Aléa Mini-course on generating trees

Mathilde Bouvel, March 2022

Slides of the mini-course (preliminary version) are available at
`http://user.math.uzh.ch/bouvel/presentations/genTrees_Alea2022.pdf`.

1 Exercises

1.1 In the Catalan space

1.1.1 Enumeration of $Av(312)$

Find a proof without generating tree that $Av(312)$ is enumerated by the Catalan numbers.

Hint: Considering the final element of a permutation avoiding 312, decompose it into two blocks which are themselves permutations avoiding 312.

1.1.2 A generating tree for $Av(321)$ (from[Wes95] or [BM02], up to symmetry)

Letting permutations grow on the right, show that the generating tree for $Av(321)$ also corresponds to the succession rule Ω_{Cat}.

1.2 In the Schröder space (from [BDLPP99])

A *Schröder path* of size n is a lattice path starting at $(0,0)$, ending at $(2n,0)$, using up steps $(1,1)$, down steps $(1,-1)$ and double flat steps $(2,0)$, and staying weakly above the x-axis.

We want to find a generating tree for these paths, a corresponding succession rule, and then derive their generating function using the kernel method.

For this purpose, it is useful to rephrase the generating tree for Dyck path seen in the course. We have seen that the children of a Dyck path are obtained by inserting a peak in every point of the last descent. It can be equivalently presented as inserting *an up step* in every point of the last descent, and a new down step at the end of the path.

1. Adapting the definition of last descent, and the possible insertions in the points of the last descent, find a way to generate *exactly once* all Schröder path of size $n+1$ from those of size n.

   Hint: Be careful on where to insert the double flat step. You should do it in a way that allows to uniquely define the parent of a path.

2. The construction in the previous question defines a generating tree for Schröder paths. Show that it is encoded by the following rewriting rule:

$$\Omega_{Sch} = \begin{cases} (2) & \\ (k) & \rightsquigarrow (k+1),(3),\dots,(k),(k+1). \end{cases}$$

3. Let $(s_{n,k})_{n\geq 0,k\geq 2}$ be the number of Schröder paths of size n and label k.
   Let $S(x,y) = \sum_{n\geq 0,k\geq 2} s_{n,k}x^n y^k$ be the bivariate generating function of Schröder paths.
   Translating the rewriting rule Ω_{Sch} into an equation for $S(x,y)$ show that

$$K(x,y)S(x,y) = y^2 - y^3 + xy^3 S(x,1),$$

   where $K(x,y) = 1 - y - xy + 2xy^2$.

4. Find the formal power series $Y(x)$ canceling the kernel $K(x,y)$. Substituting it in the equation above, show that $S(x,1)$ is the generating function of (large) Schröder numbers, namely:

$$S(x,1) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x}.$$

1.3 In the Motzkin space (from [BDLPP99])

A *Motzkin path* of size n is a lattice path starting at $(0,0)$, ending at $(\mathbf{n},\mathbf{0})$, using up steps $(1,1)$, down steps $(1,-1)$ and flat steps $(\mathbf{1},\mathbf{0})$, and staying weakly above the x-axis.

1. Adapting the construction of the previous exercise, show that a rewriting rule for Motzkin paths is

$$\Omega_{Mot} = \begin{cases} (2) \\ (k) \rightsquigarrow (1), (2) \ldots, (k-1)(k+1). \end{cases}$$

   Hint: For the size to grow of exactly 1, we need to insert only one new step. But we may also simultaneously transform an existing step and add a new one.

2. Applying the kernel method, show that the generating function for Motzkin paths is that of the Motzkin numbers, namely,

$$M(x) = \frac{1 - x - 2x^2 - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

1.4 Generating trees where labels are pairs of integers

1.4.1 Baxter permutations (from [BM02])

A *Baxter permutation* is a permutation avoiding both patterns $2\underline{41}3$ and $3\underline{14}2$.

