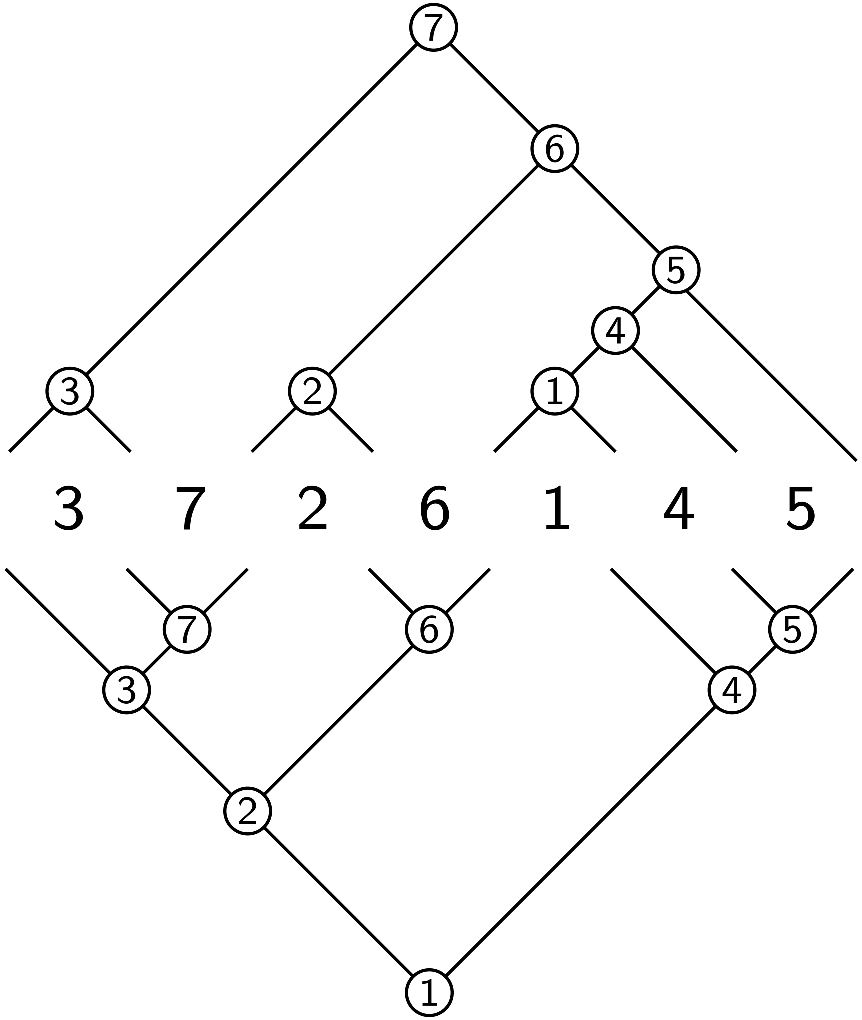
$\sigma \in \mathfrak{S}_n$ ⟶ weak rectangulation formed by max-tree(σ) min-tree(σ)

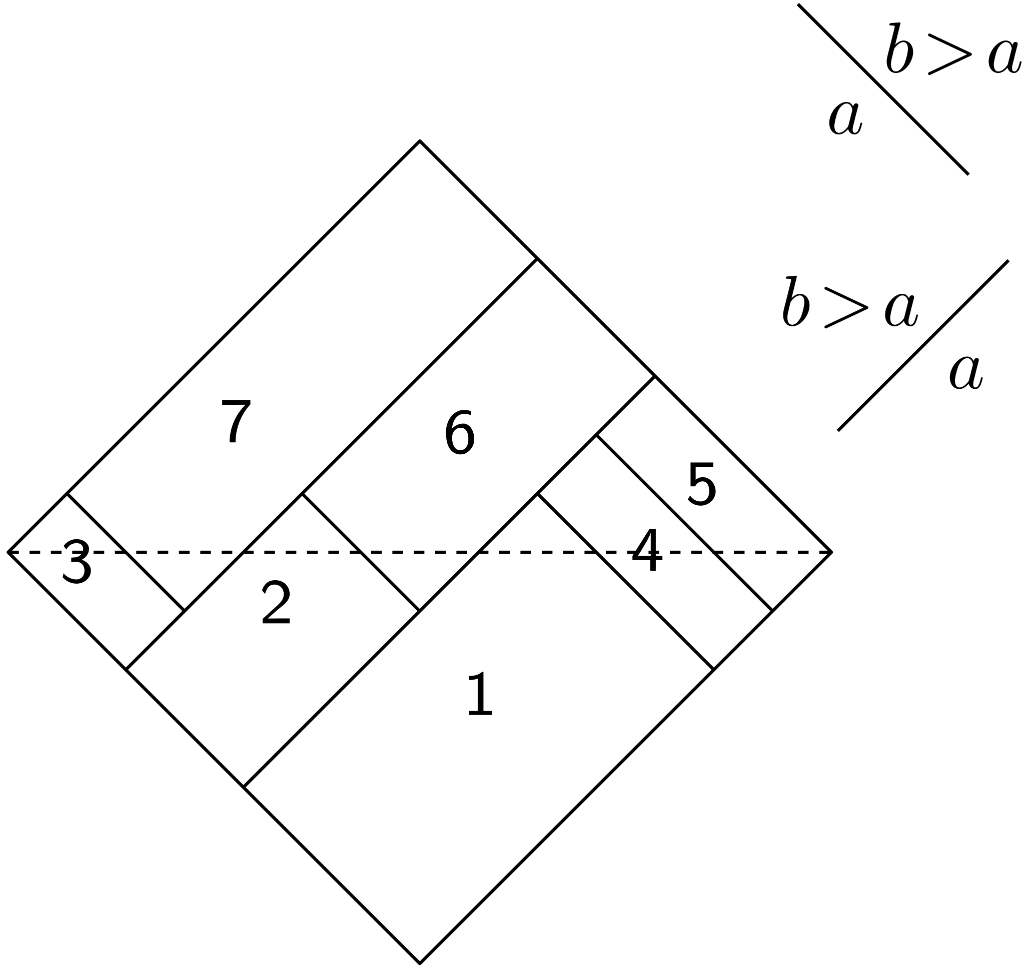




Extension to all permutations

$\sigma \in \mathfrak{S}_n \longrightarrow$ weak rectangulation formed by max-tree(σ) min-tree(σ)

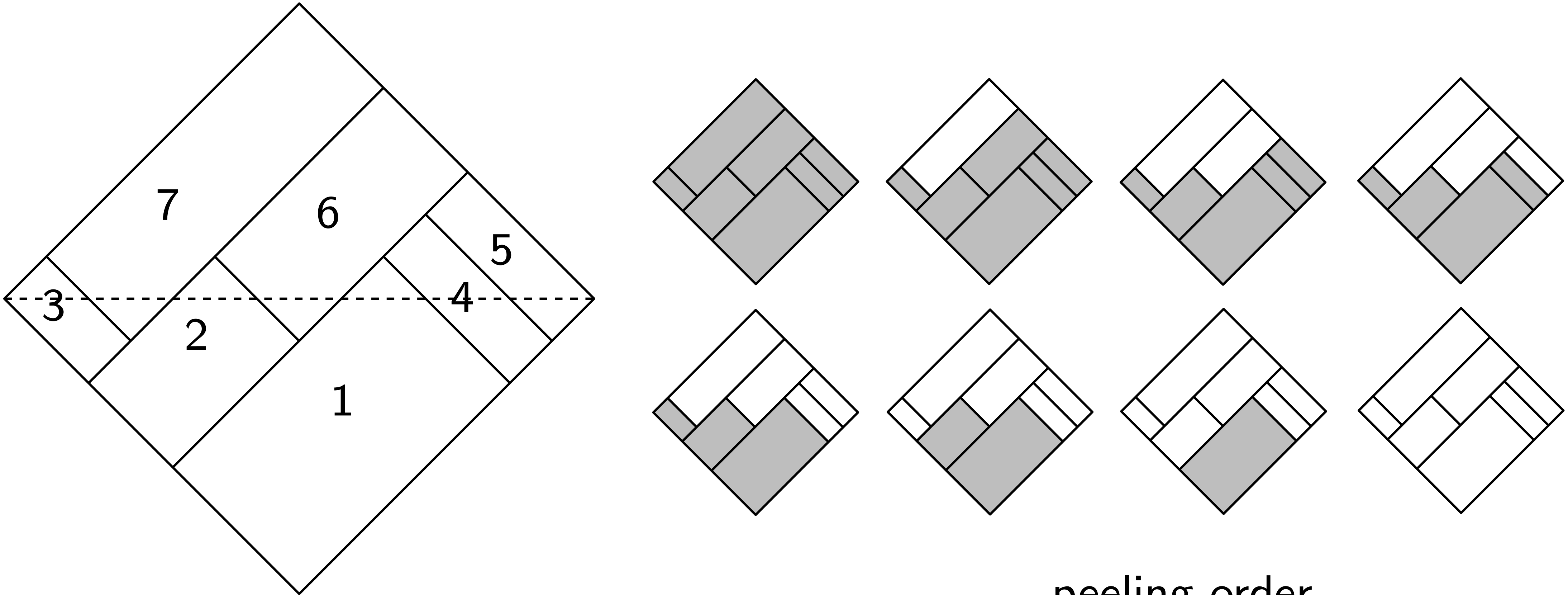




Bijection from $\mathfrak{S}_n$ to weak rectangulations + increasing labeling of regions
(in anti-diagonal representation)

Extension to all permutations

$\sigma \in \mathfrak{S}_n \longrightarrow$ weak rectangulation formed by $\begin{matrix}\text{max-tree}(\sigma) \\ \text{min-tree}(\sigma)\end{matrix}$



peeling order

$\Updownarrow$

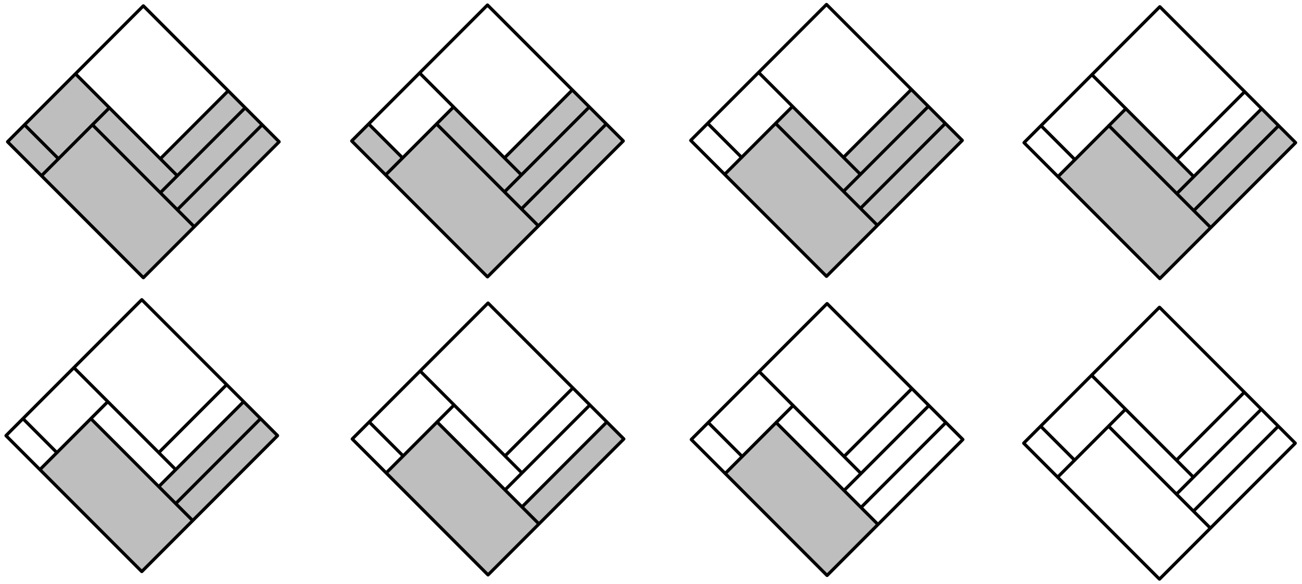
Bijection from $\mathfrak{S}_n$ to weak rectangulations + increasing labeling of regions

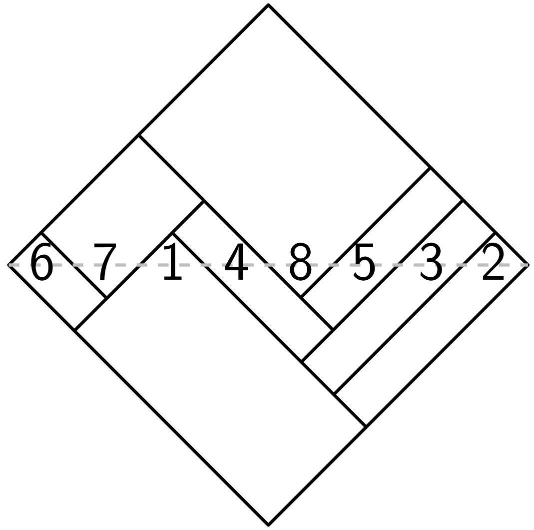
(in anti-diagonal representation)

Specialization to twisted-cotwisted Baxter

[Law, Reading'10]

Leftmost peeling $\Leftrightarrow$ σ^{-1} avoids $3\underline{14}2$, $2\underline{14}3$ (co-twisted Baxter)





Specialization to twisted-cotwisted Baxter

[Law, Reading'10]

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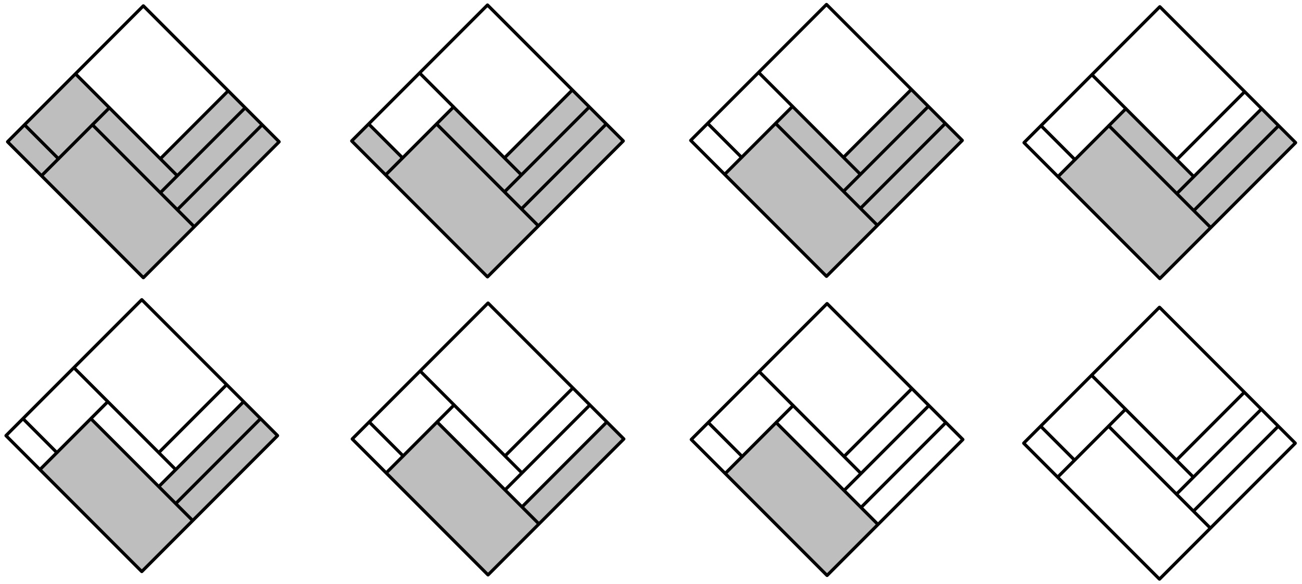
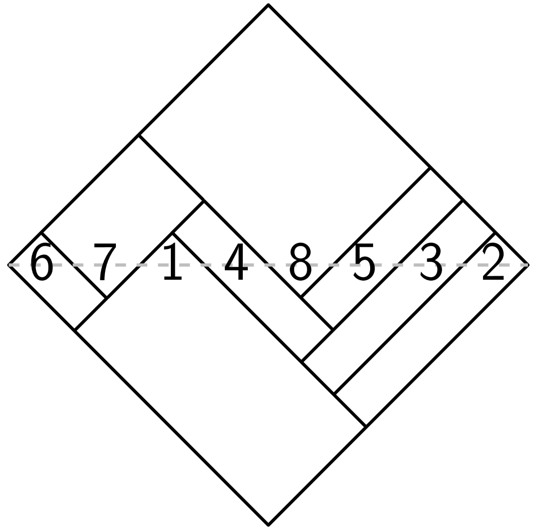
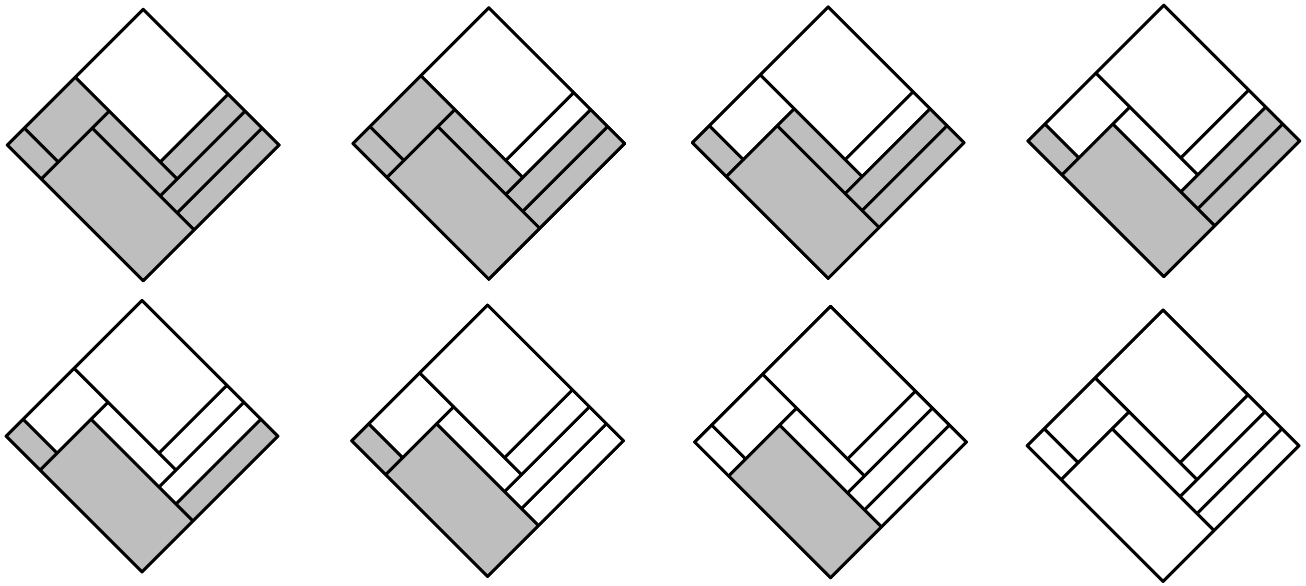
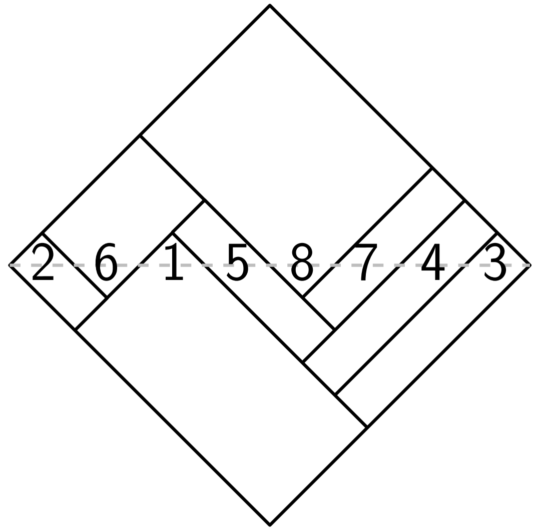




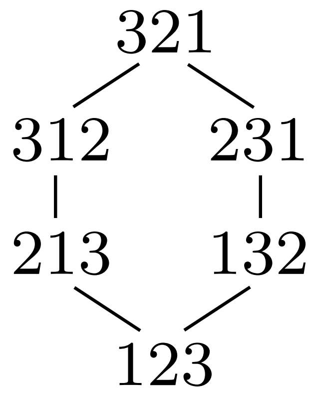
Rightmost peeling $\Leftrightarrow$ σ^{-1} avoids $3\underline{41}2$, $2\underline{41}3$ (twisted Baxter)



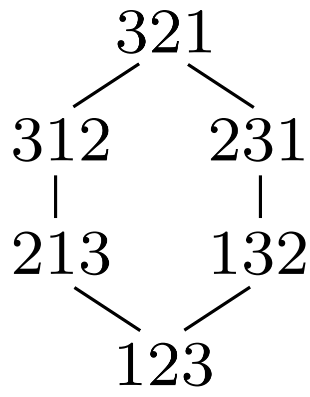


Left weak order on permutations

covering relation: $\cdots i \cdots i{+}1 \cdots \prec \cdots i{+}1 \cdots i \cdots$

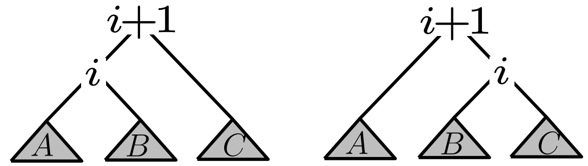


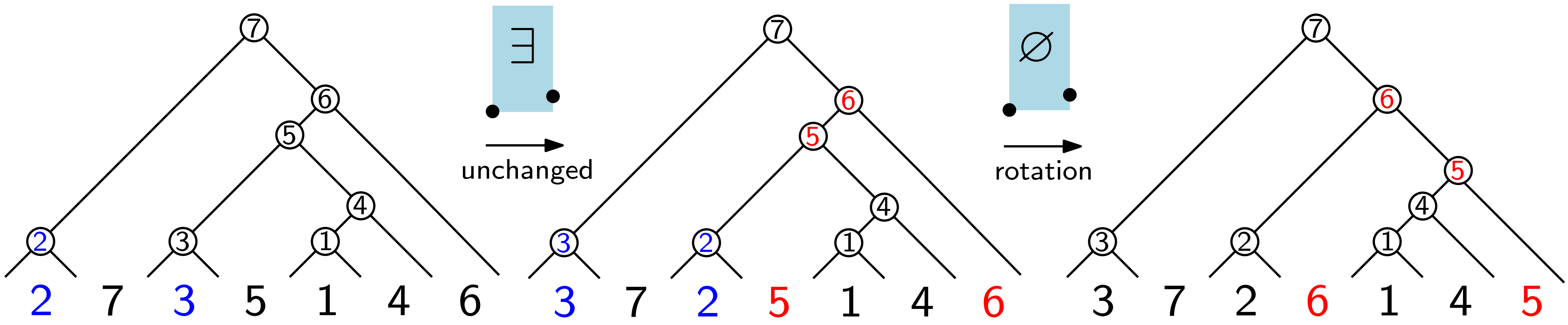
Left weak order on permutations



covering relation: $\cdots i \cdots i{+}1 \cdots \prec \cdots i{+}1 \cdots i \cdots$

