Exercises on rectangulations, Alea 2026

Exercise 1: Baxter generating tree

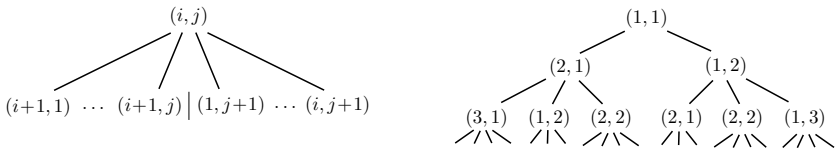

Figure 1: Left: the Baxter succession rule. Right: the first 3 levels of $\mathcal{T}_{\text{Baxt}}$.

The Baxter generating tree $\mathcal{T}_{\text{Baxt}}$ is the (infinite) rooted plane tree where every node is labeled by a pair (i, j) of positive integers, such that the root has label $(1, 1)$, and for every node of label (i, j), the labels of the children obey the succession rule of Figure 1 left. The first 3 levels of $\mathcal{T}_{\text{Baxt}}$ are shown in Figure 1 right. The number of nodes at level $n \geq 1$ is called the Baxter number b_n. The sequence starts as $1, 2, 6, 22, 92, 422, 2074, 10754, 58202, 326240, \ldots$

A combinatorial family $\mathcal{B} = \cup_{n\geq 1}\mathcal{B}_n$ is called a Baxter family if one can associate to each object $\gamma \in \mathcal{B}_n$ a parent $\gamma' \in \mathcal{B}_{n-1}$ (for $n \geq 2$), and associate two explicit parameters i, j to γ (for $n \geq 1$), such that the parameter values of the children of γ obey the Baxter succession rule.

Q1. Baxter permutations are permutations avoiding the two vincular patterns $2\underline{41}3$ and $3\underline{14}2$ (in a pattern occurence the two middle elements of the pattern have to be consecutive). Show that Baxter permutations form a Baxter family.

(hint: one can try with i the number of left-to-right maxima and j the number of right-to-left maxima)

Solution. The parent is obtained by deleting n from the permutation, e.g. the parent of 41352 is 4132. One has to check that the parent of a Baxter permutation is still a Baxter permutation. This is done by an easy case inspection. The crucial point is that we delete the largest element (on the other hand, deleting an element that is not maximal nor minimal could make the obtained permutation non-Baxter, e.g. deleting 3 from the above Baxter permutation yields 4152 which normalizes as 3142, a non-Baxter permutation).

To find the children of a Baxter permutation σ, one has to characterize the so-called active spots, i.e., the spots where the insertion of $n+1$ keeps the permutation Baxter (i.e., does not create one of the two forbidden patterns). One checks that the active spots are those that are either just before a left-to-right maximum, or just after a right-to-left maximum: inserting in these spots does not creates a forbidden pattern, while inserting in the other spots creates one of the two forbidden patterns (see figure below). As also illustrated in the figure,

if we label each Baxter permutation by the pair (i, j) where i is the number of left-to-right maxima and j is the number of right-to-left maxima, then the labels of the children obey the Baxter succession rule.