1. Adapting the construction for $2\underline{41}3$-avoiding permutations, find a generating tree for Baxter permutations. Show that the rewriting rule encoding this generating tree is

$$\Omega_{Bax} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1), \ldots, (h,k+1) \\ \qquad\qquad (h+1,1), \ldots, (h+1,k). \end{cases}$$

2. Can you find an equivalent characterization of the active sites (hence of the labels)? *(Note: answer to 2 is not necessary to solve 3!)*

   Hint: Use the characterization of the active sites in $Av(312)$ as inspiration.

3. Translate the rewriting rule into a functional equation for the trivariate generating function of Baxter permutations. Put this equation in kernel form.

4. Apply the obstinate kernel method to obtain an expression for the bivariate generating function of Baxter permutation according to the size and the number of RtoL-maxima.

1.4.2 Strong-Baxter permutations (from [BGRR18])

A *strong-Baxter permutation* is a permutation avoiding all three patterns $2\underline{41}3$, $3\underline{14}2$ and $3\underline{41}2$.

1. Similarly to the case of Baxter permutations, find a generating tree for these permutations. Find the rewriting rule encoding this generating tree.

2. Translate the rewriting rule into a functional equation for the trivariate generating function of $Av(2\underline{41}3, 3\underline{14}2, 3\underline{41}2)$. Put this equation in kernel form.

3. What do you observe when looking for the involutions leaving the kernel invariant?

1.5 Encoding of uniform pattern-avoiding permutations as conditioned random walks (from [Bor21a])

Recall that permutations in $Av(1423, 4123)$ grow according to the rewriting rule

$$\Omega_{Sch} = \left\{ \begin{array}{ll} (2) & \\ (k) & \rightsquigarrow (k+1)^B, (3), \dots, (k), (k+1)^T. \end{array} \right.$$

Therefore, there is a bijective correspondence between permutations σ of size n in $Av(1423, 4123)$ and sequences of n labels $(\ell_1, \ell_2, \dots, \ell_n)$ such that $\ell_1 = 2$ and ℓ_{i+1} is among the children produced from label ℓ_i according to Ω_{Sch}. Noticing that every label at least 3 produces exactly one child with label 3, we can equivalently encode σ of size ≥ 2 as the sequence of $n+1$ labels $(\ell_1, \ell_2, \dots, \ell_n, \ell_{n+1})$ with $\ell_{n+1} = 3$. We set $F(\sigma) = (\ell_1, \ell_2, \dots, \ell_n, \ell_{n+1})$.

Consider the set of jumps $\mathcal{J} = \mathbb{Z}_{\leq 0} \cup \{(+1)^B, (+1)^T\}$. Consider also the probability distribution $(\xi_y)_{y \in \mathcal{J}}$ on the jumps given by $\xi_y = p \cdot q^y$ for $p = 3 - 2\sqrt{2}$ and $q = \frac{1}{2-\sqrt{2}}$.

Let $\boldsymbol{\sigma_n}$ be a uniform random permutation among those of size n avoiding 4123 and 1423.

Show that $F(\boldsymbol{\sigma_n})$ has the same distribution as the random walk $(\boldsymbol{X}_i)_{i \geq 1}$ for the jump distribution given above, conditioned to take value 2 at time 1, value 3 at time $n+1$ and to take values at least 3 at any time $2 \leq t \leq n$.

2 Non-exhaustive list of references

- Early articles of J. West: [Wes95, Wes96], see also his PhD thesis [Wes90].
- Early articles of the Florentine group (R. Pinzani, E. Barcucci, A. Del Lungo, . . .): [BDLPP95, BDLPP98, BDLP99, BDLPP99], see also the survey [Pin05] (even just the one-page abstract which gives historical context).
- Examples with complicated rewriting rules (labels of varrying size): [DGW96, DGG98, Sta94, BG14]
- An (implicit) generating-tree bijection (between plane bipolar orientations and Baxter permutations) made explicit: [BBMF08].
- Enumeration by generating trees of many Schröder permutation classes: [Kre00, Kre03].
- Link between type of rewriting rule and nature of the generating function: [BBMD+02].
- Enumeration using generating trees where labels are pairs of integers: [BM02, BGRR18]
- Probabilistic results involving generating trees with one-dimensional labels: [Bor21a]
- Probabilistic results involving generating trees with two-dimensional labels: [BM22] (although the encoding as two-dimensional walks chosen is not described by generating trees) and [Bor21c, Bor21b]

References

[BBMD+02] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Disc. Math.*, 246:29–55, 2002.