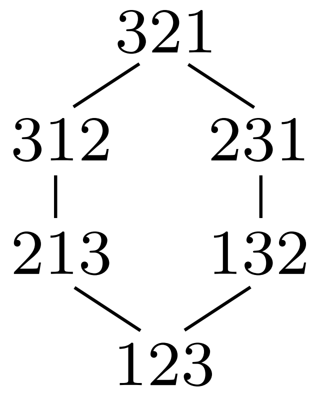
Effect on max-tree



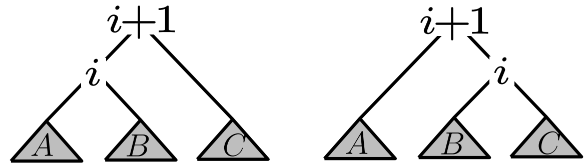


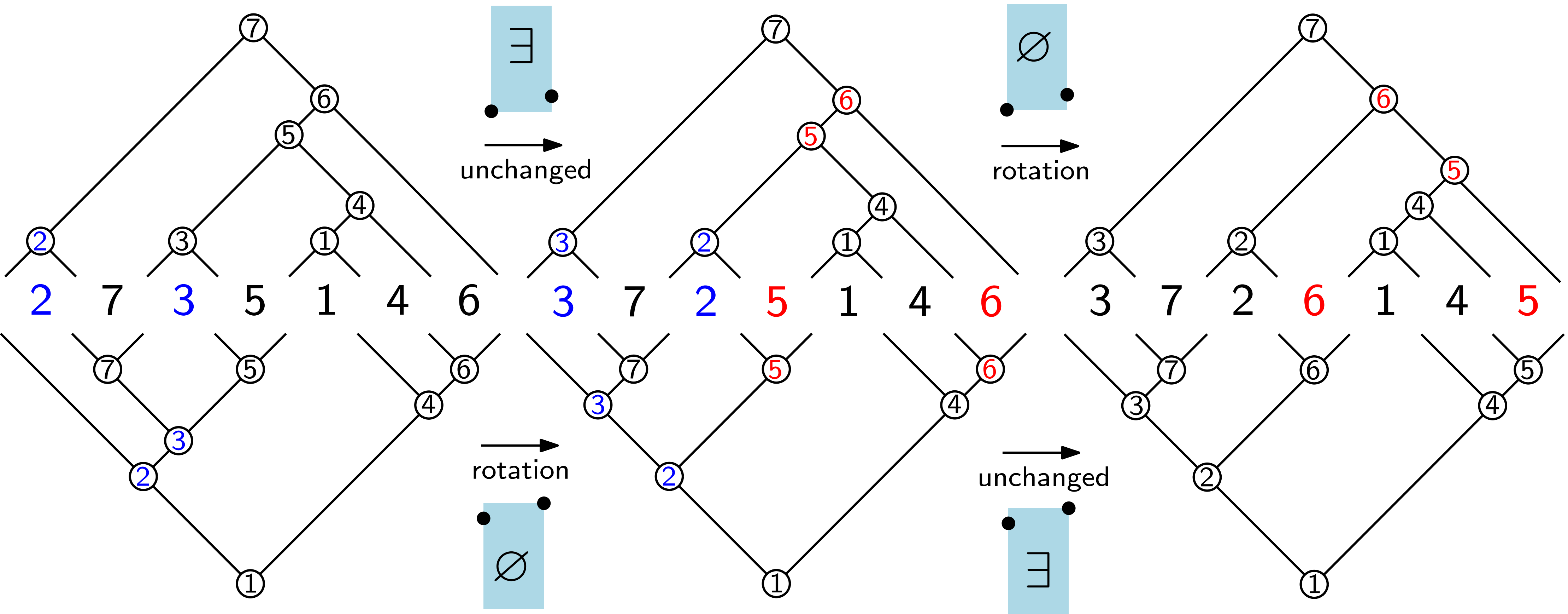
Left weak order on permutations

covering relation: $\cdots i \cdots i{+}1 \cdots \;\prec\; \cdots i{+}1 \cdots i \cdots$



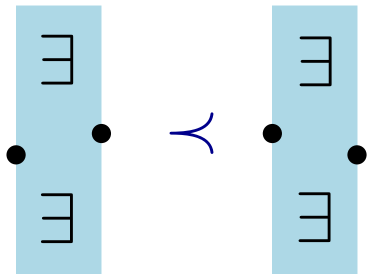
Effect on max-tree





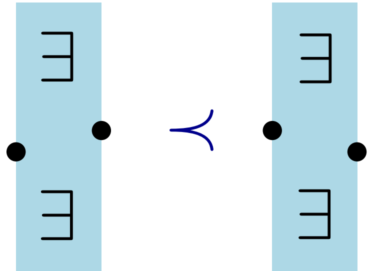
Effect on min-tree

Preimages of a rectangulation in the weak order

covering relations are  (preserve rectangulation)

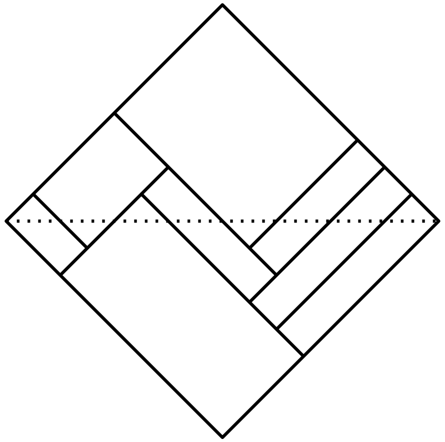
Preimages of a rectangulation in the weak order

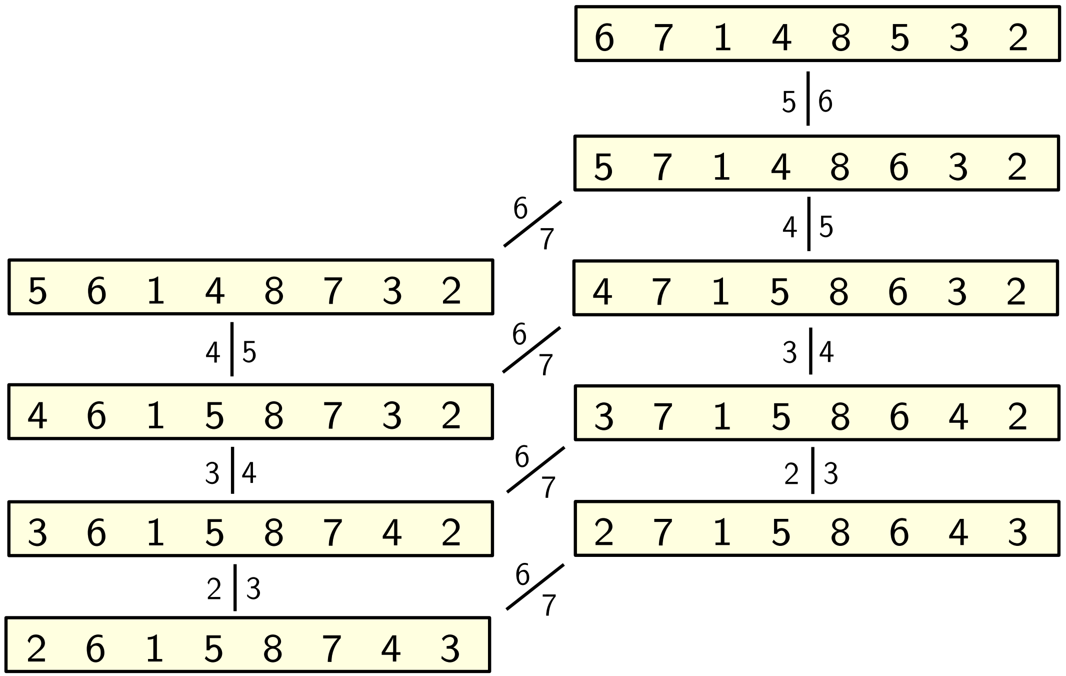
covering relations are



(preserve rectangulation)

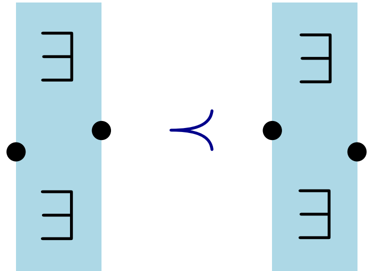
premiages form an interval:



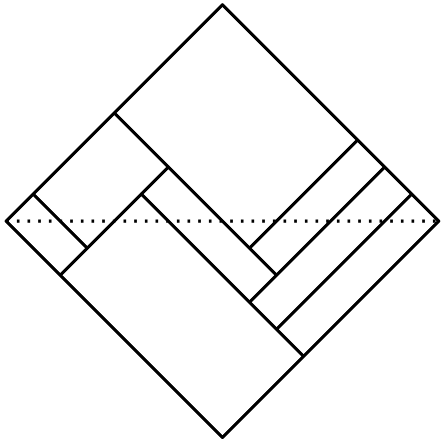


[Law-Reading'10]
[Giraudo'10]

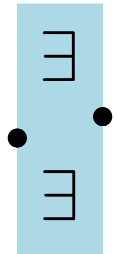
Preimages of a rectangulation in the weak order

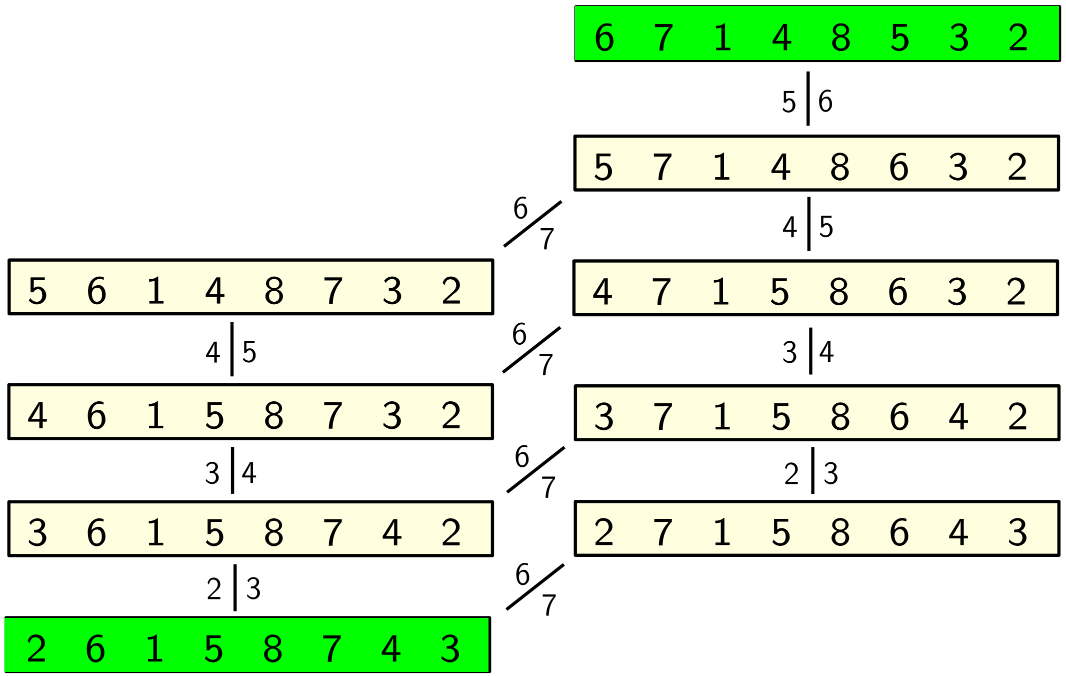
covering relations are  (preserve rectangulation)

premiages form an interval:

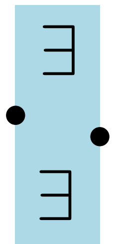


(inv) co-twisted Baxter

avoids 



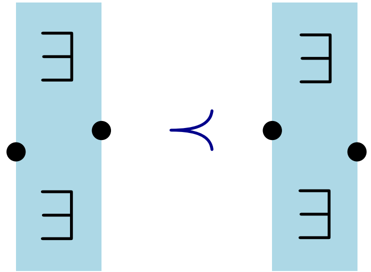
(inv) twisted Baxter

avoids 

[Law-Reading'10]
[Giraudo'10]

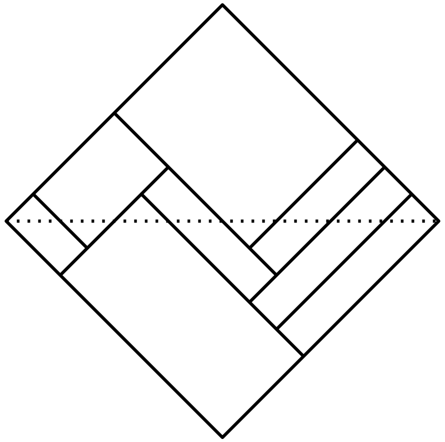
Preimages of a rectangulation in the weak order

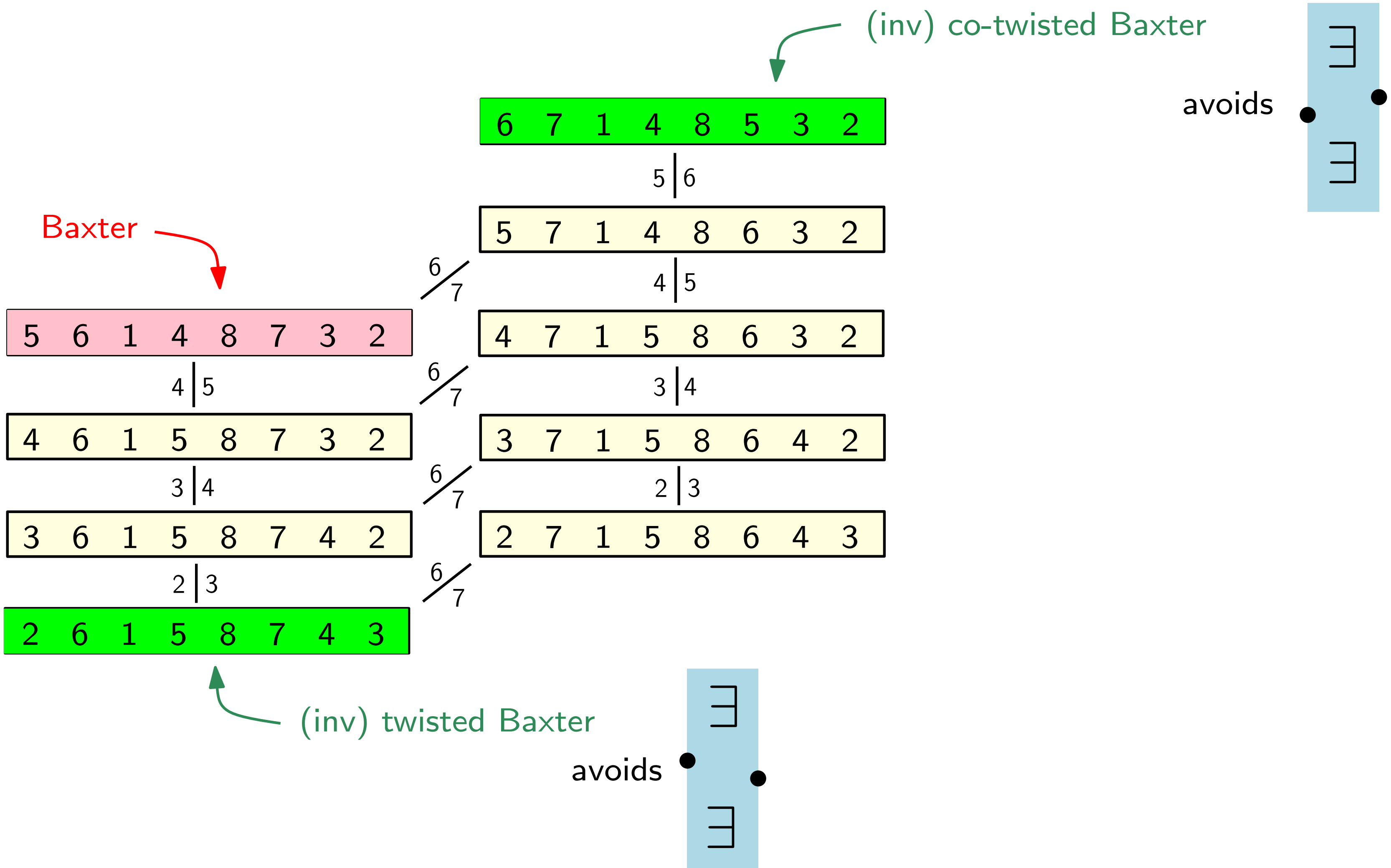
covering relations are



(preserve rectangulation)

premiages form an interval:

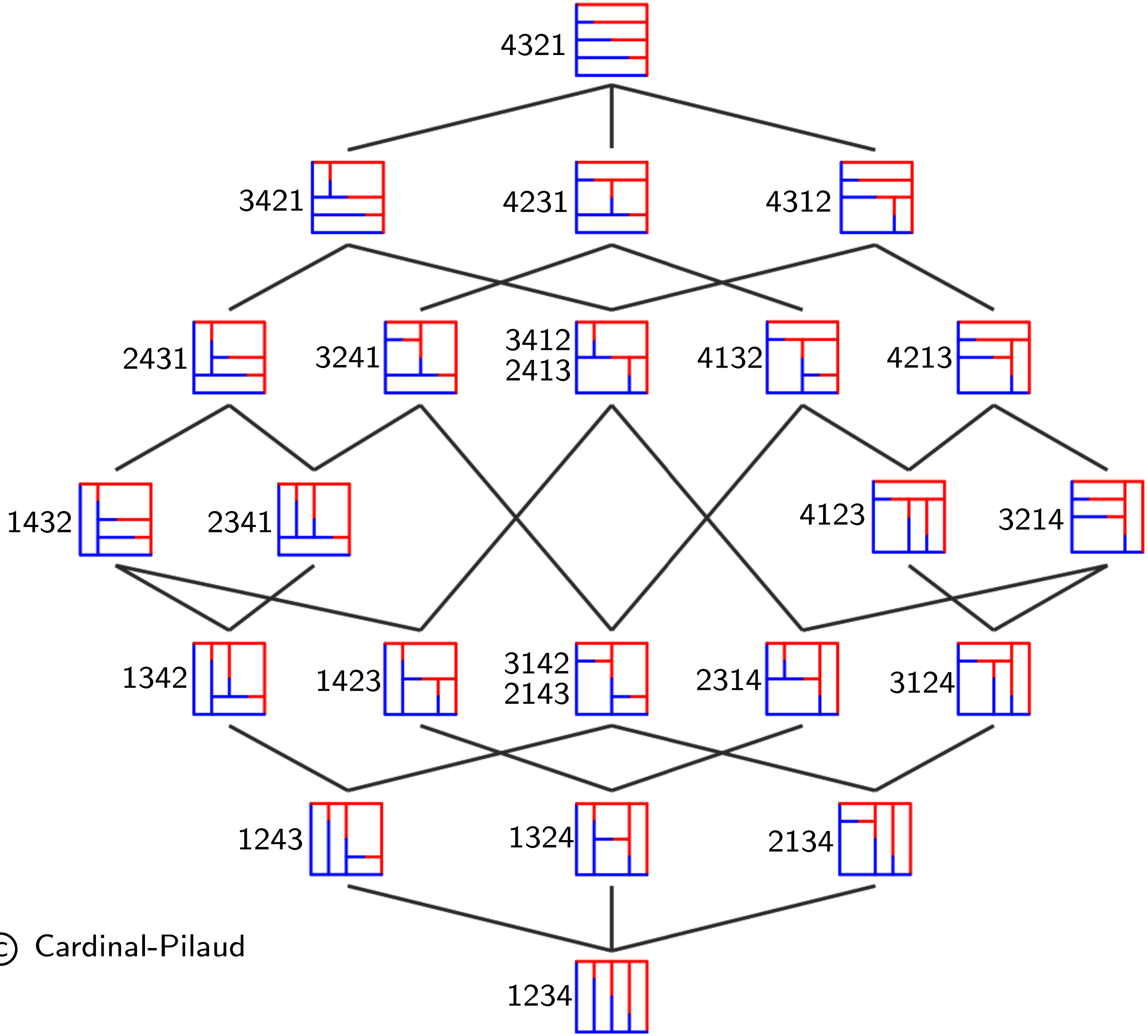




[Law-Reading'10]
[Giraudo'10]

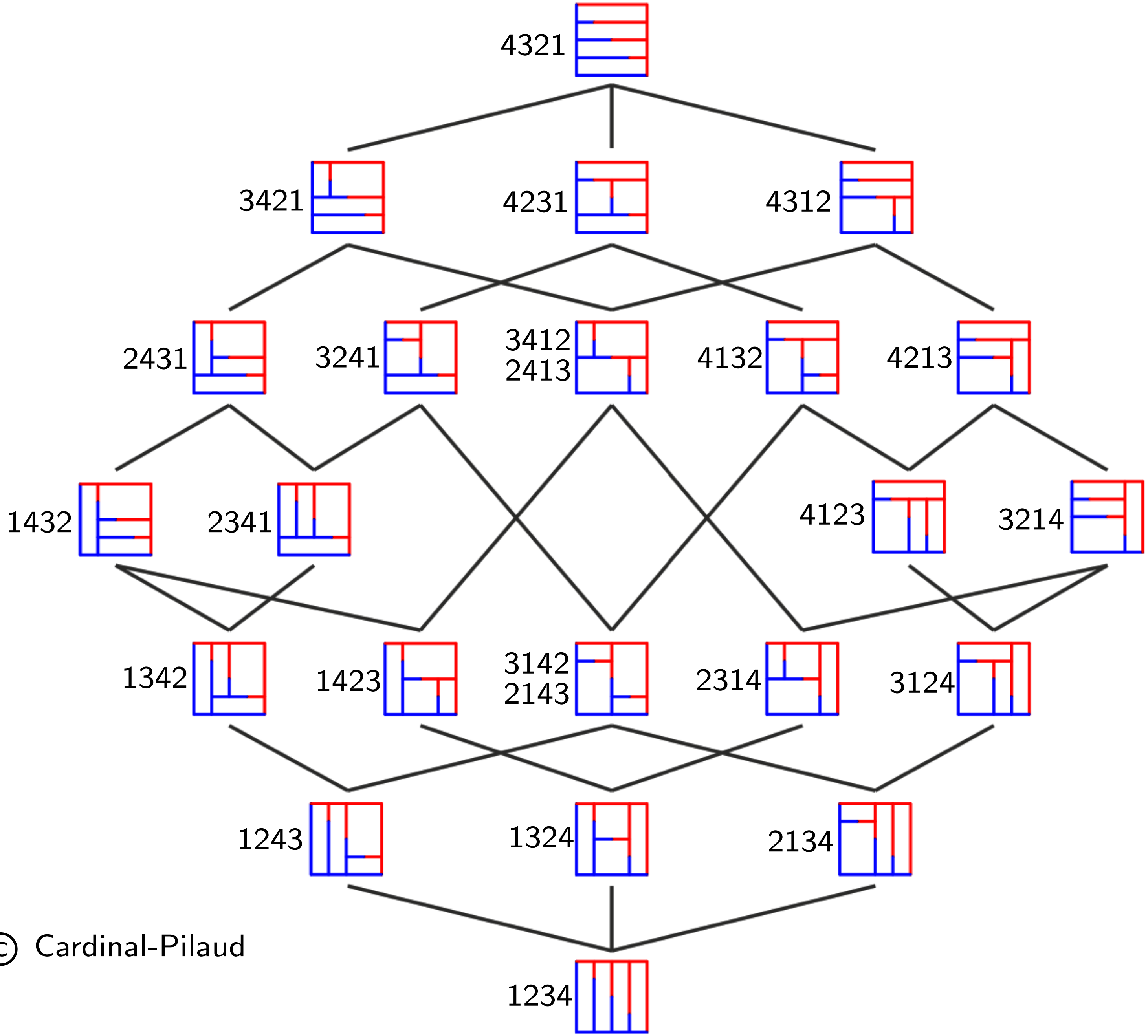
Quotient lattice of weak rectangulations