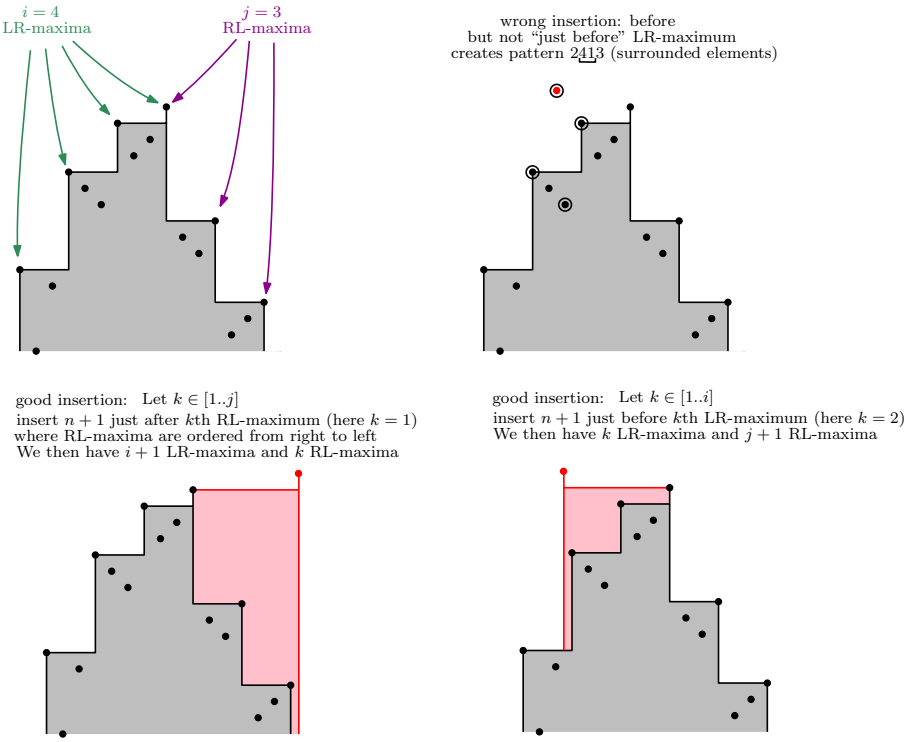

Figure 2: Situations when inserting $n+1$ to a spot in a Baxter permutation.

Q2. Show that weak rectangulations (with n the number of inner regions) are a Baxter family.

(hint: to define the parent, one can consider the rectangulation obtained by contracting the inner region incident to the top-left corner)

Solution. It is shown in Figure 3.

Q3. Let $\mathcal{P}_n$ be the set of triples of lattice walks P_1, P_2, P_3 each having $n-1$ steps that are East or North, have same starting point (the origin) and same ending point, and where P_2 (the middle walk) is weakly above P_1 (the lower walk) and weakly below P_3 (the upper walk). Show that $\mathcal{P} = \cup_n \mathcal{P}_n$ is a Baxter family.

Solution. The parent is obtained as follows. Delete the last step s of P_2. If s is East, we delete the last East step in P_1 and P_3. In P_3 it has to be the last step

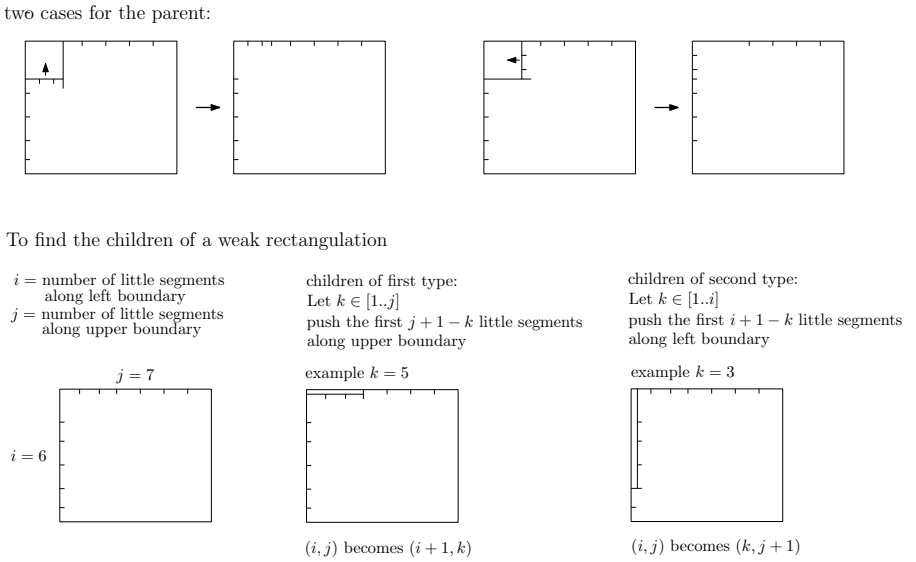

Figure 3: Baxter succession rule for weak rectangulations.