[BBMF08] N. Bonichon, M. Bousquet-Mélou, and É. Fusy. Baxter permutations and plane bipolar orientations. *Séminaire Lotharingien de Combinatoire*, 61A, 2008. article [B61Ah].

[BDLP99] Elena Barcucci, Alberto Del Lungo, and Elisa Pergola. Random generation of trees and other combinatorial objects. *Theoretical Computer Science*, 218(2):219–232, 1999.

[BDLPP95] Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. A construction for enumerating k-coloured Motzkin paths. *Lecture Notes in Comput. Sci.*, 959:254–263, 1995.

[BDLPP98] Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. A methodology for plane tree enumeration. *Discrete Mathematics*, 180(1–3):45–64, 1998.

[BDLPP99] Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. ECO: A methodology for the enumeration of combinatorial objects. *J. Difference Equ. Appl.*, 5(4–5):435–490, 1999.

[BG14] Mathilde Bouvel and Olivier Guibert. Refined enumeration of permutations sorted with two stacks and a d_8-symmetry. *Annals of Combinatorics*, 18(2):199–232, 2014.

[BGRR18] Mathilde Bouvel, Veronica Guerrini, Andrew Rechnitzer, and Simone Rinaldi. Semi-Baxter and Strong-Baxter: two relatives of the Baxter sequence. *SIAM J. Discrete Math.*, 32(4):2795–2819, 2018.

[BM02] Mireille Bousquet-Mélou. Four classes of pattern-avoiding permutations under one roof: Generating trees with two labels. *Electr. J. Comb.*, on(2), 2002.

[BM22] Jacopo Borga and Mickaël Maazoun. Scaling and local limits of baxter permutations and bipolar orientations through coalescent-walk processes. *Ann. Probab.*, 2022. to appear.

[Bor21a] Jacopo Borga. Asymptotic normality of consecutive patterns in permutations encoded by generating trees with one-dimensional labels. *Random Struct Alg.*, 59:339–375, 2021.

[Bor21b] Jacopo Borga. The permuton limit of strong-baxter and semi-baxter permutations is the skew brownian permuton. Arxiv preprint, 2021.

[Bor21c] Jacopo Borga. The skew brownian permuton: a new universality class for random constrained permutations. Arxiv preprint, 2021.

[DGG98] S. Dulucq, S. Gire, and O. Guibert. A combinatorial proof of J. West's conjecture. *Discrete Mathematics*, 187:71–96, 1998.

[DGW96] S. Dulucq, S. Gire, and J. West. Permutations with forbidden subsequences and nonseparable planr maps. *Discrete Mathematics*, 153:85–103, 1996.

[Kre00] D. Kremer. Permutations with forbidden subsequences and a generalized Schröder number. *Discrete Mathematics*, 218(1–3):121–130, 2000.

[Kre03] D. Kremer. Postscript: "permutations with forbidden subsequences and a generalized Schröder number". *Discrete Mathematics*, 270(1–3):333–334, 2003.

[Pin05] Renzo Pinzani. ECO's carol. Invited talk at FPSAC 2005, 2005.

[Sta94] Zvezdelina E. Stankova. Forbidden subsequences. *Discrete Math.*, 132:291–316, 1994.

[Wes90] Julian West. *Permutations with forbidden subsequences and stack-sortable permutations*. PhD thesis, MIT, 1990.

[Wes95] Julian West. Generating trees and the Catalan and Schröder numbers. *Discrete Mathematics*, 146:247–262, 1995.

[Wes96] Julian West. Generating trees and forbidden subsequences. *Discrete Mathematics*, 157:363–374, 1996.