[Law-Reading'10]
[Giraudo'10]



Quotient lattice of weak rectangulations

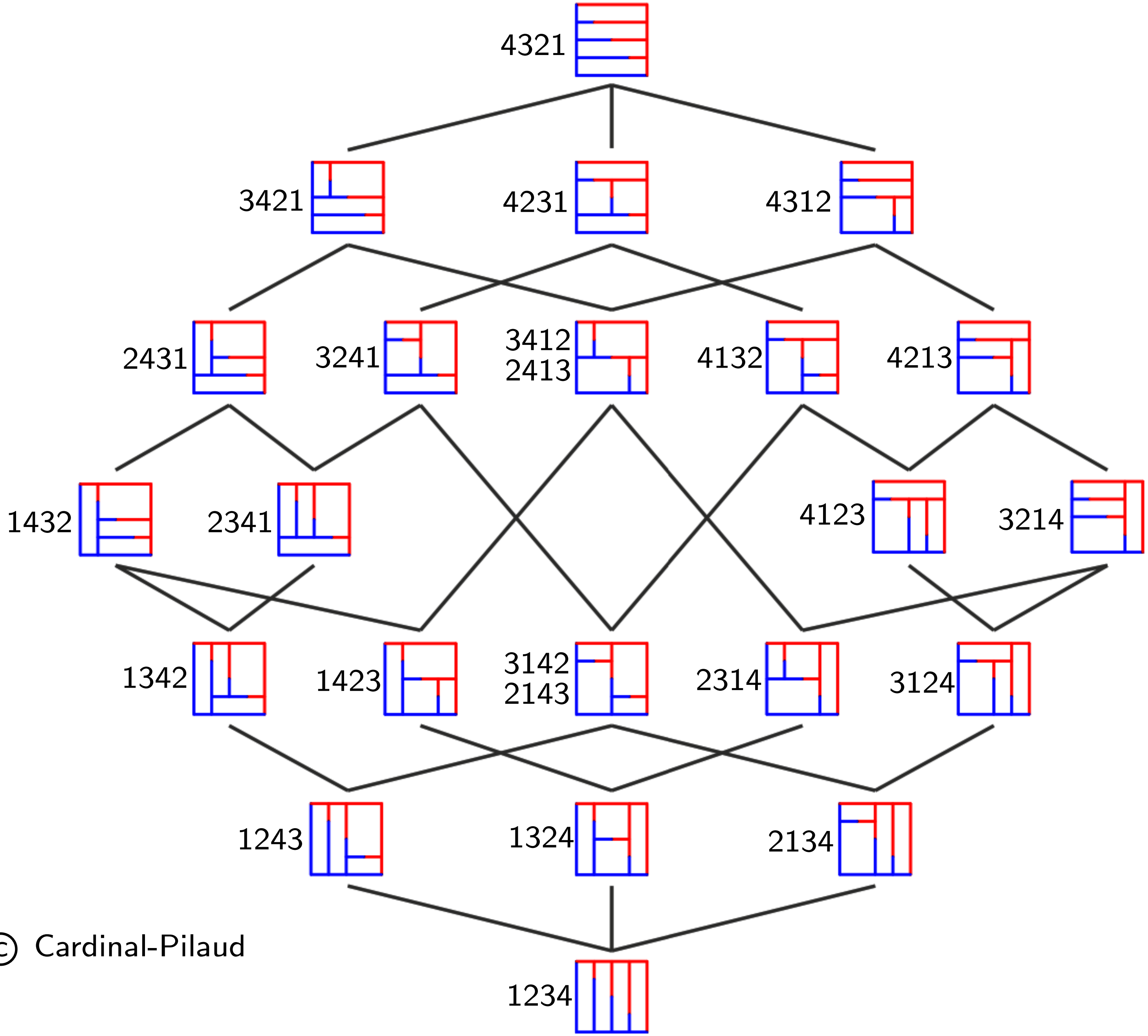
[Law-Reading'10]
[Giraudo'10]



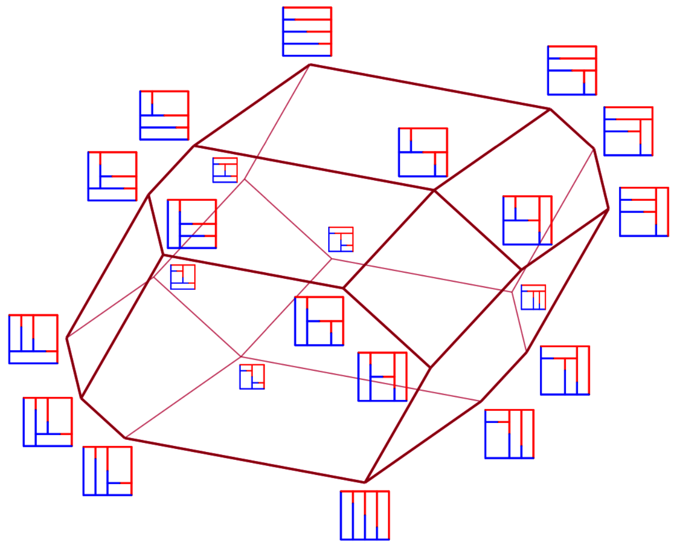
Rk: Permutation grouped by max-tree → Tamari lattice (associahedron)

Quotient lattice of weak rectangulations

[Law-Reading'10]
[Giraudo'10]



© Cardinal-Pilaud



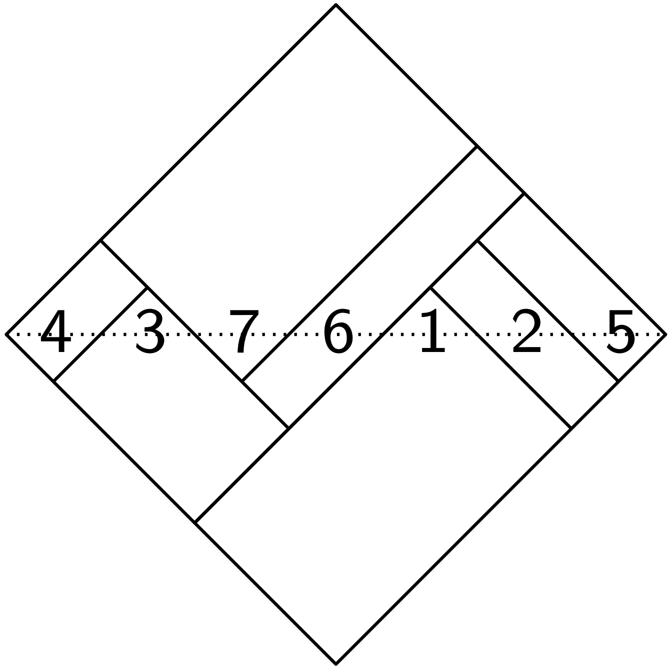
[Cardinal-Pilaud'24]
rectangulotope

Rk: Permutation grouped by max-tree → Tamari lattice (associahedron)

Permutation → strong rectangulation

[Fujimaki, Takahashi'07]
[Reading'12]

4 3 7 6 1 2 5 →

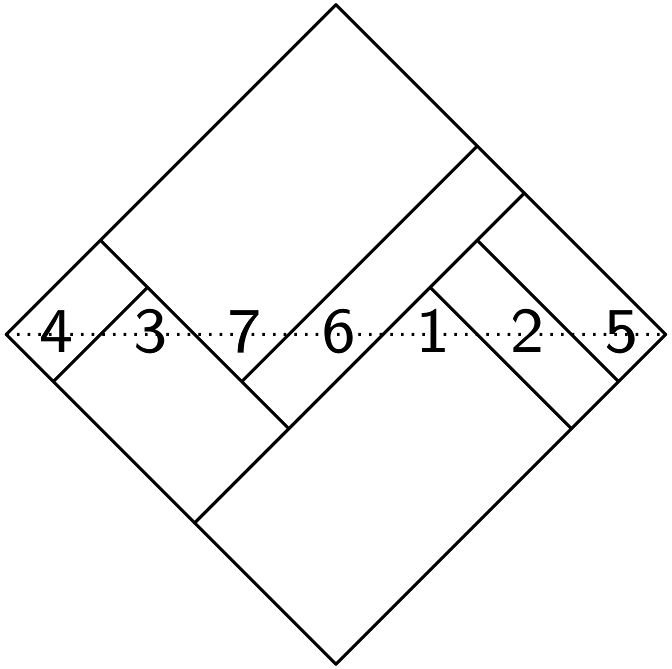


weak rectangulation

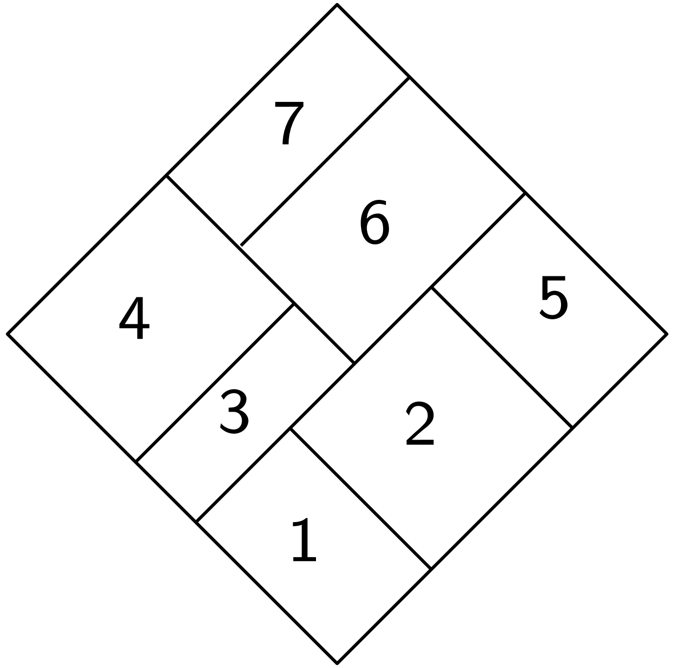
Permutation → strong rectangulation

[Fujimaki, Takahashi'07]
[Reading'12]

4 3 7 6 1 2 5



weak rectangulation

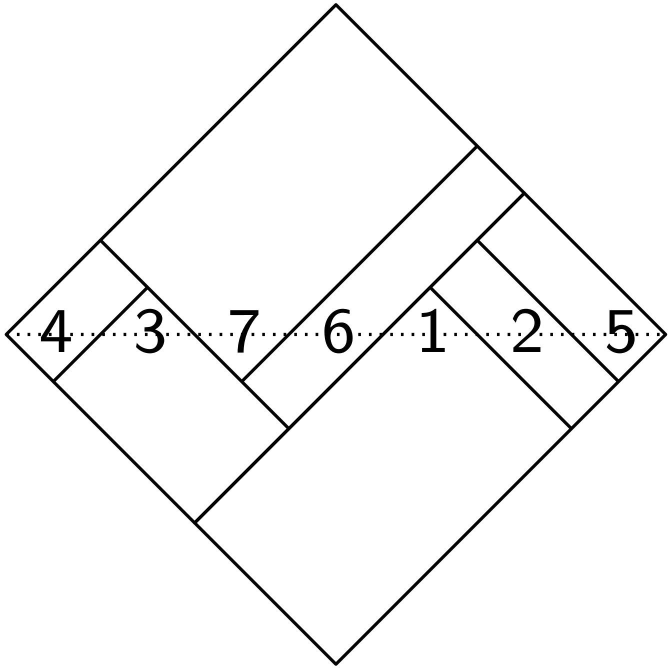


strong rectangulation

Permutation → strong rectangulation

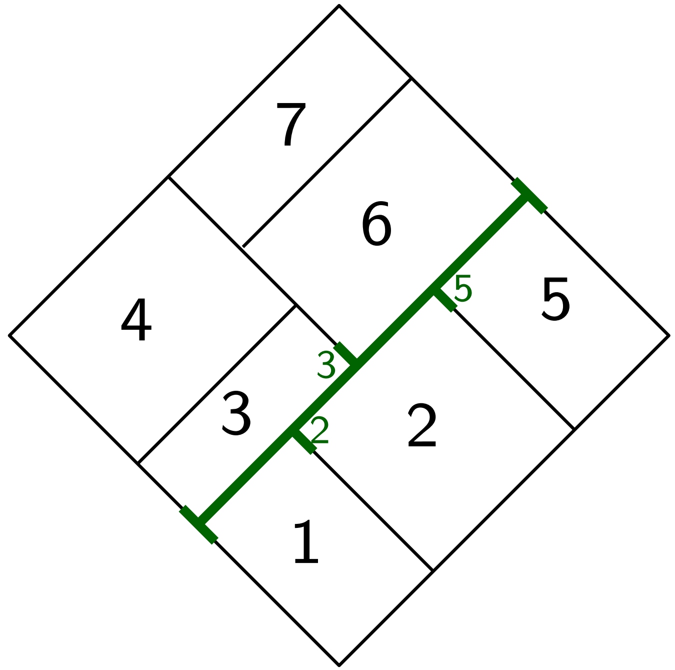
[Fujimaki, Takahashi'07]
[Reading'12]

4 3 7 6 1 2 5 →



weak rectangulation

→

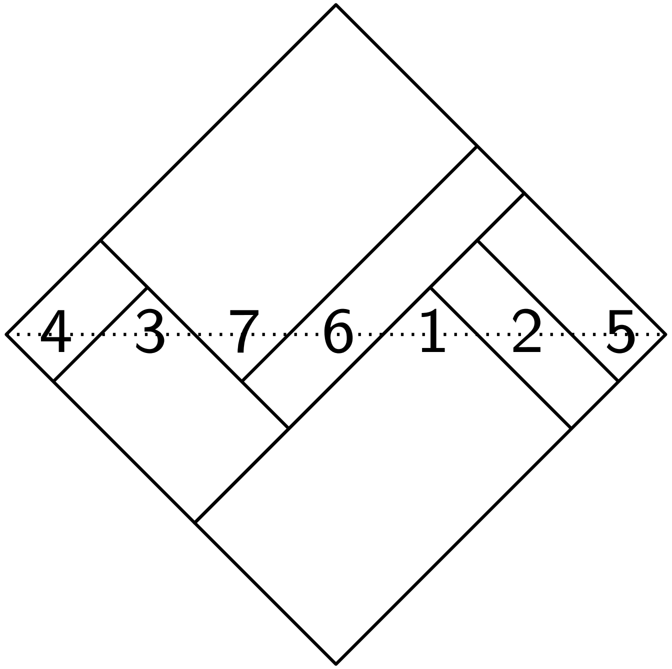


strong rectangulation

Permutation → strong rectangulation

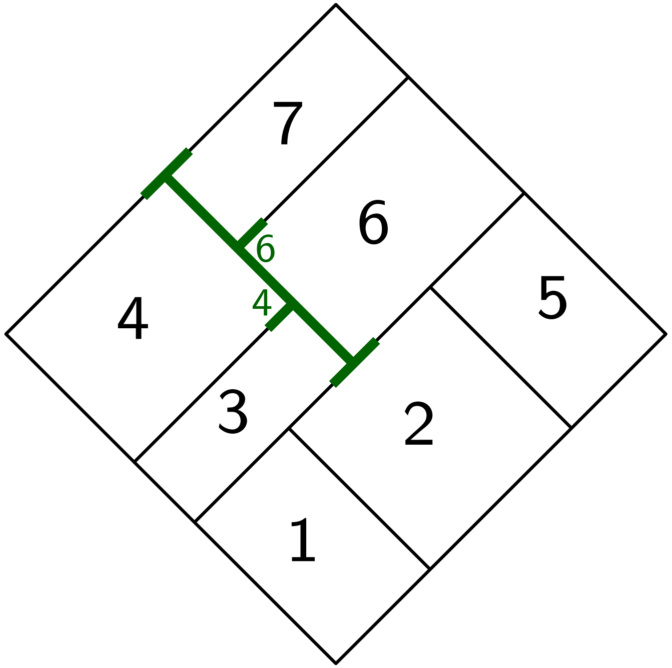
[Fujimaki, Takahashi'07]
[Reading'12]

4 3 7 6 1 2 5 →



weak rectangulation

→

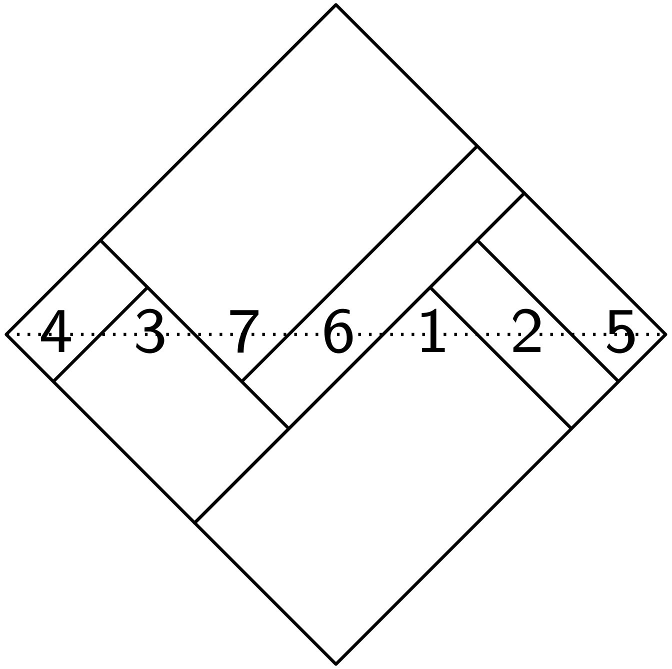


strong rectangulation

Permutation → strong rectangulation

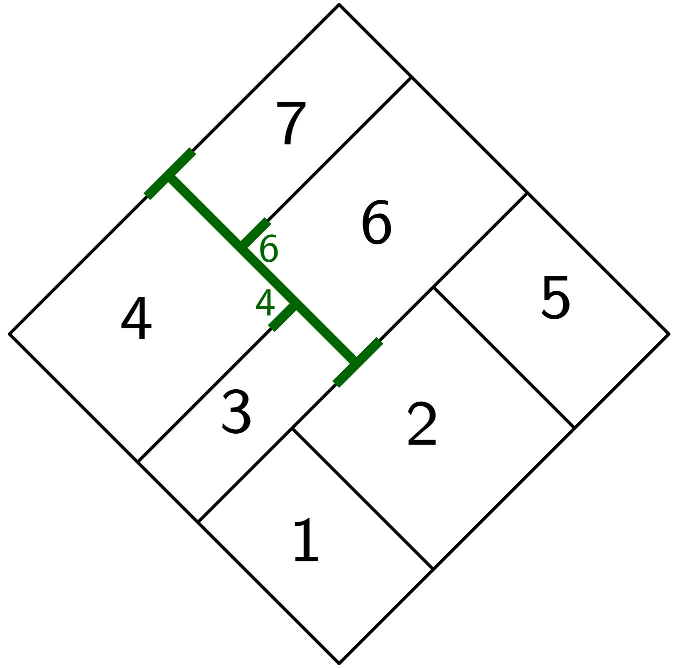
[Fujimaki, Takahashi'07]
[Reading'12]