(since P_3 is weakly above P_2). In P_1 the last East step may not be the last step, it can be followed by a run of North steps. Similarly, if s is North, we delete the last North step in P_1 and P_3. In P_1 it has to be the last step (since P_1 is weakly below P_2). In P_3 the last North step may not be the last step, it can be followed by a run of East steps.

To find the children, given $(P_1, P_2, P_3) \in \mathcal{P}_n$, we let $i-1$ be the length of the final run of East steps in P_3, and let $j-1$ be the length of the final run of North steps in P_1. Children of the first type are obtained as follows. Let $k \in [1..j]$. Then we add a final East step to both P_2 and P_3. And in P_1 we insert an East step just before the last $k-1$ steps in the final run of North steps. In that way we obtain a triple in $\mathcal{P}_{n+1}$, and parameters (i, j) have become $(i+1, k)$. Children of the second type are obtained as follows. Let $k \in [1..i]$. Then we add a final North step to both P_1 and P_2. And in P_3 we insert a North step just before the last $k-1$ steps in the final run of East steps. In that way we obtain a triple in $\mathcal{P}_{n+1}$, and parameters (i, j) have become $(k, j+1)$.

Note: From the fact that $\mathcal{P}$ is a Baxter family, and the Gessel-Viennot lemma, one gets $b_n = \frac{2}{n(n+1)^2} \sum_{r=1}^{n} \binom{n+1}{r-1}\binom{n+1}{r}\binom{n+1}{r+1}$.

Exercise 2: Pattern avoidance

In a (weak) rectangulation, a horizontal cut is an inner horizontal segment connecting the left and right outer boundaries; a vertical cut is an inner vertical segment connecting the lower and top outer boundaries. A rectangulation is guillotine if either it is empty or it contains an horizontal or vertical cut, and the rectangulations in the two subregions (on each side of the cut) are guillotine. A windmill is a subset of 4 segments whose incidences are either ⊤⊥ or ⊥⊤.

Q1. Show that a rectangulation is guillotine iff it has no windmill.

Solution. We first check that a guillotine rectangulation R has no windmill. By definition it has to cut into two subrectangulations A and B (either separated by a vertical cut or by a horizontal cut). Clearly, if R has a windmill, it can not overlap over A and B, hence either A or B has it. By recursion on the size of guillotine rectangulations, we conclude that they can not have a windmill.

Conversely, assume there is a non-guillotine rectangulation having no windmill. We can consider a minimal such example. It has no vertical nor horizontal cut (indeed if it has such a cut, then on at least one side of the cut the subrectangulation is not guillotine, and would give a smaller example). Consider a lower contact whose vertical segment goes the highest among lower contacts. Let s_1 be this vertical segment. Its upper incident (horizontal) segment s_2 is not the upper boundary (since we have no vertical cut). Since there is no horizontal cut, at the left or right extremity s_2 touches a vertical segment s_3 that is not on the boundary. Assume without loss of generality that s_3 is at the left extremity of s_2. Then at the bottom s_3 touches an horizontal segment s_4 that is not the bottom boundary (otherwise s_4 would start from a lower contact and go higher than s_1). The configuration of s_1, s_2, s_3, s_4 at this stage is called a possibly uncomplete ⊥⊤. If the right extremity of s_4 touches s_1 then s_1, s_2, s_3, s_4 form a ⊥⊤. Otherwise it touches a vertical segment $s_5 \neq s_1$, but then s_2, s_3, s_4, s_5 form a possibly uncomplete ⊥⊤ (rotated by 90 degrees). This continues into a spiral which has to stop (since the size is finite), ending at a configuration where the 4 segments form a ⊥⊤. We reach a contradiction. Hence non-guillotine rectangulations have to contain a windmill.

For two permutations σ, σ' of respective sizes n, n', let $\tau = \sigma \oplus \sigma'$ be the permutation of size $n + n'$ given by $\tau(i) = \sigma(i)$ for $i \in [1..n]$ and $\tau(i) = \sigma'(i) + n$ for $i \in [n + 1, n + n']$; and let $\tau = \sigma \ominus \sigma'$ be the permutation of size $n + n'$ given by $\tau(i) = \sigma(i) + n'$ for $i \in [1..n]$ and $\tau(i) = \sigma'(i)$ for $i \in [n + 1, n + n']$. A permutation τ is called separable if it can be written $\tau = \sigma \oplus \sigma'$ or $\tau = \sigma \ominus \sigma'$ such that σ and σ' are separable.