4 3 7 6 1 2 5 →



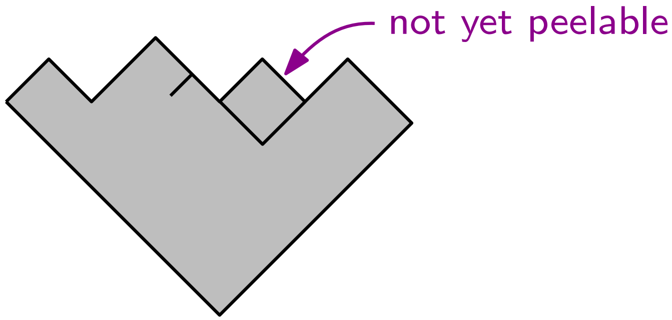
weak rectangulation

→



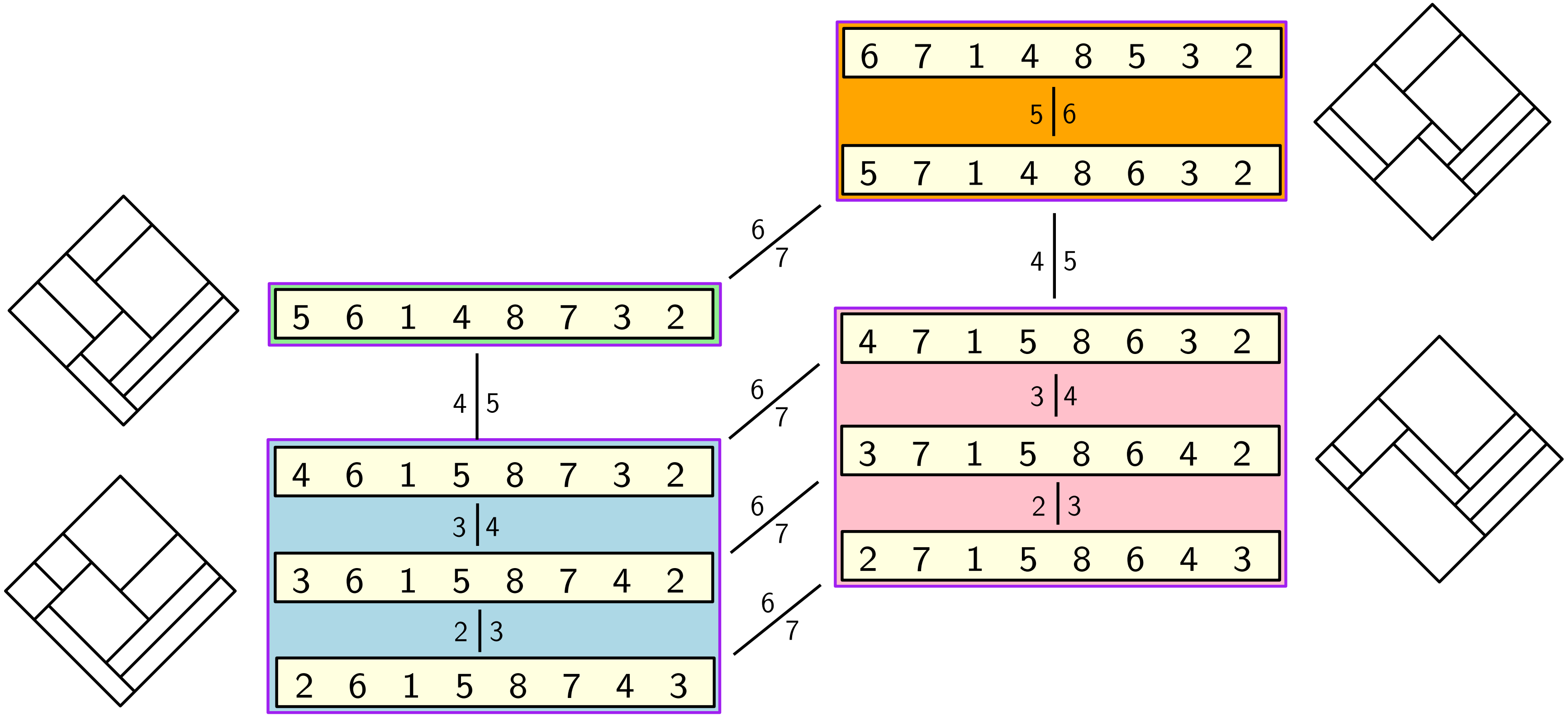
strong rectangulation

Rk: Bijection from $\mathfrak{S}_n$ to strong rectangulations + adapted peeling

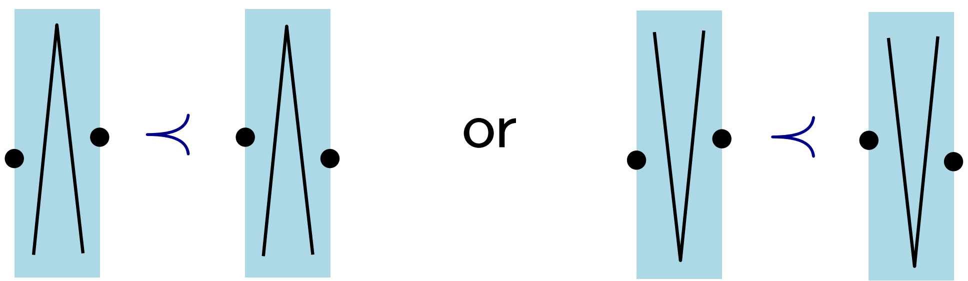


Preimages of a rectangulation: refined picture

[Reading'12]

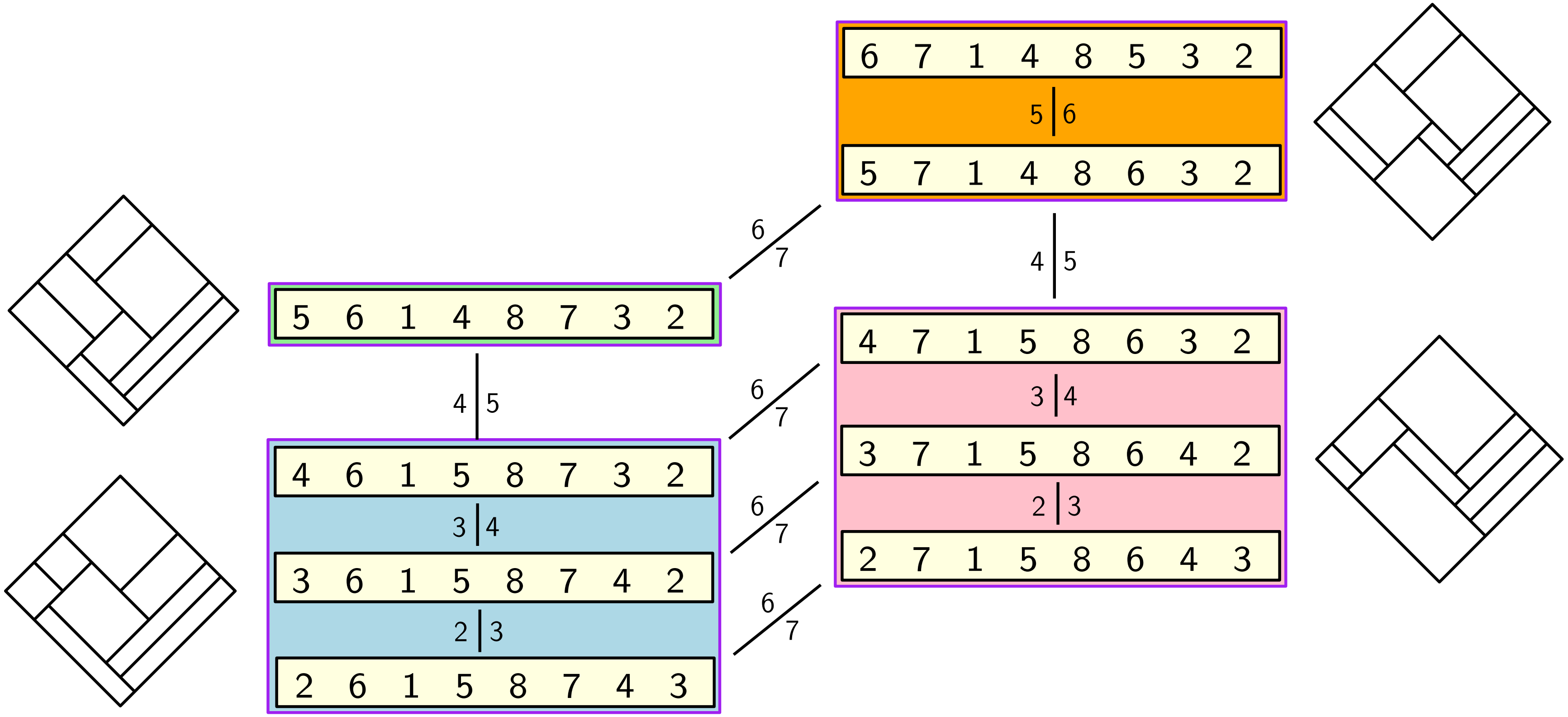


coverings preserving strong rect.:

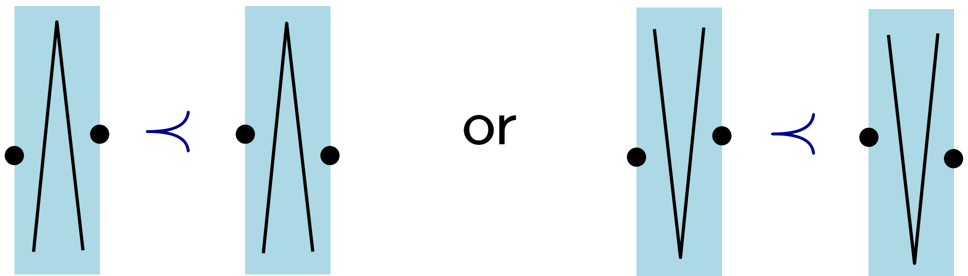


Preimages of a rectangulation: refined picture

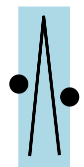
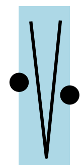
[Reading'12]

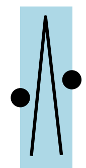
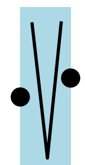


coverings preserving strong rect.:



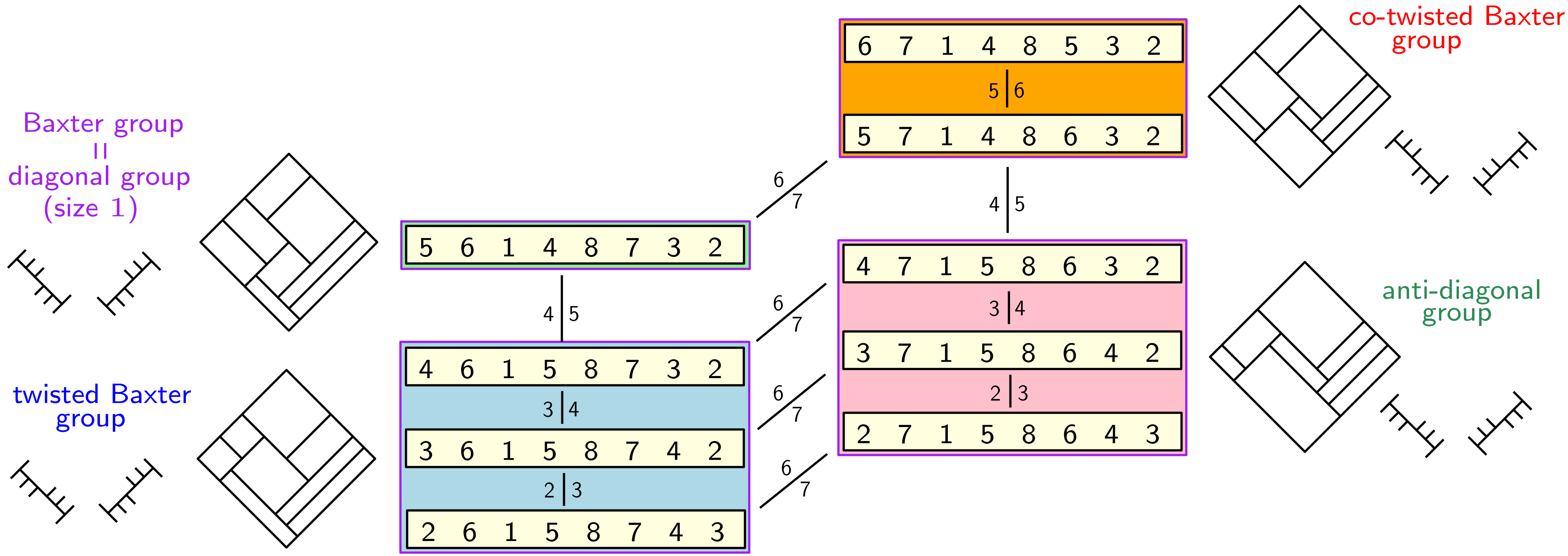
in each strong group:

minimal element avoids  and 
(2-clumped permutation)

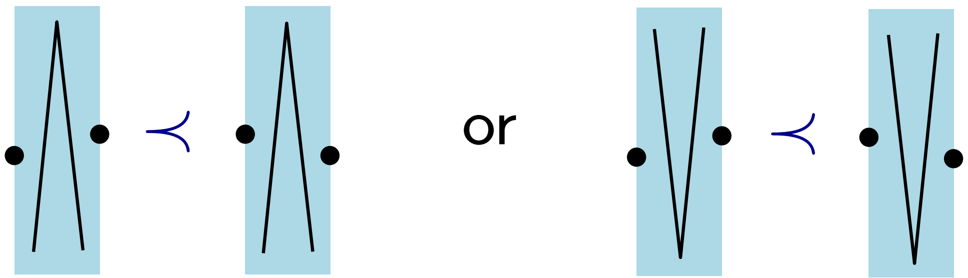
maximal element avoids  and 
(co-2-clumped permutation)

Preimages of a rectangulation: refined picture

[Reading'12]



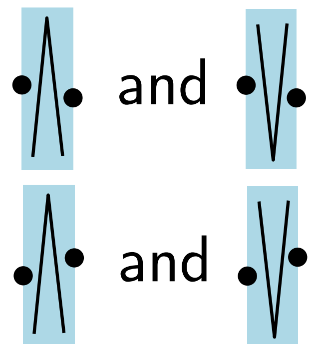
coverings preserving strong rect.:



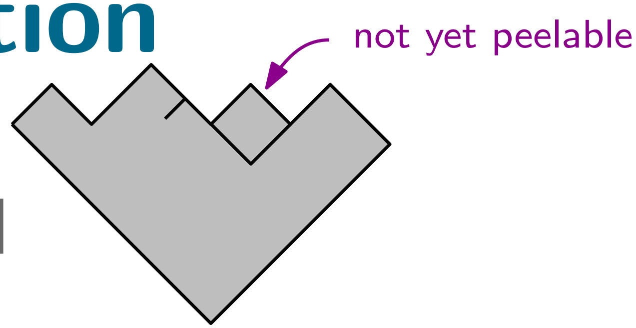
in each strong group:

minimal element avoids (2-clumped permutation) and

maximal element avoids (co-2-clumped permutation) and



Bijection in incremental formulation



$\sigma \in \mathfrak{S}_n \to$ strong rectangulation + adapted peeling

incrementally, via gapped permutations [Françon-Viennot'79]

every step: choice of gap to insert i

$\cdots \bullet \cdots \bullet \cdots \bullet \cdots$

4 ways to insert:

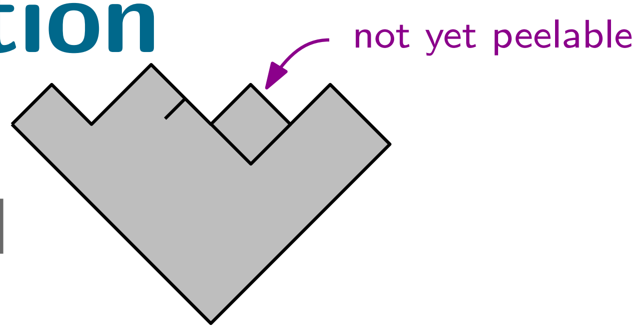
$\bullet \;\to\; \bullet\, i\, \bullet$ or $\bullet\, i$ or $i\, \bullet$ or i

Example: 4 3 7 6 1 2 5

$\bullet \qquad \bullet\, 1\, \bullet \qquad \bullet\, 1\, 2\, \bullet \qquad \bullet\, 3\, \bullet\, 1\, 2\, \bullet$

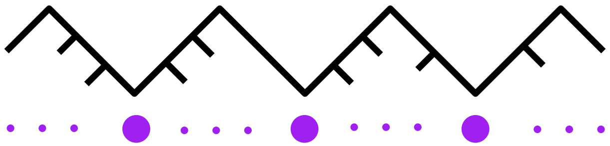
$4\, 3\, \bullet\, 1\, 2\, \bullet \qquad 4\, 3\, \bullet\, 1\, 2\, 5 \qquad 4\, 3\, \bullet\, 6\, 1\, 2\, 5 \qquad 4\, 3\, 7\, 6\, 1\, 2\, 5$

Bijection in incremental formulation

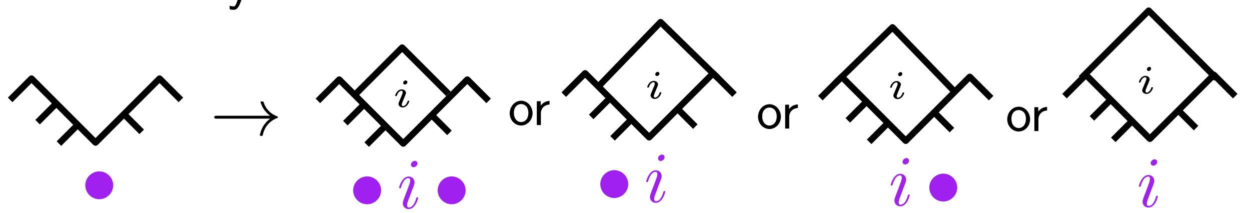


$\sigma \in \mathfrak{S}_n \to$ strong rectangulation + adapted peeling
incrementally, via gapped permutations [Françon-Viennot'79]

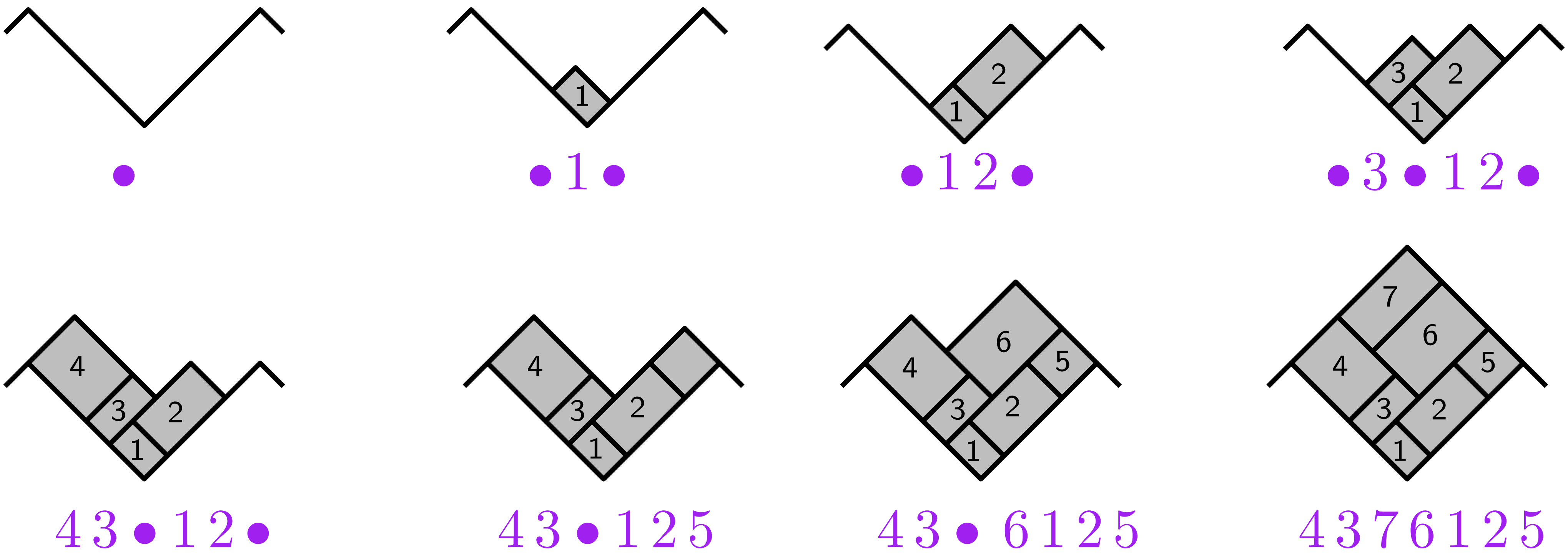
every step: choice of valley/gap to insert i



4 ways to insert:

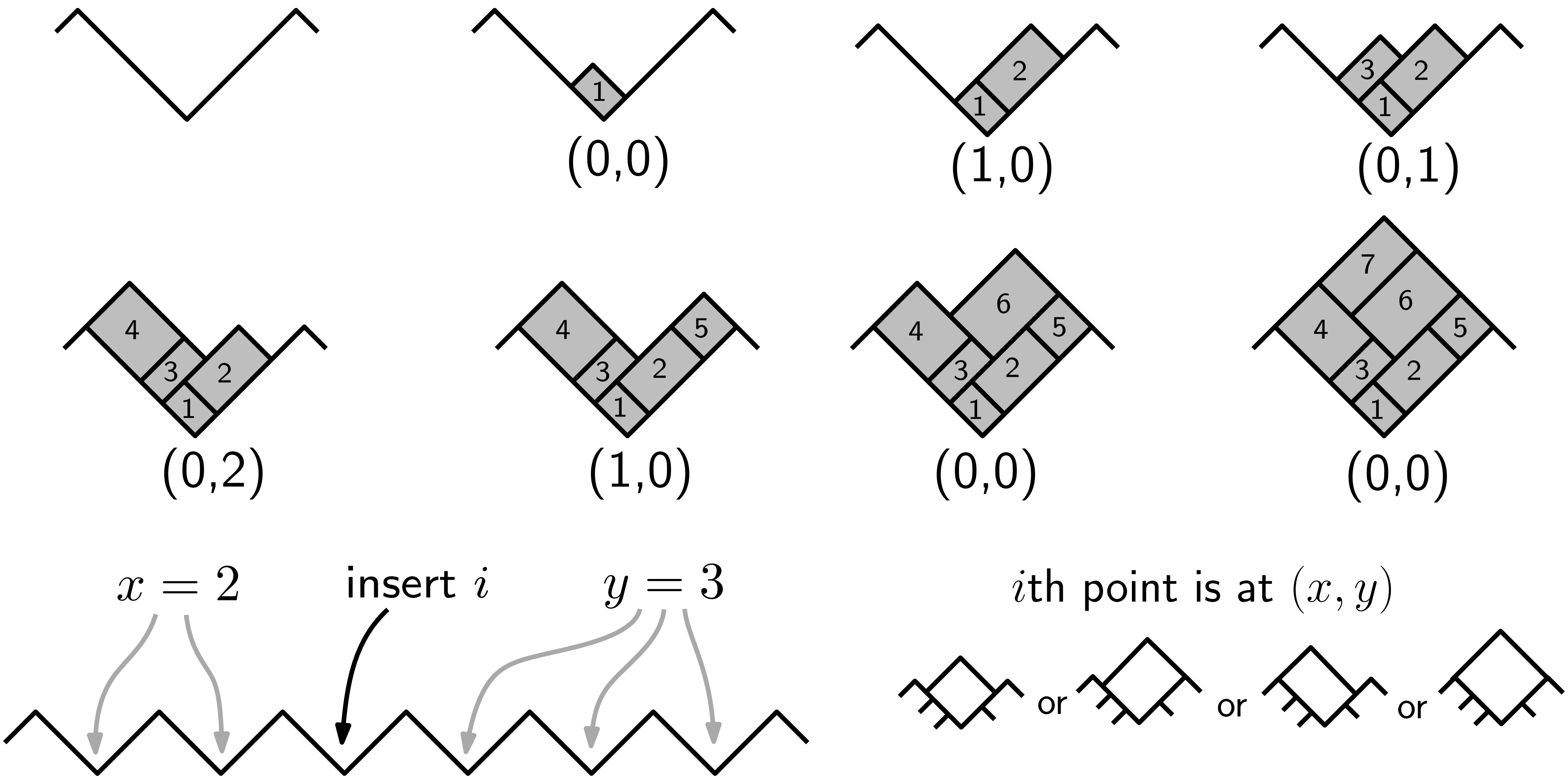


Example: 4 3 7 6 1 2 5



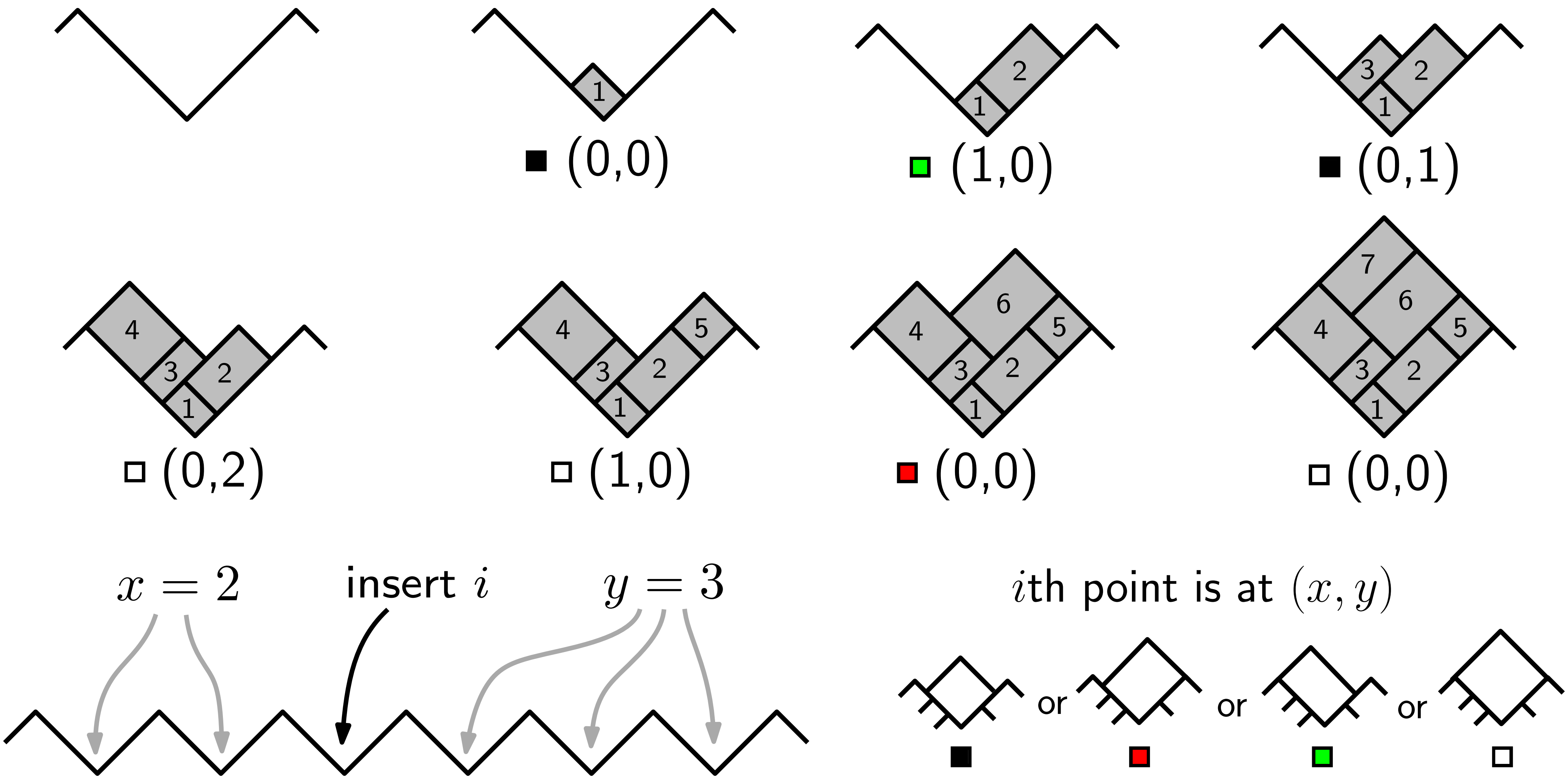
Quadrant walk encoding

Example:



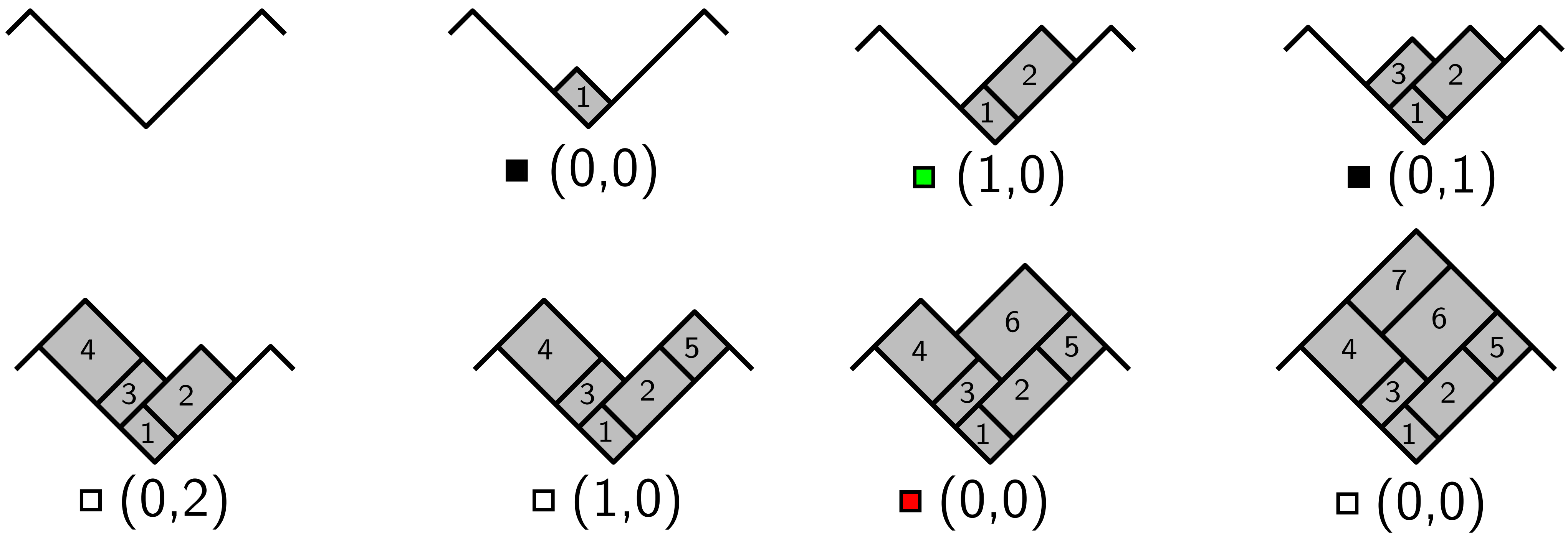
Quadrant walk encoding

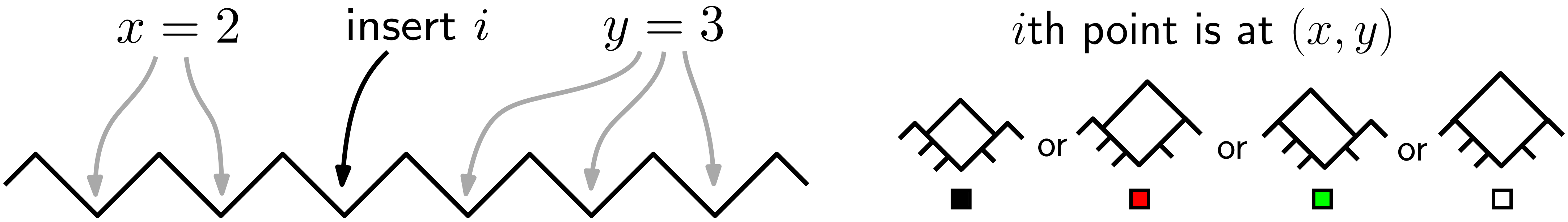
Example:



Quadrant walk encoding

Example:





Rk: sequence of values of $x + y$ gives Motzkin walk
$\rightarrow$ Laguerre history of associated permutation [Françon-Viennot'79] [Flajolet'80]

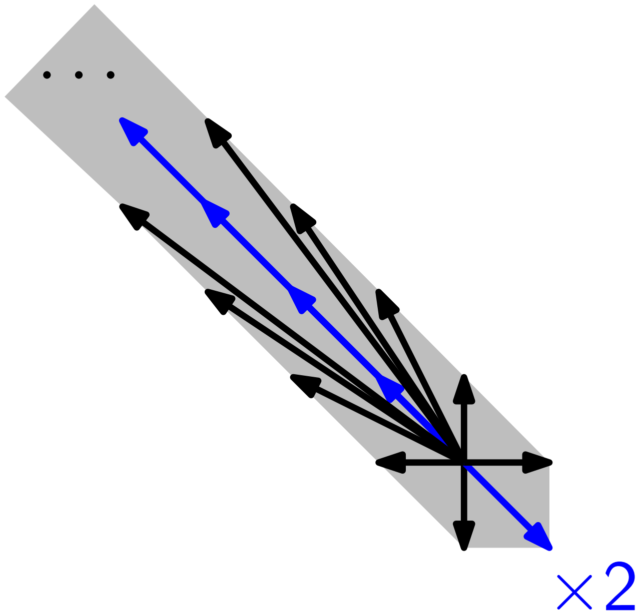
Specializations

[Inoue, Takahashi, Fujikami'09] counting

[Takahashi, Fujikami, Inoue'09] coding

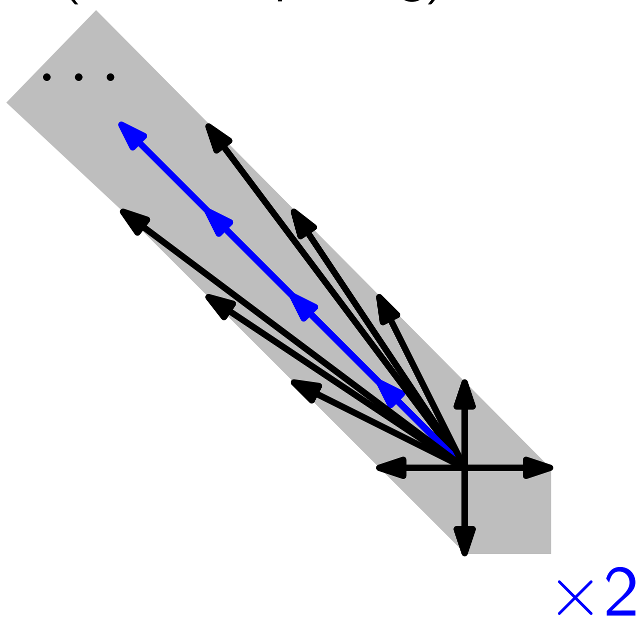
step set

strong rect.
(leftmost peeling)



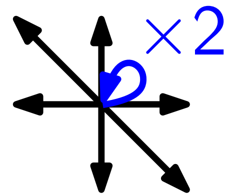
2-clump permutations

weak rect.
(leftmost peeling)



twisted Baxter

weak rect.
(diagonal peeling)



Baxter
[Viennot'81]

Specializations

[Inoue, Takahashi, Fujikami'09] counting

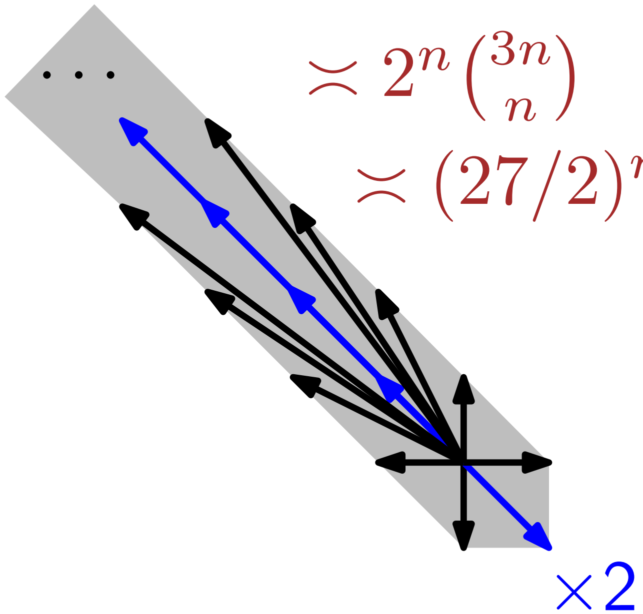
[Takahashi, Fujikami, Inoue'09] coding

step set

strong rect.
(leftmost peeling)

$\asymp 2^n \binom{3n}{n}$

$\asymp (27/2)^n$

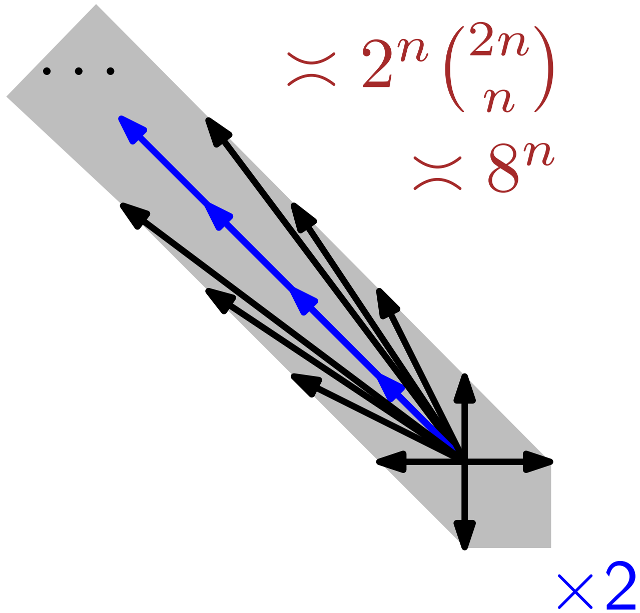


2-clump permutations

weak rect.
(leftmost peeling)

$\asymp 2^n \binom{2n}{n}$

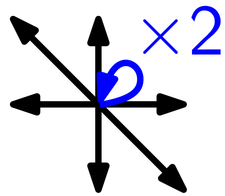
$\asymp 8^n$



twisted Baxter

weak rect.
(diagonal peeling)

$\asymp 8^n$



Baxter
[Viennot'81]

Specializations

[Inoue, Takahashi, Fujikami'09] counting

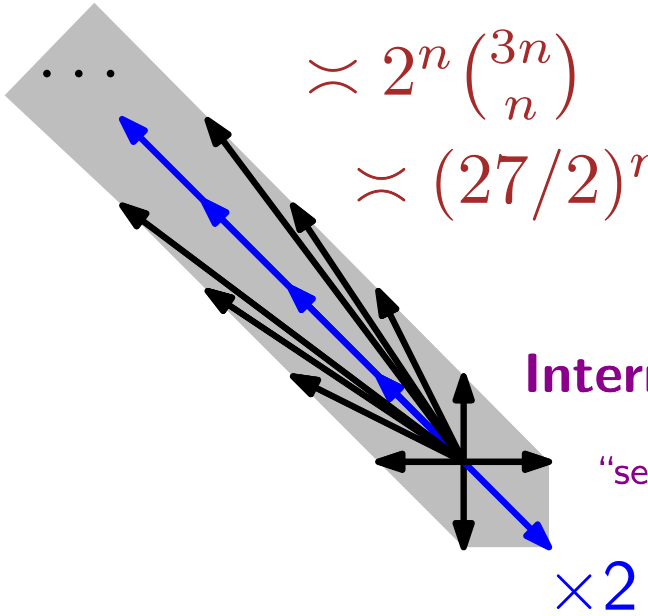
[Takahashi, Fujikami, Inoue'09] coding

step set

strong rect. (leftmost peeling)

$\asymp 2^n \binom{3n}{n}$

$\asymp (27/2)^n$



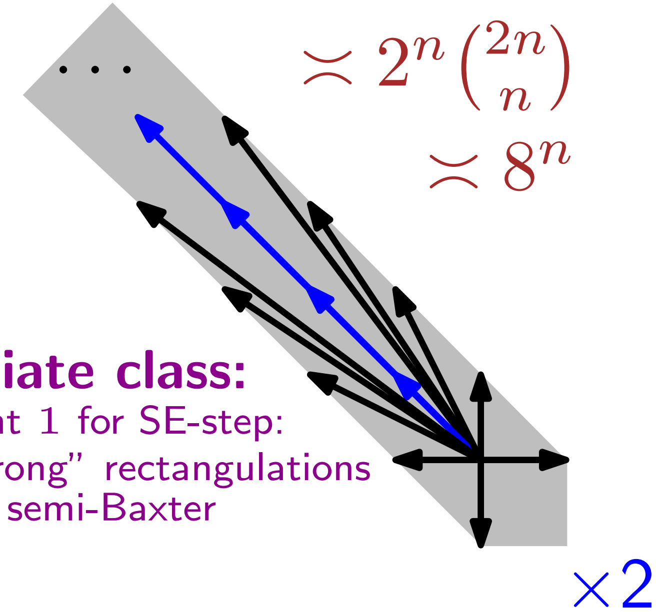
2-clump permutations

Intermediate class:
weight 1 for SE-step:
"semi-strong" rectangulations
semi-Baxter

weak rect. (leftmost peeling)

$\asymp 2^n \binom{2n}{n}$

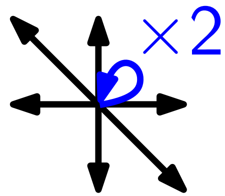
$\asymp 8^n$



twisted Baxter

weak rect. (diagonal peeling)

$\asymp 8^n$



Baxter
[Viennot'81]

Specializations

[Inoue, Takahashi, Fujikami'09] counting

[Takahashi, Fujikami, Inoue'09] coding

strong rect. (leftmost peeling)

$\asymp 2^n \binom{3n}{n}$

$\asymp (27/2)^n$

weak rect. (leftmost peeling)

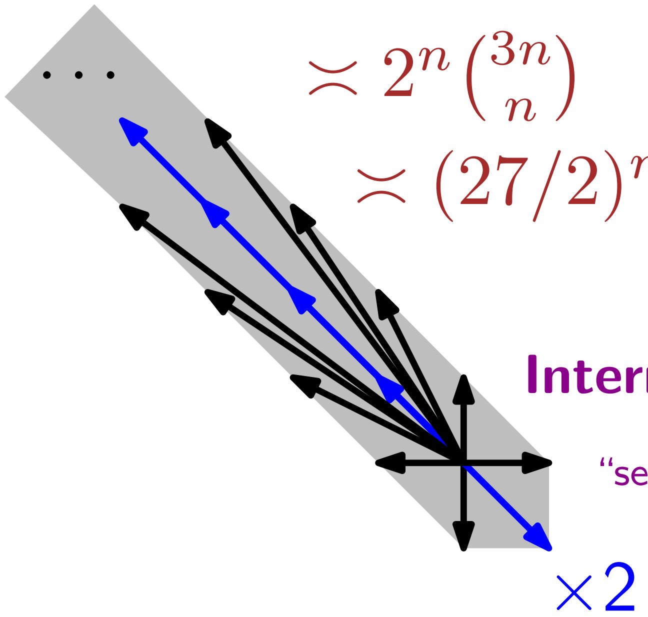
$\asymp 2^n \binom{2n}{n}$

$\asymp 8^n$

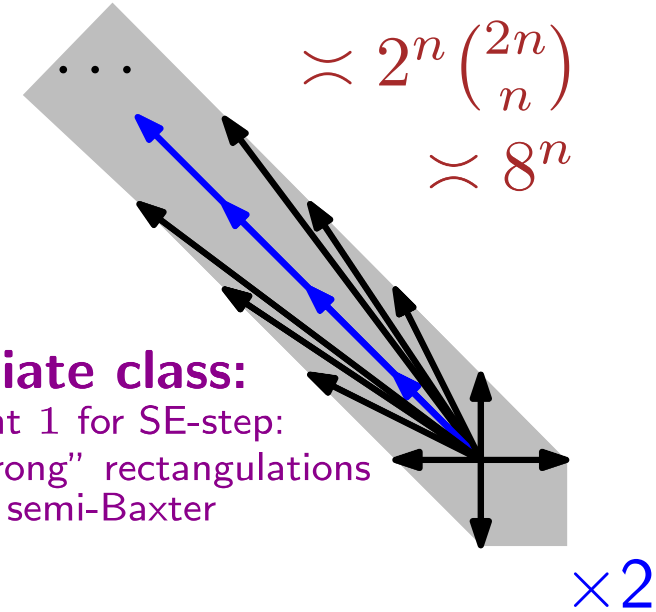
weak rect. (diagonal peeling)

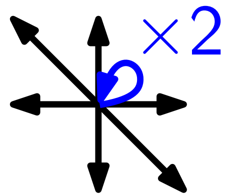
$\asymp 8^n$

step set



Intermediate class:
weight 1 for SE-step:
"semi-strong" rectangulations
semi-Baxter





2-clump permutations

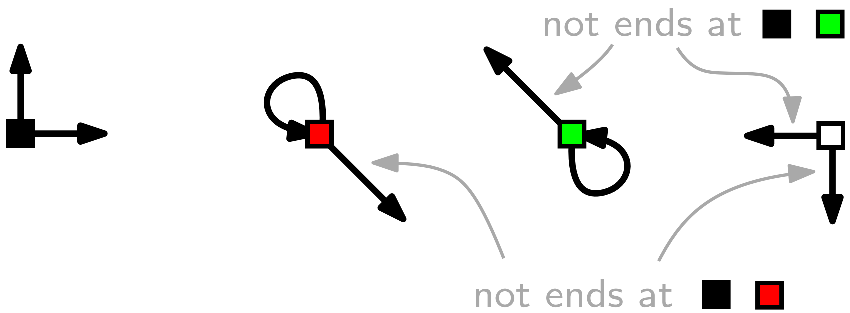
twisted Baxter

Baxter [Viennot'81]

- 1-sided rect. (= weak rect. with unique peeling)

[Asinowski, Cardinal, Felsner, F'25]

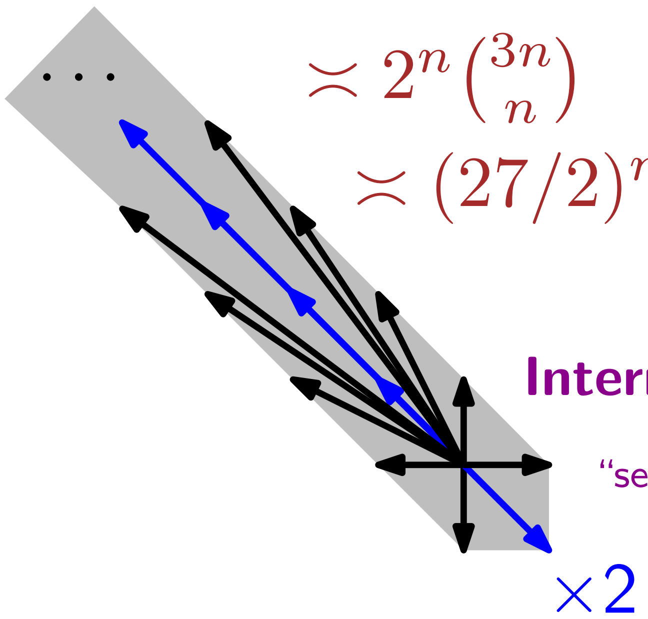
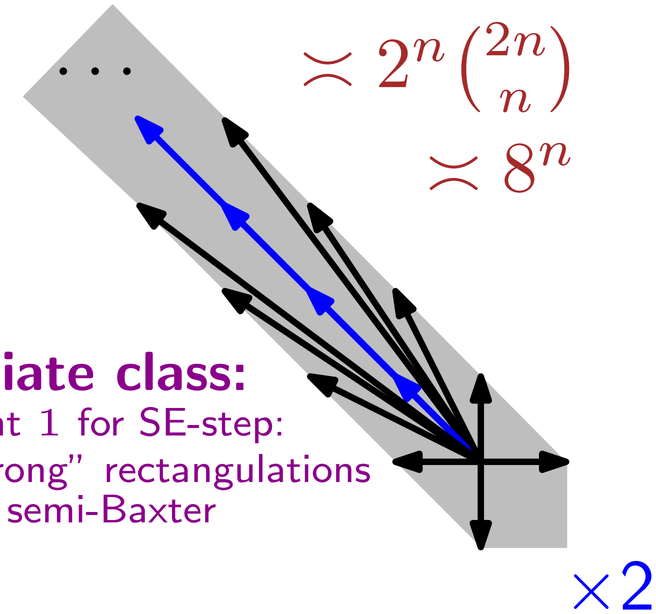
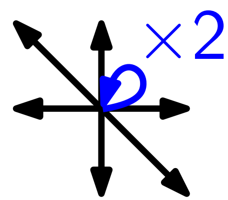
allowed steps after