Q2. Show that a permutation is separable iff it avoids the patterns 2413 and 3142.

Solution. Let τ be a separable permutation, hence there exist two permutations σ, σ' such that $\tau = \sigma \oplus \sigma'$ or $\tau = \sigma \ominus \sigma'$. If τ has a pattern 2413 or 3142, clearly the pattern can not overlap over σ and σ'. Hence it is either in σ or in

σ'. By recursion on the size we conclude that separable permutations have no pattern 2413 nor 3142.

Conversely, assume there is a non-separable permutation τ having no pattern 2413 nor 3142, and take a minimal such example. Deleting the rightmost entry x thus gives a separable permutation $\overline{\tau}$. Assume $\overline{\tau}$ decomposes with $\oplus$, then we can write $\overline{\tau}$ as $\overline{\tau} = \sigma \oplus \sigma'$, where σ is either of size 1 or decomposes with a $\ominus$. One easily checks that the only possibilities for the value of x to have τ nonseparable is that x forms a pattern 312 with two entries in σ. But if we add an arbitrary entry from σ' to these three entries, we obtain a pattern 3142, reaching a contradiction. Similarly, if $\overline{\tau}$ decomposes with $\ominus$ then we can find a pattern 2413 containing x. We conclude that non-separable permutations have to contain 3142 or 2413.

Exercise 3: Non-generic rectangulations

In the strong setting (i.e., taking into account the adjacencies of regions) a non-generic rectangulation[1] is allowed to have vertices where 4 rectangles meet, these are called cross-vertices. Let $a_{n,k}$ be the number of non-generic rectangulations with n inner regions and k cross-vertices. Let $A_n(v) = \sum_k a_{n,k} v^k$ (note that $A_n(0) = a_{n,0}$ gives the number of strong rectangulations of size n). For instance we have $A_8(v) = 26194 + 9220v + 770v^2 + 10v^3$.

Q1. Show that the polynomial $A_n(v-2)$ has non-negative coefficients (for instance $A_8(v-2) = 10754 + 6260v + 710v^2 + 10v^3$), that $A_n(-2)$ gives the number of weak rectangulations of size n, and that $A_n(-1)$ is the counting sequence for an explicit subfamily of strong rectangulations of size n.

(hint: it can help to consider reduction operations where certain pairs of adjacent vertices of degree 3 are merged into a cross-vertex)

Solution. We define a *reduction* as an operation turning a into or turning into , by contraction of the little segment in the middle. These operations can be performed independently and simultaneously (performing a reduction does not create new possible reductions); we obtain a non-generic rectangulation of the same size n (the number of inner regions does not change), and with no nor . Conversely, for a non-generic rectangulation with no nor , to find all the preimages, we may choose, for each , whether to keep it unchanged, or expand it into a , or expand it into a .

Thus, if we let $B_n(v)$ be the restriction of $A_n(v)$ to the non-generic rectangulations with no nor , then we have $A_n(v) = B_n(v+2)$, so that $A_n(v-2) = B_n(v)$.

[1] It should rather be called non-necessarily generic, but we use the shorter terminology.

Then $A_n(-2) = B_n(0)$ is the number of generic strong rectangulations with no $\begin{smallmatrix} \cdot & \cdot \\ & \cdot \end{smallmatrix}$ nor $\begin{smallmatrix} \cdot & \cdot \\ \cdot & \end{smallmatrix}$, which means that there is a canonical shuffle order for the contacts along horizontal and vertical segments (for vertical segments, the left contacts are below the right contacts, and for horizontal segments the upper contacts are left of the lower contacts). So $A_n(-2)$ is the number of weak rectangulations of size n.