Specializations

[Inoue, Takahashi, Fujikami'09] counting

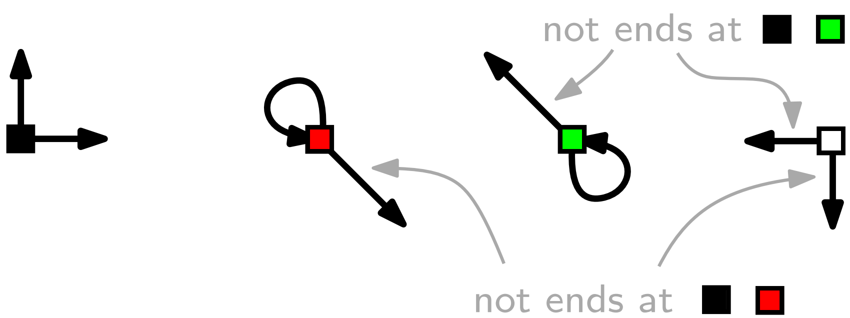
[Takahashi, Fujikami, Inoue'09] coding

	strong rect. (leftmost peeling)	weak rect. (leftmost peeling)	weak rect. (diagonal peeling)
	$\asymp 2^n \binom{3n}{n}$ $\asymp (27/2)^n$	$\asymp 2^n \binom{2n}{n}$ $\asymp 8^n$	$\asymp 8^n$
step set			
	2-clump permutations	twisted Baxter	Baxter [Viennot'81]

Intermediate class:
weight 1 for SE-step:
"semi-strong" rectangulations
semi-Baxter

- 1-sided rect. (= weak rect. with unique peeling) [Asinowski, Cardinal, Felsner, F'25]

allowed steps after



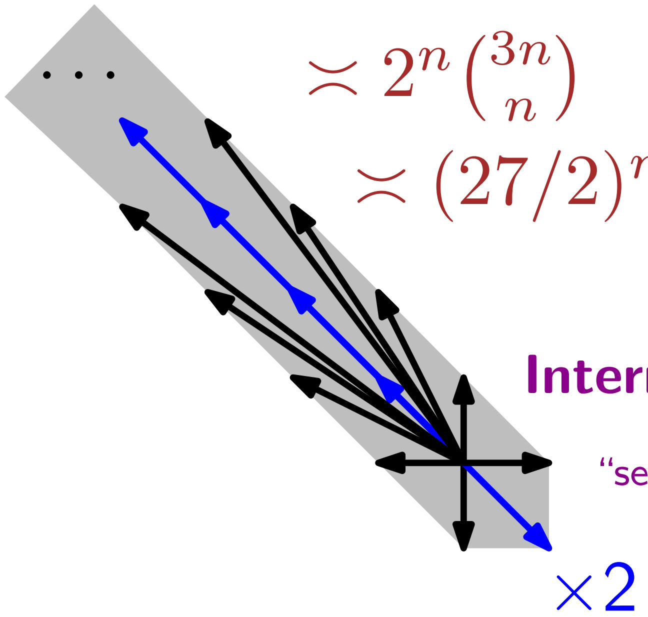
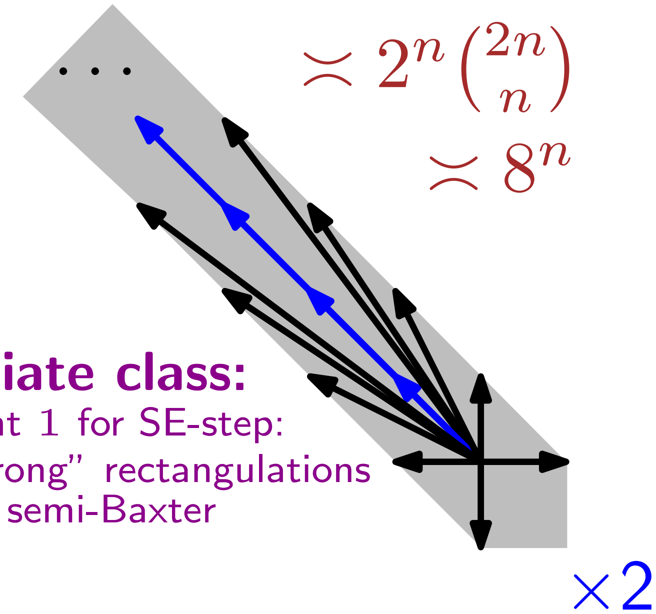
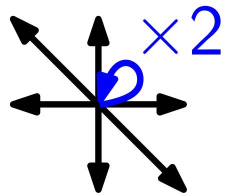
$\asymp \gamma^n$

with $\gamma = \frac{1}{2}(7+\sqrt{17})$
≈ 5.56

Specializations

[Inoue, Takahashi, Fujikami'09] counting

[Takahashi, Fujikami, Inoue'09] coding

	strong rect. (leftmost peeling)	weak rect. (leftmost peeling)	weak rect. (diagonal peeling)
	$\asymp 2^n \binom{3n}{n}$ $\asymp (27/2)^n$	$\asymp 2^n \binom{2n}{n}$ $\asymp 8^n$	$\asymp 8^n$
step set			
	2-clump permutations	twisted Baxter	Baxter [Viennot'81]

Intermediate class:
weight 1 for SE-step:
"semi-strong" rectangulations
semi-Baxter

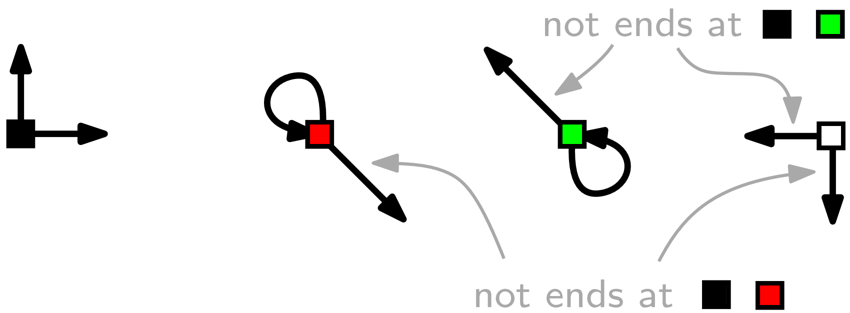
- 1-sided rect. (= weak rect. with unique peeling)

[Asinowski, Cardinal, Felsner, F'25]

$\sim c\,\gamma^n\, n^{-\alpha}$ $\quad \alpha \notin \mathbb{Q}$

with $\gamma = \frac{1}{2}(7+\sqrt{17})$
≈ 5.56

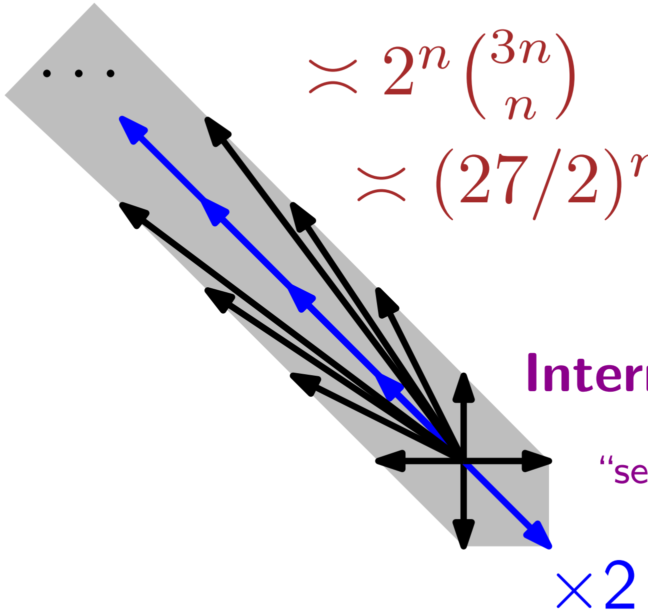
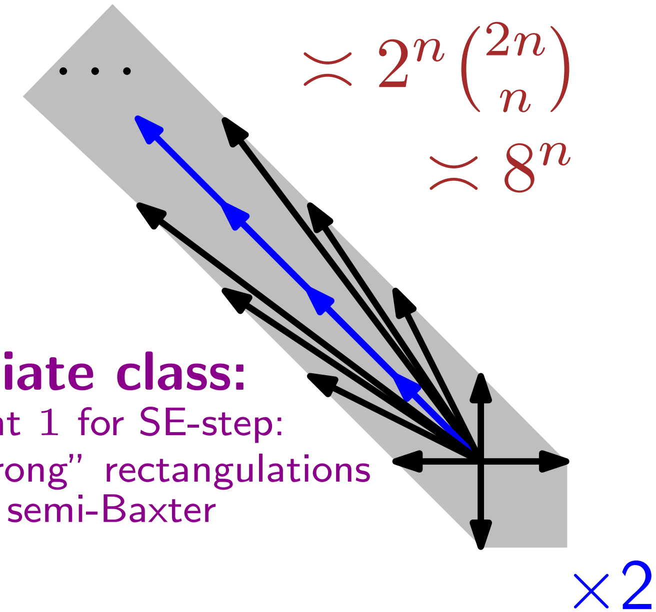
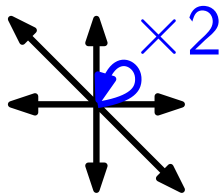
allowed steps after



Specializations

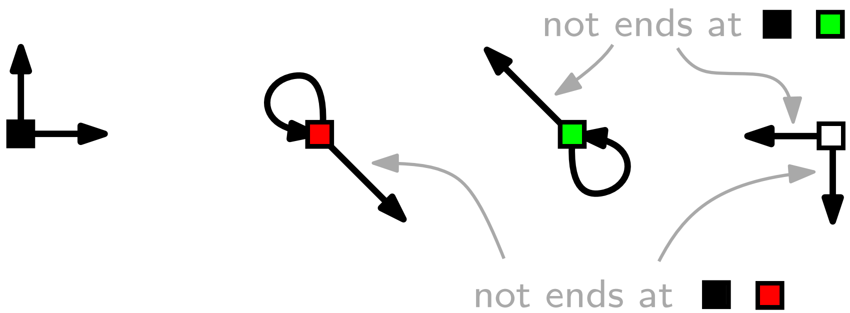
[Inoue, Takahashi, Fujikami'09] counting

[Takahashi, Fujikami, Inoue'09] coding

	strong rect. (leftmost peeling)	weak rect. (leftmost peeling)	weak rect. (diagonal peeling)
	$\asymp 2^n \binom{3n}{n}$	$\asymp 2^n \binom{2n}{n}$	$\asymp 8^n$
	$\asymp (27/2)^n$	$\asymp 8^n$	
step set			
	2-clump permutations	twisted Baxter	Baxter [Viennot'81]

Intermediate class:
weight 1 for SE-step:
"semi-strong" rectangulations
semi-Baxter

- 1-sided rect. (= weak rect. with unique peeling) [Asinowski, Cardinal, Felsner, F'25]

allowed steps after 

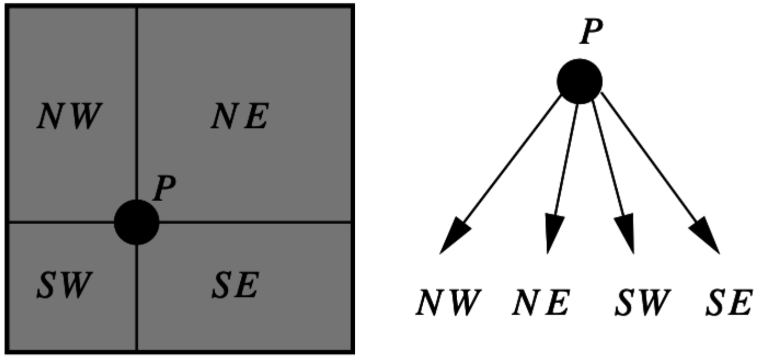
fully Baxter [Bouvel, Guerrini, Rinaldi'19]

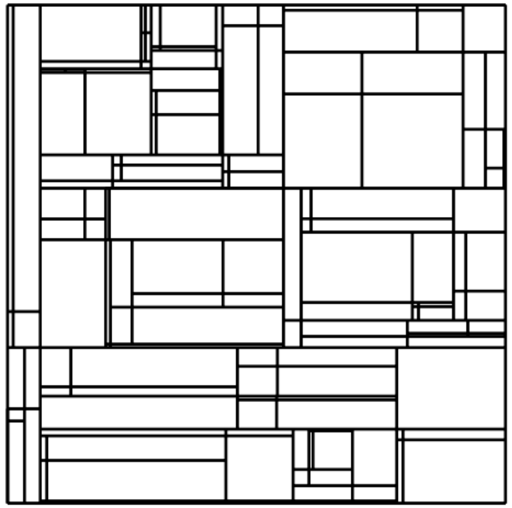
Random rectangulations

Rectangulations from random point-sets

- quadtrees

Flajolet-Sedgewick





Rectangulations from random point-sets

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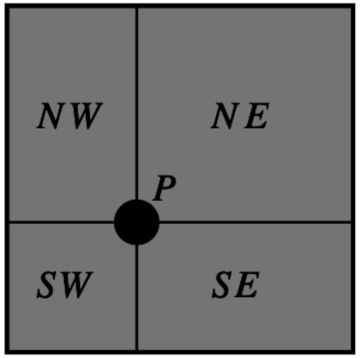
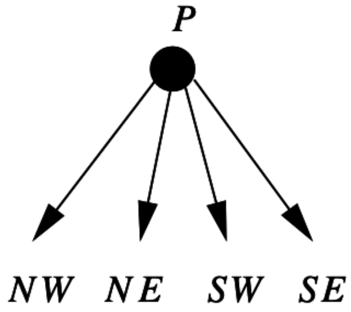
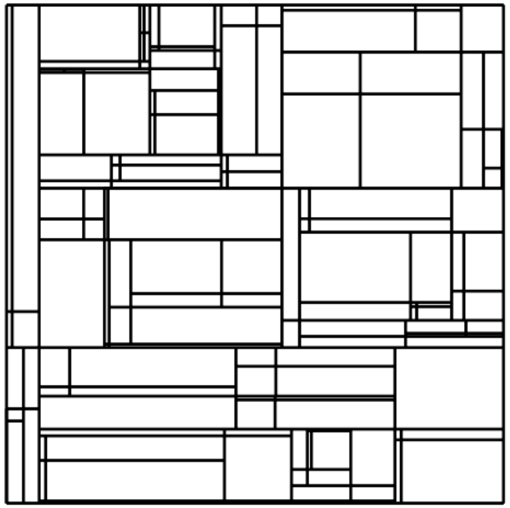


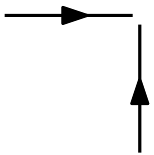


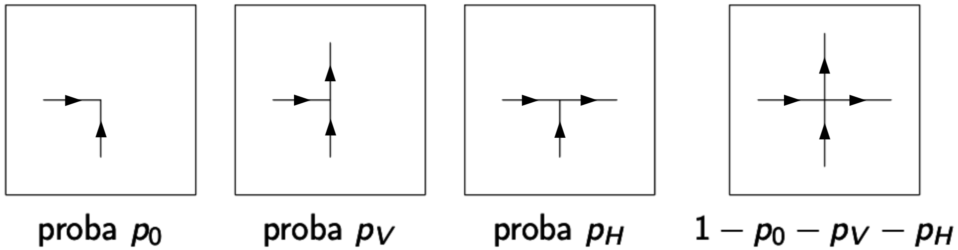


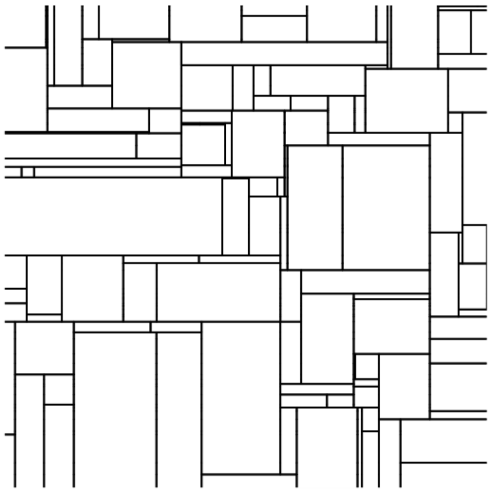
- balistic models

[Briquet,Chassaing,Gerin,Krikun,Popov'15] Alea'15
[Casse'25] Alea'25

colliding particles



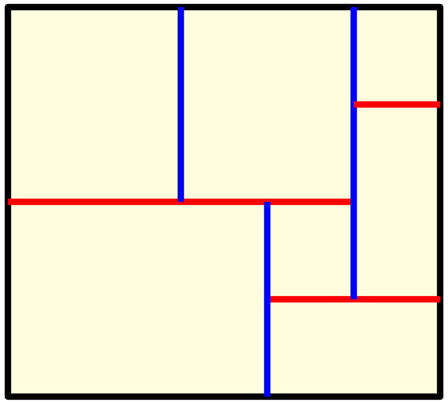


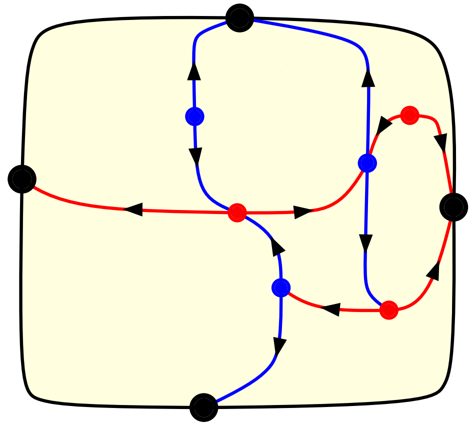


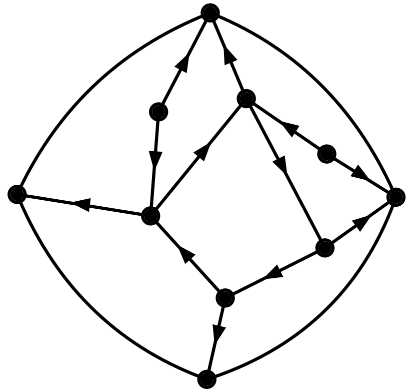
$p_V = p_H = 1/2$

Random combinatorial rectangulations

- **Weak** rectangulations: contact-systems of **segments**

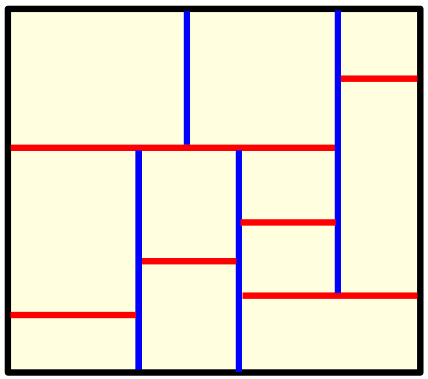


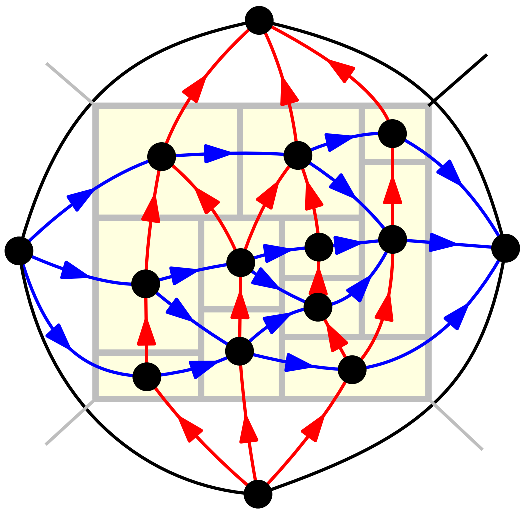


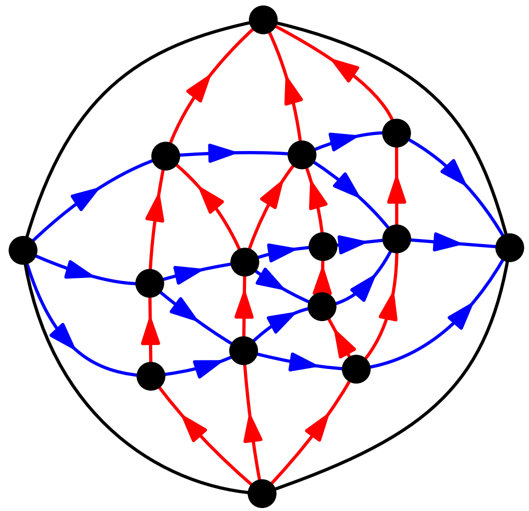


simple quadrangulation
+ 2-orientation

- **Strong** rectangulations: contact-systems of **regions**



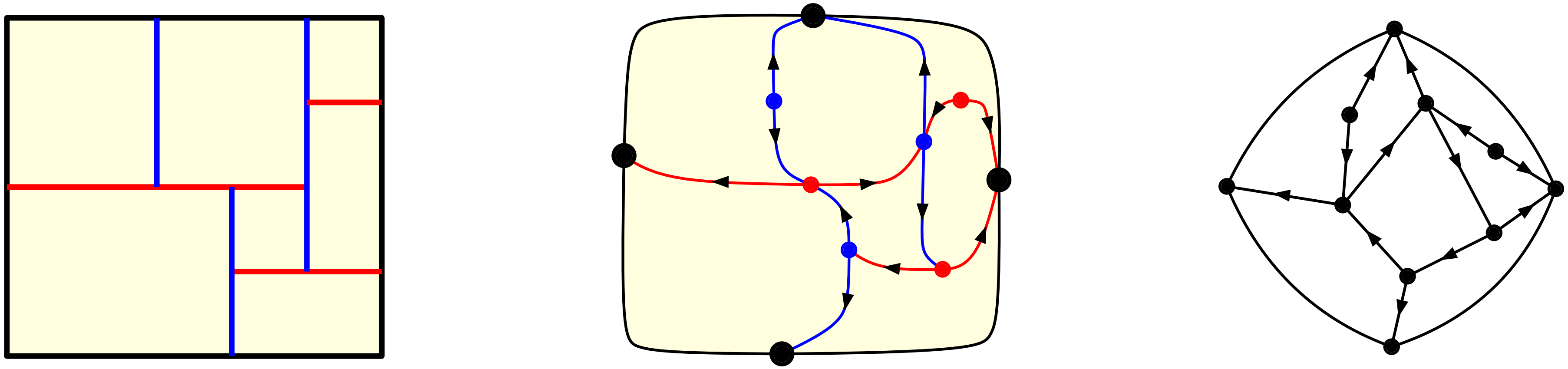




irreducible triangulation
+ transversal structure

Random combinatorial rectangulations

- **Weak** rectangulations: contact-systems of **segments**



simple quadrangulation
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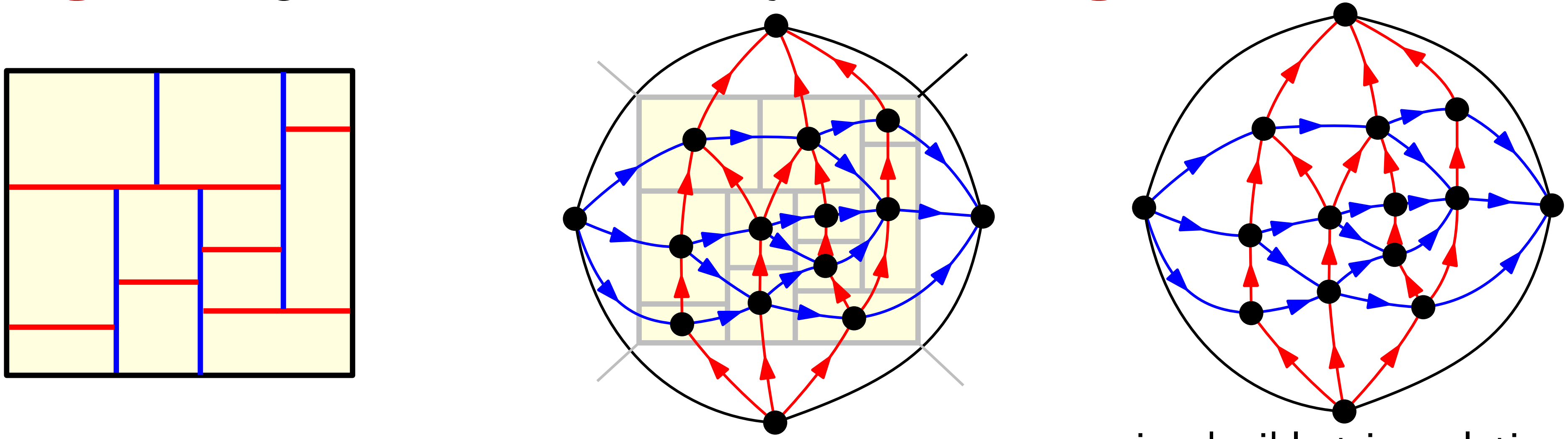


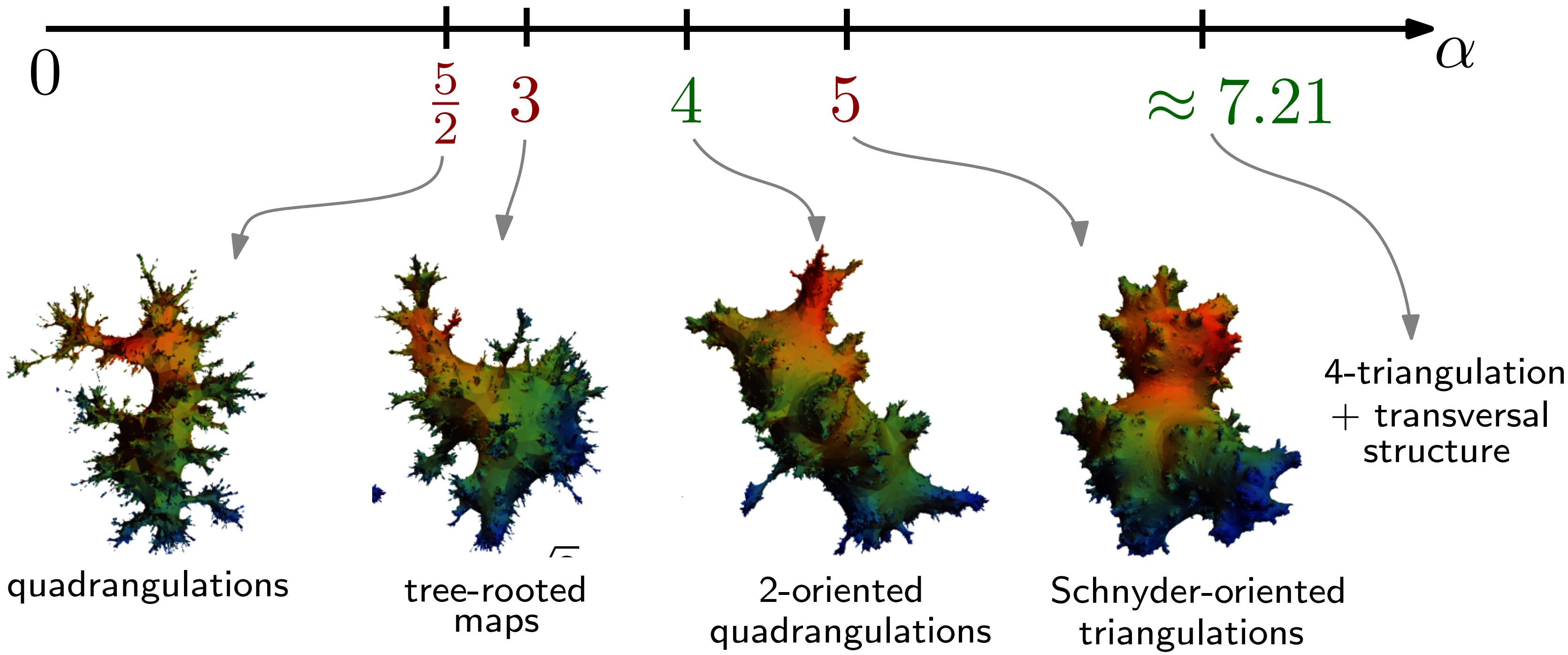
irreducible triangulation
+ transversal structure

⇒ incidences of random rectangulation form a random (decorated) planar map

Scaling order for distances

Universality class (central charge) indicated by **subexponential exponent**

$$a_n \sim \; c\,\gamma^n\, n^{-\alpha}$$



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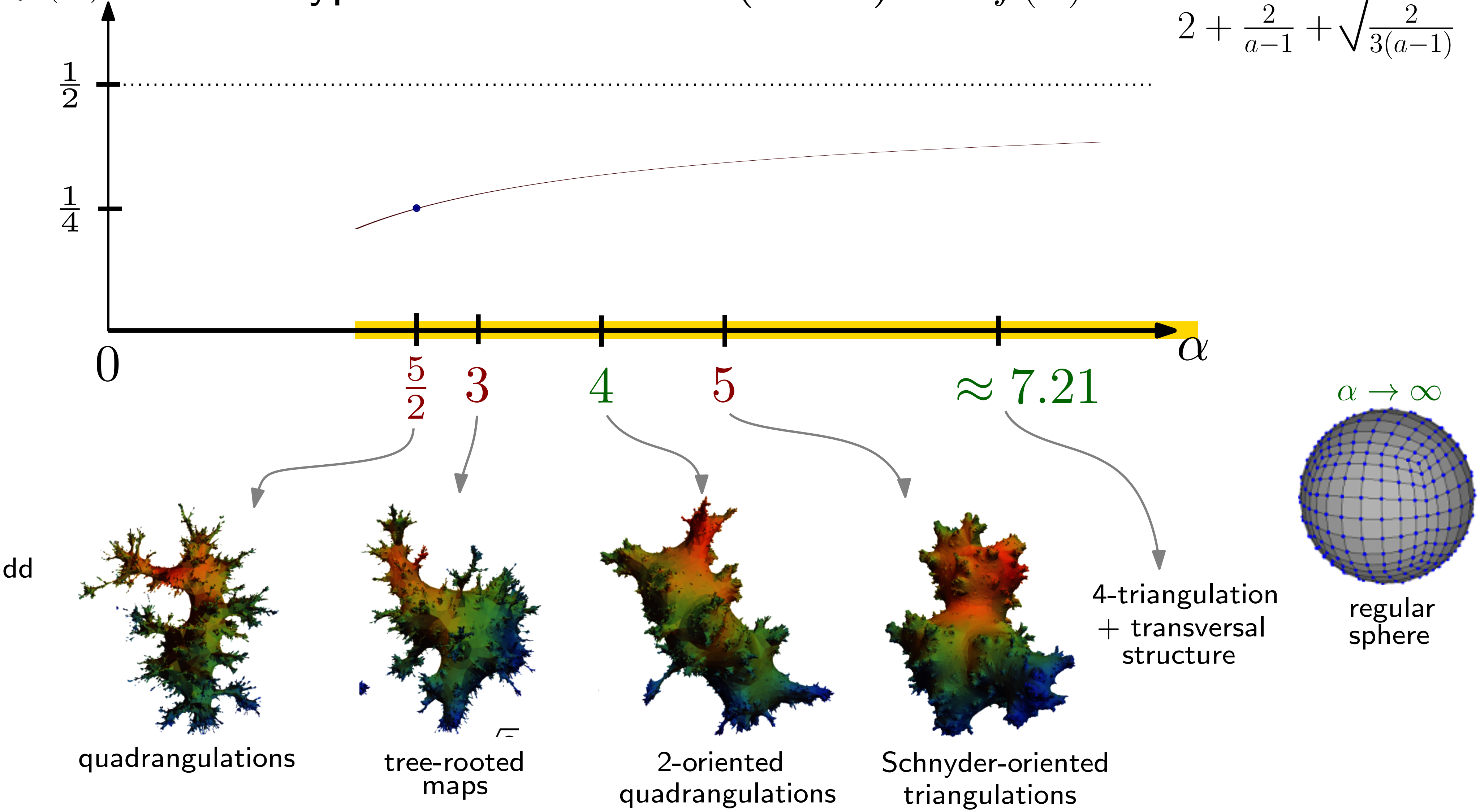
Prediction (& bounds): [Watabiki'93], [Ding, Gwynne'18], [Barkley, Budd'19]

typical distances $= \Theta(n^{f(\alpha)})$

$$f(\alpha) = \frac{1}{2 + \frac{2}{a-1} + \sqrt{\frac{2}{3(a-1)}}}$$



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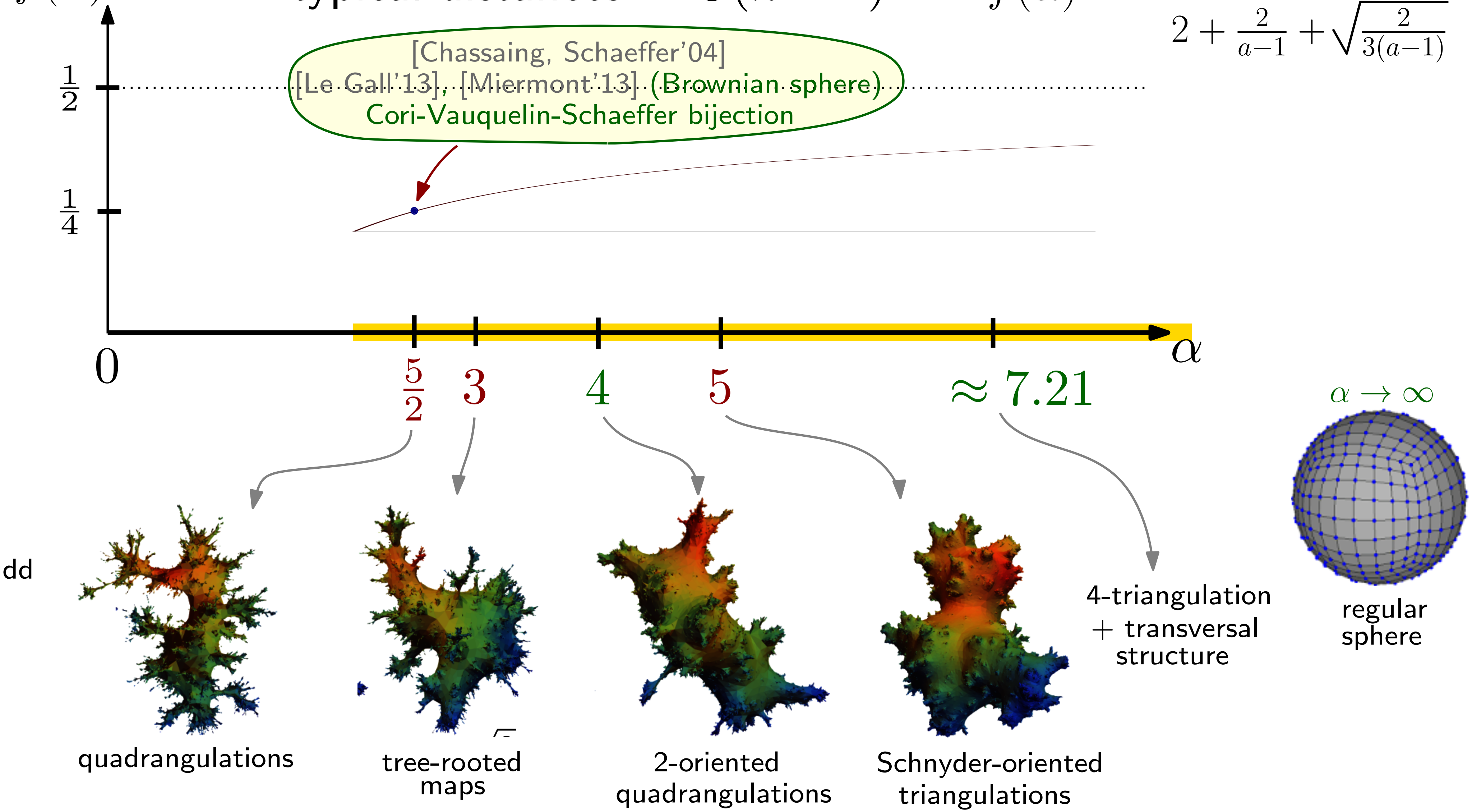

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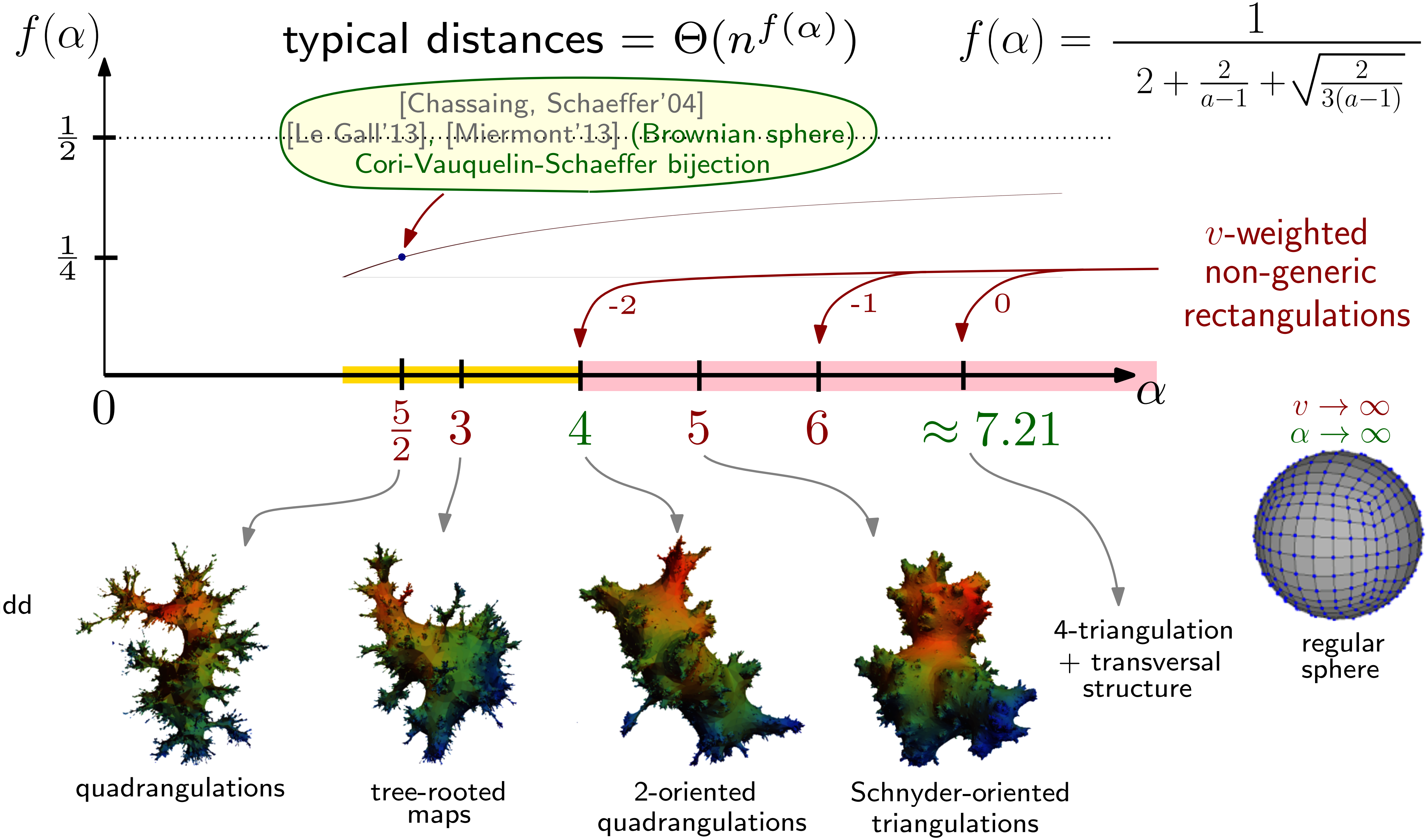
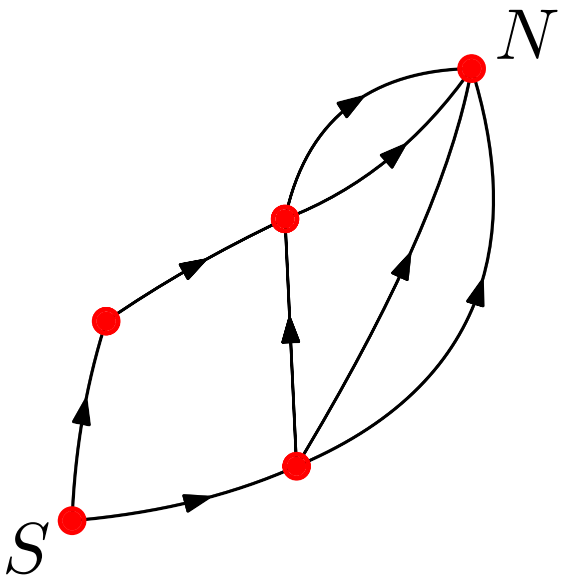


Scaling order for longest path length

In size n

X_n = **length longest walk** from S to N in random bipolar orientation



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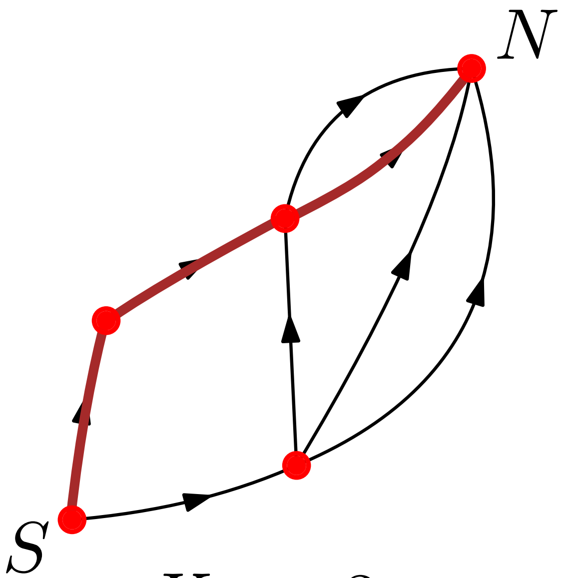

$X_n = 3$

Scaling order for longest path length

In size n

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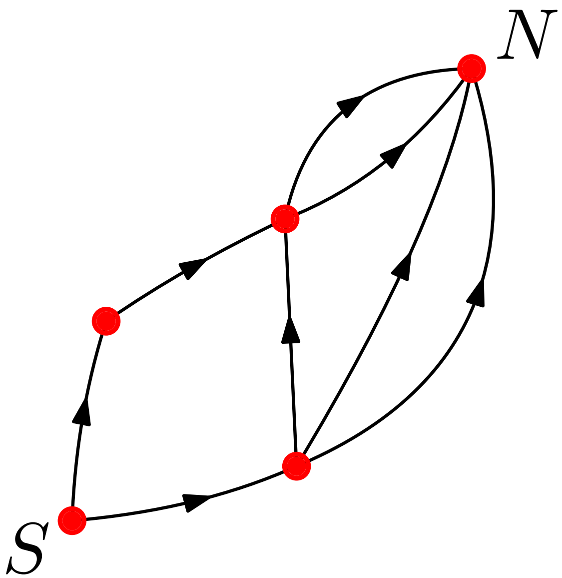
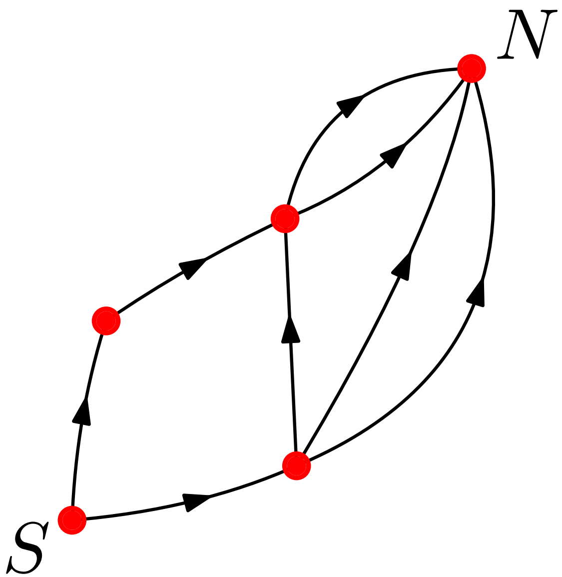