Finally, $A_n(-1) = B_n(1)$ is the number of non-generic strong rectangulations with no $\begin{smallmatrix} \cdot & \cdot \\ & \cdot \end{smallmatrix}$ nor $\begin{smallmatrix} \cdot & \cdot \\ \cdot & \end{smallmatrix}$. We may then turn any $+$ into $\begin{smallmatrix} \cdot & \cdot \\ & \cdot \end{smallmatrix}$. This gives a bijection with generic strong rectangulations with no $\begin{smallmatrix} \cdot & \cdot \\ \cdot & \end{smallmatrix}$. Here we have canonical shuffle order for the contacts along the horizontal segments only (the upper contacts are left of the lower contacts). We can call these semi-strong generic rectangulations.

Exercise 4: Square tiling with a prescribed dual map

As shown in the figure below, a square tiling of a rectangle gives rise either to:

- a 2-connected map M by associating a vertex to each maximal horizontal segment, and an edge to each region (connecting the vertices at the two horizontal boundaries of the region).
- a triangulation T of the 4-gon by taking the dual map.

Using the interpretation as an electrical network one shows that a 2-connected map M has a unique square tiling representation (computed using Kirchhoff's laws).

This exercise considers the analogous question for triangulations, i.e., for T a triangulation of the 4-gon, with W, N, E, S its four outer vertices, realize T as the dual of a square tiling. The exercise aims at showing that the square tiling is unique and characterized in terms of a certain metric optimization problem.

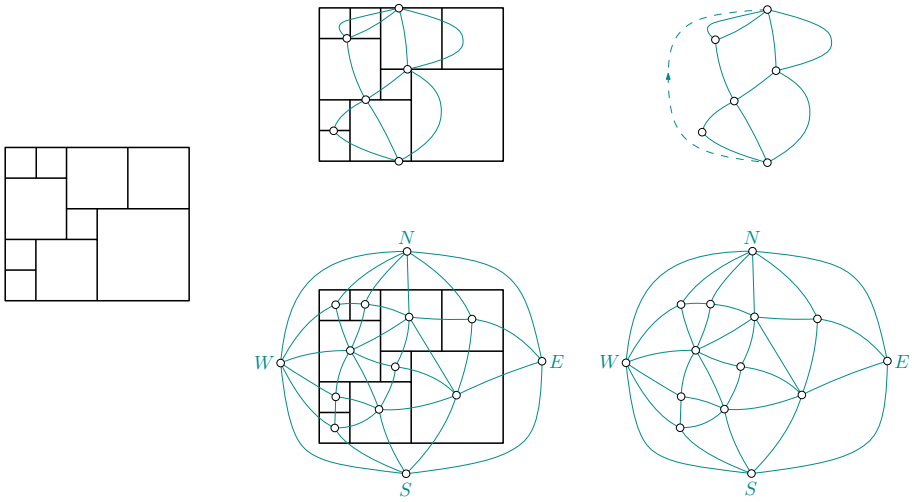

Figure 4: Left: a square tiling. Top: the associated 2-connected map. Bottom: the dual triangulation of the 4-gon.

Let V be the set of internal vertices of T. A *metric* on T is a mapping $m(\cdot)$ from V to non-negative values that are not all zero. The norm $||m||$ of m is given by $||m||^2 = \sum_{v \in V} m(v)^2$. An (S, N)-path is a path γ from S to N passing by internal vertices only, and the m-length of γ, denoted $L_m(\gamma)$, is the sum of the m-values of the visited internal vertices. The (S, N)-length of m is then defined as

$$\ell_m := \min L_m(\gamma),$$

where the minimum is taken over all (S, N)-paths. The metric is called optimal if it maximizes $\ell_m / ||m||$.

Q1. Show that there is a unique optimal metric up to scaling, i.e., up to multiplication of all values by a same constant.

(Hint: consider the problem of minimizing $||m||$ under the constraint $\ell_m \geq 1$)

Solution. The domain $\{\ell_m \geq 1\}$ is a non-empty polyhedron (hence convex), and the function $||m||^2$ is convex, hence has a unique minimum on $\{\ell_m \geq 1\}$. This minimum has to satisfy $\ell_m = 1$ (if $\ell_m = a > 1$ one can divide all $m(.)$ values by a and obtain another solution for which $||m||$ is divided by a, contradicting the minimality). Hence it is the unique minimum of $||m||$ over $\ell_m = 1$. Equivalently it is the unique minimum up to scaling of the $m(.)$ values.