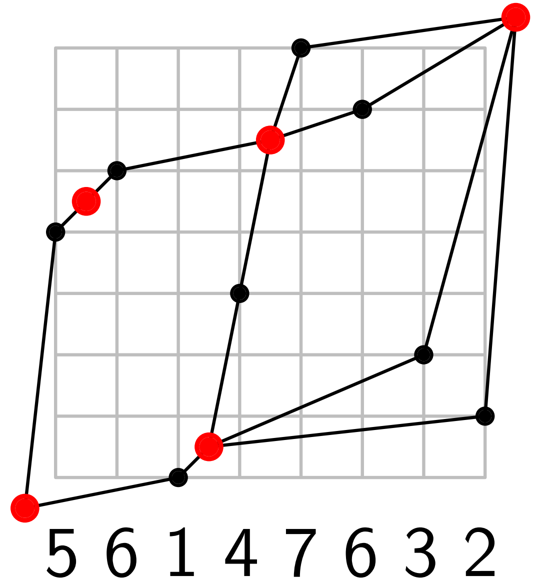
Scaling order for longest path length

In size n

X_n = **length longest walk** from S to N in random bipolar orientation

= size **longest increasing subsequence** in random Baxter permutation

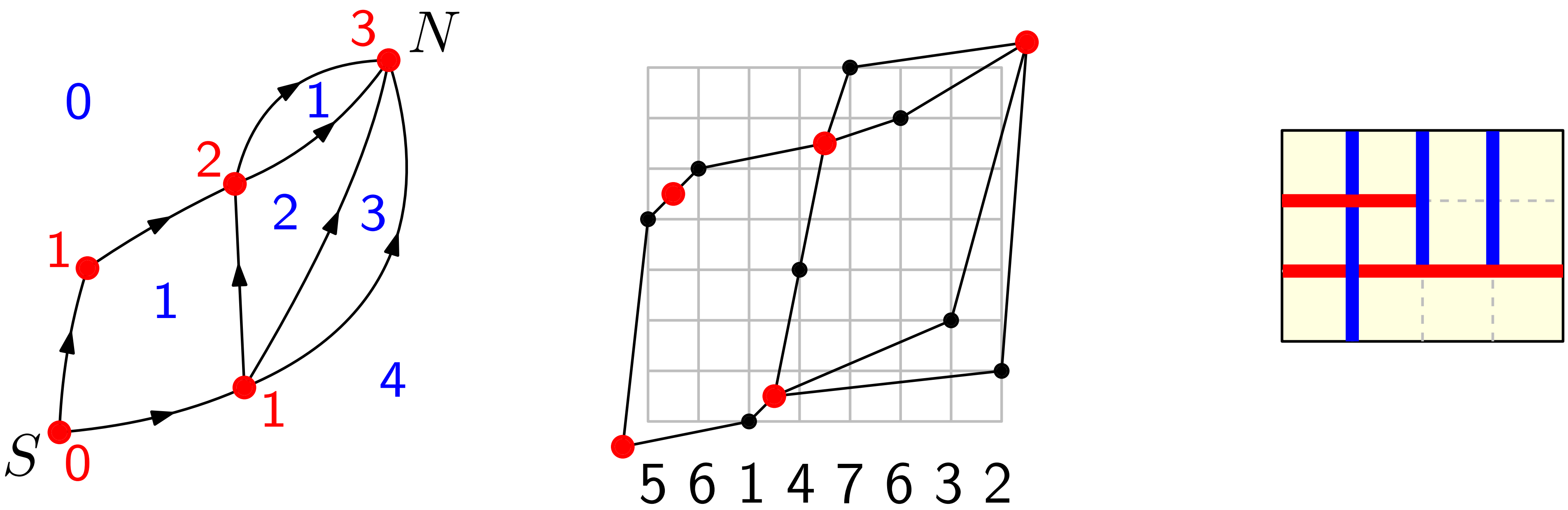




Scaling order for longest path length

In size n

X_n = **length longest walk** from S to N in random bipolar orientation
= size **longest increasing subsequence** in random Baxter permutation
= **height of most compact drawing** of random weak rectangulation



Scaling order for longest path length

In size n

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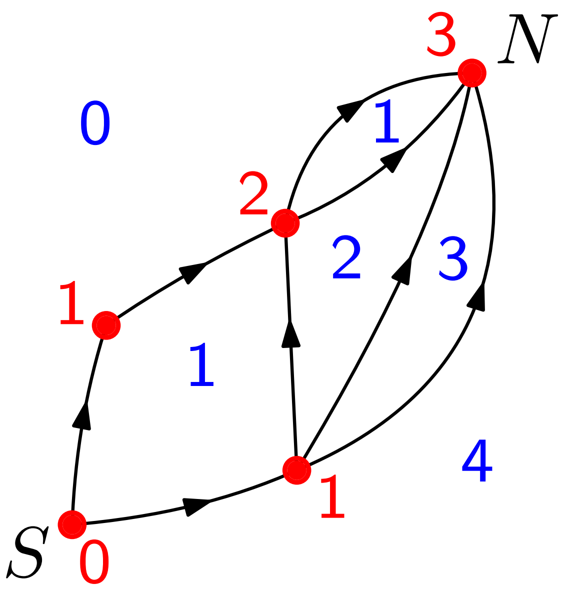

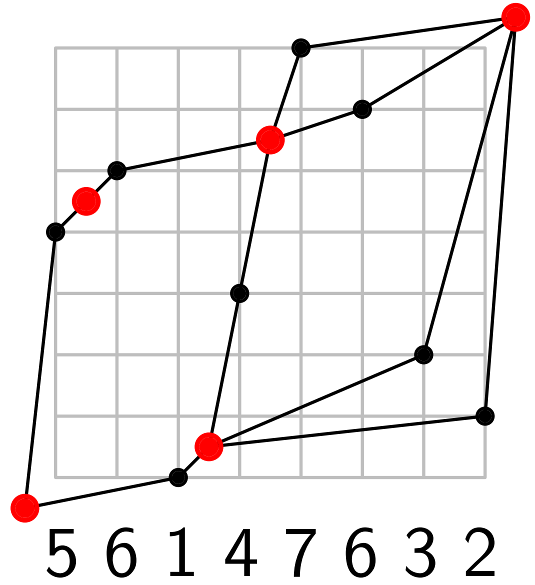

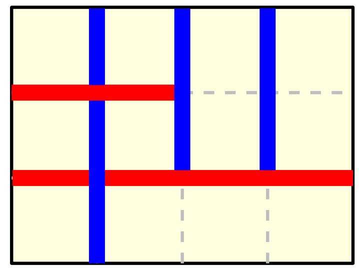

Conjecture: [Borga, Gwynne'25]

$X_n = \Theta(n^{3/4})$ in probability (proved in closely related models)

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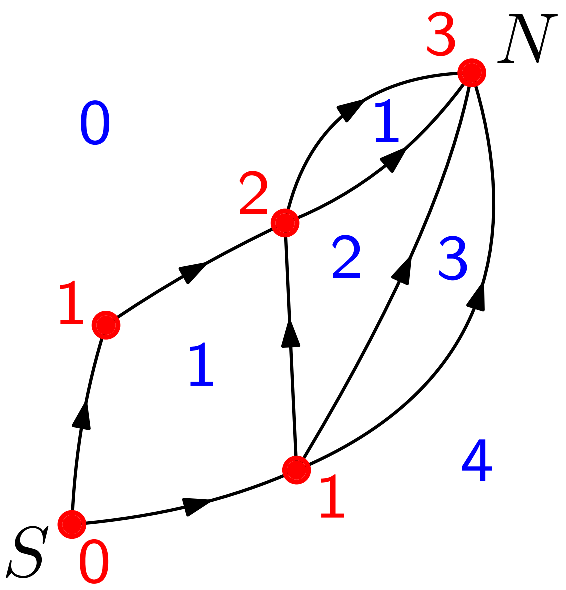

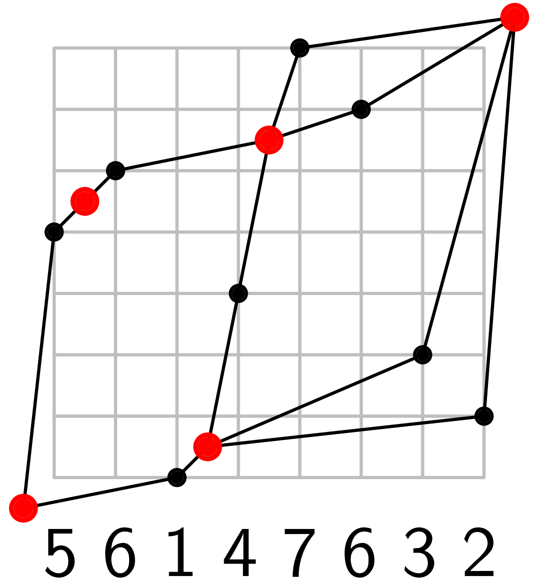

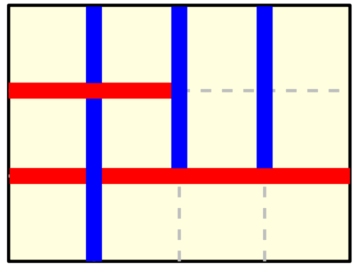

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$X_n = \Theta(n^{3/4})$ in probability (proved in closely related models)

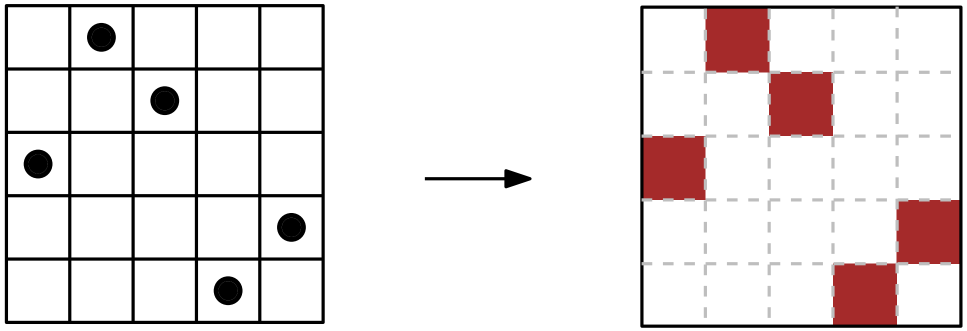
Conjecture: [Adhikari, Borga, Budzinski, Da Silva, Sénizergues'25]

for random separable permutation, $\frac{X_n}{n^\alpha} \to X$ for $\alpha \approx 0.81522$ (analytic expression)

proved for n-permutation sampled from separable permuton

Permuton limits

Permuton = measure on $[0,1]^2$ whose x- and y-marginals are uniform on $[0,1]$



Permuton limits

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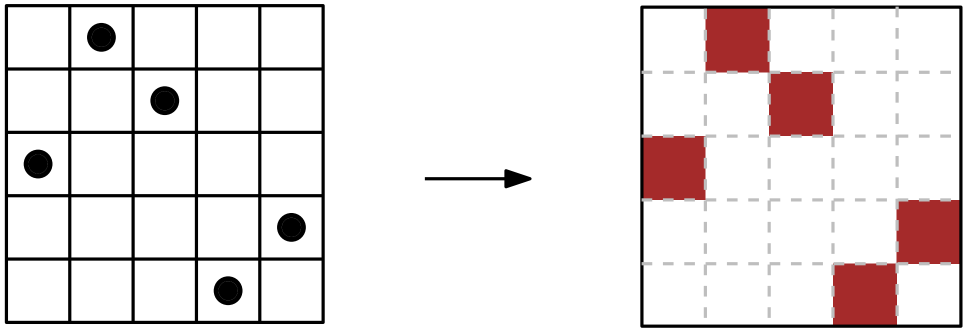
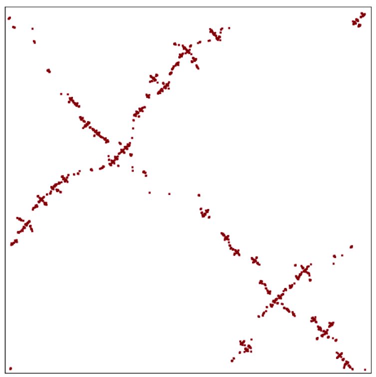


Explicit permuton limits for

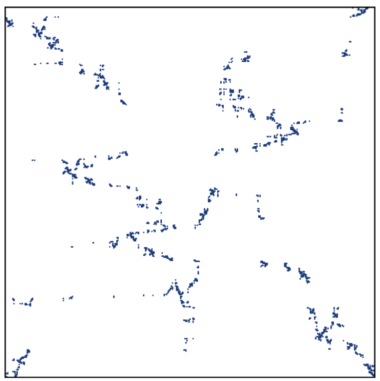
random separable permutations

[Bassino et al.'18]



random Baxter permutations

[Borga, Maazoun'22]



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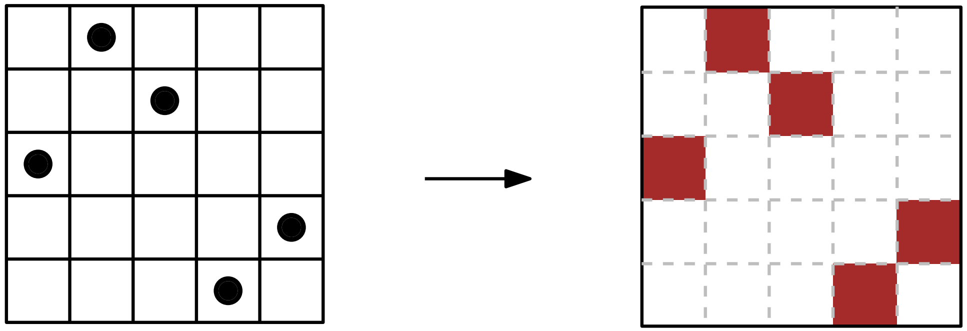

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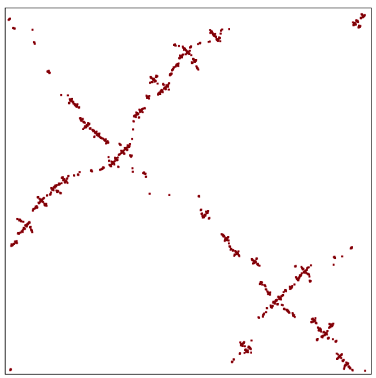
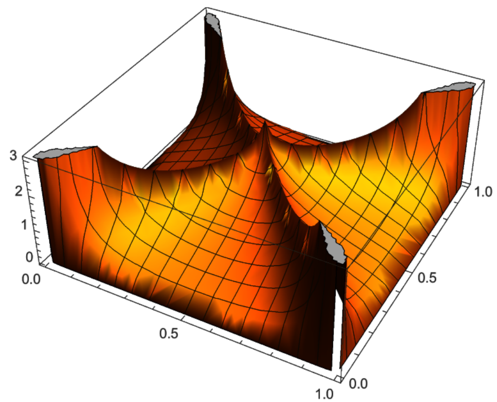
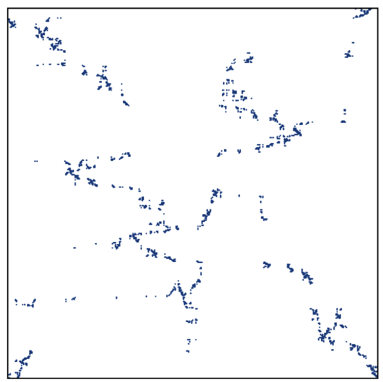
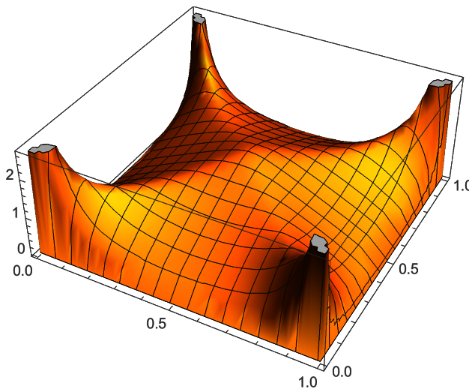




[Maazoun'20]

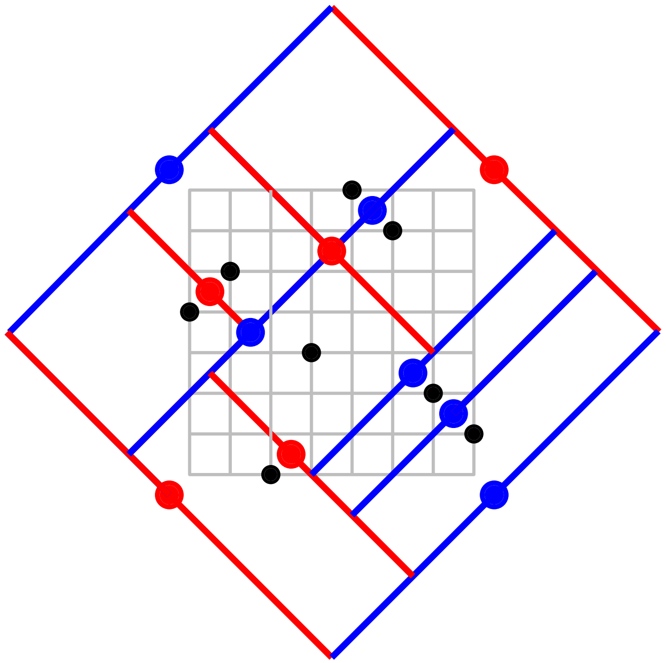




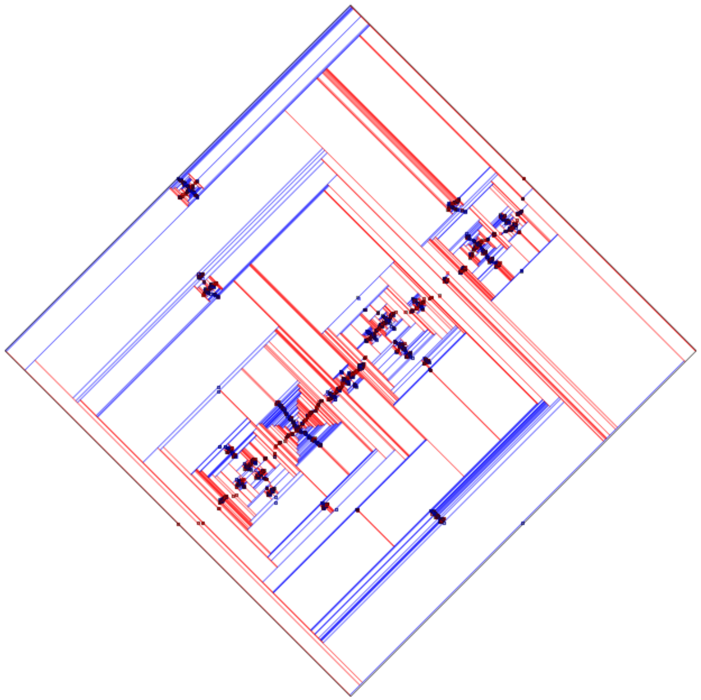
[Borga, Holden, Sun, Yu'23]

Expected occupancy density have explicit expressions

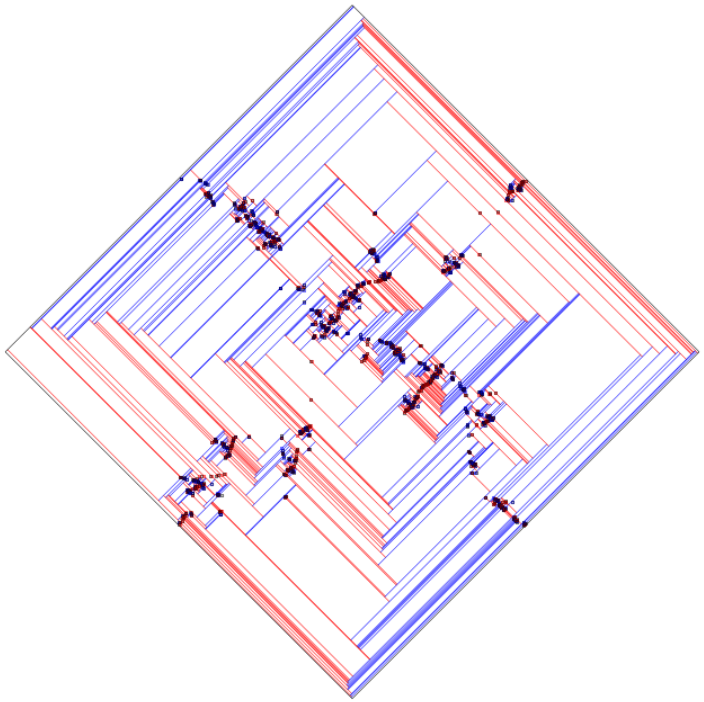
Simulations



random guillotine rectangulation



random weak rectangulation



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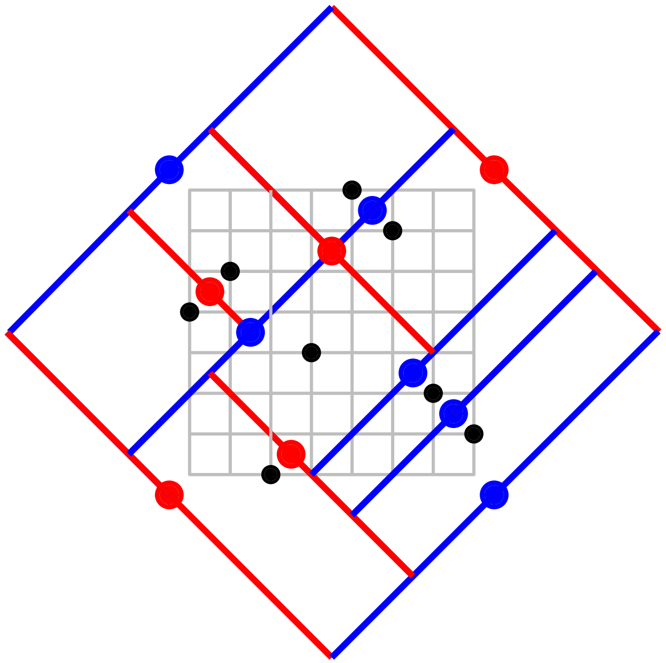

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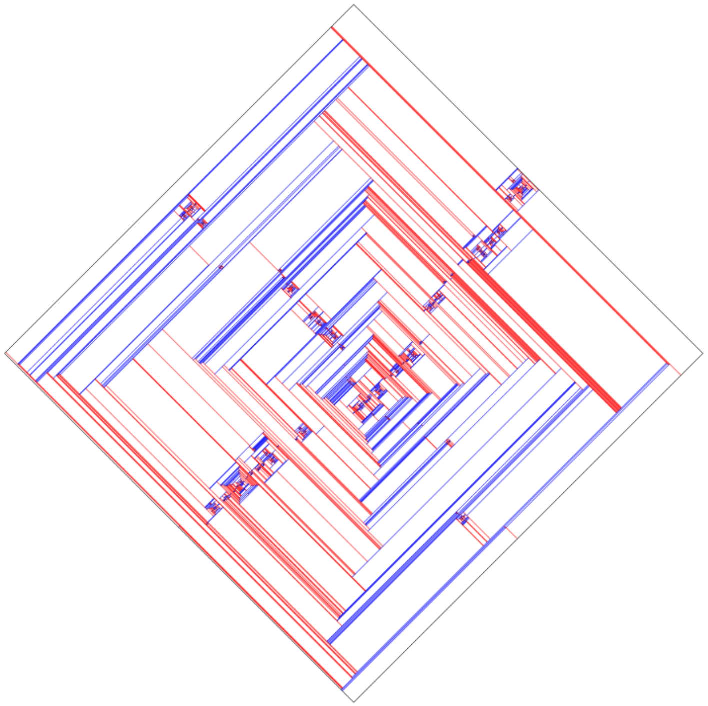

random weak rectangulation