We now assume there is a square tiling with T as its dual. Let w be its width and h its height (the tiling spans $\{0 \leq x \leq w, \quad 0 \leq y \leq h\}$). Let $s(\cdot)$ be the metric assigning to each internal vertex the length of the associated square.

Q2. Show that $\ell_s = h$.

Solution. For $0 \leq t \leq w$ consider the (S, N)-path γ_t consisting of vertices that are dual to the squares intersected by the vertical segment $\{x = t\}$ (assuming t avoids the exceptional values where the line $\{x = t\}$ traverses a vertical side of a region). Then clearly $L_s(\gamma_t) = h$. Hence $\ell_s \leq h$. Consider any (S, N)-path $\gamma = (v_0 = S, \ldots, v_k = N)$, and for $1 \leq j \leq k-1$ let I_j be the vertical segment that is the projection of the square dual to v_j to the y-axis. We have that I_1 has its lower extremity at $y = 0$, that I_{k-1} has its upper extremity at $y = h$, and that two consecutive segments I_j, I_{j+1} have to at least touch (or overlap). Hence the sum of the lengths of $I_1, \ldots, I_{k-1}$ is at least h. But the sum of these lengths is also $L_s(\gamma)$. Hence $\ell_s \geq h$. Hence $\ell_s = h$.

Q3. Let m be a metric on T. For $t \in [0, w]$, consider the (S, N)-path γ_t consisting of vertices that are dual to the squares intersected by the vertical segment $\{x = t\}$ (assuming t avoids the exceptional values where the line $\{x = t\}$ traverses a vertical side of a region). Show that

$$\int_0^w L_m(\gamma_t) = \sum_{v \in V} m(v)s(v).$$

Solution. Let v be an inner vertex of T. Then $m(v)$ contributes to $L_m(\gamma_t)$ over an interval of length $s(v)$ (the interval of values of t where $\{x = t\}$ meets the square dual to v). Hence v has contribution $m(v)s(v)$ to the integral.

Q4. Show that s is the optimal metric of T.

Solution. We have $\int_0^w L_m(\gamma_t) \geq \ell_m \cdot w$. By Cauchy-Schwartz inequality, we have $\sum_{v \in V} m(v)s(v) \leq ||m|| \cdot ||s||$. Hence $\ell_m \cdot w \leq ||m|| \cdot ||s||$, so that $w \cdot \ell_m / ||m|| \leq ||s||^2 / ||s|| = wh/||s||$, hence $\ell_m / ||m|| \leq \ell_s / ||s||$, so that $s(\cdot)$ is optimal.

Q5. Show that the dual square tiling representation is unique (for a given triangulation T of the 4-gon).

Solution. For each inner vertex v of T, the so-called upper neighbours of v are those dual to squares that touch the square of v from above. These can be read

from the optimal metric $s(\cdot)$ as those that are just after v on an (S, N)-path of length ℓ_s. Similarly the lower neighbours of v are those dual to squares that touch the square of v from below. These can be read from the optimal metric $s(\cdot)$ as those that are just before v on an (S, N)-path of length ℓ_s. Then the left neighbours of v are those between the group of lower neighbours and the group of upper neighbours in clockwise order around v, and the right neighbours of v are those between the group of upper neighbours and the group of lower neighbours in clockwise order around v. Hence, the optimal metric (which is unique) does not only determine the size of the squares dual to the inner vertices, but also determines at which side they touch. Hence the positions of the squares propagate in a unique way (e.g. starting from those incident to the lower boundary).

Note: This exercise is extracted from the article *Square tilings with prescribed combinatorics* by Oded Schramm, where it is shown that the optimal metric gives rise to a valid square tiling representation (with possible degeneracies, squares reduced to a point). This approach can also be applied to obtain the square tiling of a 2-connected map, by assigning weights to edges instead of vertices. However the approach based on electrical networks is simpler, as the solution is found by solving a linear system.