Around the Plancherel measure on integer partitions
(an introduction to Schur processes without Schur functions)

Jérémie Bouttier

A subject which I learned with Dan Betea, Cédric Boutillier, Guillaume Chapuy, Sylvie Corteel, Sanjay Ramassamy and Mirjana Vuletić

Institut de Physique Théorique, CEA Saclay
Laboratoire de Physique, ENS de Lyon

Aléa 2019, 20-21 mars
Part I

20 March 2019
What these lectures are about

In these lectures I present a very condensed version of some material which form the second part of a M2 course I gave in Lyon.
What these lectures are about

In these lectures I present a very condensed version of some material which form the second part of a M2 course I gave in Lyon.

This course was roughly based on Chapters 1 and 2 of Dan Romik’s beautiful book *The surprising mathematics of longest increasing subsequences* (available online).
What these lectures are about

In these lectures I present a very condensed version of some material which form the second part of a M2 course I gave in Lyon.

This course was roughly based on Chapters 1 and 2 of Dan Romik’s beautiful book *The surprising mathematics of longest increasing subsequences* (available online).

In the second part (Chapter 2) I somewhat diverged from the book by following my own favorite approach (developed mostly by Okounkov), based on fermions and saddle point computations for asymptotics.
What these lectures are about

In these lectures I present a very condensed version of some material which form the second part of a M2 course I gave in Lyon.

This course was roughly based on Chapters 1 and 2 of Dan Romik’s beautiful book *The surprising mathematics of longest increasing subsequences* (available online).

In the second part (Chapter 2) I somewhat diverged from the book by following my own favorite approach (developed mostly by Okounkov), based on fermions and saddle point computations for asymptotics.

This is the material I would like to present here: fermions because of physics, saddle point computations because, well, we are in Aléa!
Integer partitions and Young diagrams/tableaux

An (integer) partition $\lambda$ is a finite nonincreasing sequence of positive integers called parts:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0.$$ 

Its size is $|\lambda| := \sum \lambda_i$ and its length is $\ell(\lambda) := \ell$ (by convention $\lambda_n = 0$ for $n > \ell$).
Integer partitions and Young diagrams/tableaux

An (integer) partition $\lambda$ is a finite nonincreasing sequence of positive integers called parts:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0.$$ 

Its size is $|\lambda| := \sum \lambda_i$ and its length is $\ell(\lambda) := \ell$ (by convention $\lambda_n = 0$ for $n > \ell$). It may be represented by a Young diagram, e.g. for $\lambda = (4, 2, 2, 1)$:

```
1 2 5 7 3 6 9 4 8
```
Integer partitions and Young diagrams/tableaux

An (integer) partition $\lambda$ is a finite nonincreasing sequence of positive integers called parts:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0.$$ 

Its size is $|\lambda| := \sum \lambda_i$ and its length is $\ell(\lambda) := \ell$ (by convention $\lambda_n = 0$ for $n > \ell$). It may be represented by a Young diagram, e.g. for $\lambda = (4, 2, 2, 1)$:

A standard Young tableau (SYT) of shape $\lambda$ is a filling of the Young diagram of $\lambda$ by the integers $1, \ldots, |\lambda|$ that is increasing along rows and columns. We denote by $d_\lambda$ the number of SYTs of shape $\lambda$. 

\[ \begin{array}{cccc}
\Box & \Box & \Box & \Box \\
\Box & \Box & \Box & \Box \\
\Box & \Box & \Box & \Box \\
\Box & \Box & \Box & \Box \\
\end{array} \quad \begin{array}{cccc}
8 & 4 & 3 & 1 \\
9 & 6 & 2 & 5 \\
7 & & & \\
& & & \\
\end{array} \]
Plancherel measure

The Plancherel measure on partitions of size $n$ is the probability measure such that

$$\text{Prob}(\lambda) = \begin{cases} \frac{d^2_\lambda}{n!} & \text{if } \lambda \vdash n, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\lambda \vdash n$ is a shorthand symbol to say that $\lambda$ is partition of size $n$. 
Plancherel measure

The Plancherel measure on partitions of size $n$ is the probability measure such that

$$\text{Prob}(\lambda) = \begin{cases} \frac{d_\lambda^2}{n!} & \text{if } \lambda \vdash n, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\lambda \vdash n$ is a shorthand symbol to say that $\lambda$ is partition of size $n$. It is a probability measure because of the “well-known” identity

$$n! = \sum_{\lambda \vdash n} d_\lambda^2$$

which has (at least) two classical proofs:
Plancherel measure

The Plancherel measure on partitions of size $n$ is the probability measure such that

$$\text{Prob}(\lambda) = \begin{cases} \frac{d^2_\lambda}{n!} & \text{if } \lambda \vdash n, \\ 0 & \text{otherwise}. \end{cases}$$

Here $\lambda \vdash n$ is a shorthand symbol to say that $\lambda$ is partition of size $n$. It is a probability measure because of the “well-known” identity

$$n! = \sum_{\lambda \vdash n} d^2_\lambda$$

which has (at least) two classical proofs:

- representation theory: $n!$ is the dimension of the regular representation of the symmetric group $S_n$, and $d_\lambda$ is the dimension of its irreducible representation indexed by $\lambda$,
Plancherel measure

The Plancherel measure on partitions of size $n$ is the probability measure such that

\[
\text{Prob}(\lambda) = \begin{cases} 
\frac{d_\lambda^2}{n!} & \text{if } \lambda \vdash n, \\
0 & \text{otherwise.}
\end{cases}
\]

Here $\lambda \vdash n$ is a shorthand symbol to say that $\lambda$ is partition of size $n$. It is a probability measure because of the “well-known” identity

\[n! = \sum_{\lambda \vdash n} d_\lambda^2\]

which has (at least) two classical proofs:

- **representation theory**: $n!$ is the dimension of the regular representation of the symmetric group $S_n$, and $d_\lambda$ is the dimension of its irreducible representation indexed by $\lambda$,

- **bijection**: the Robinson-Schensted correspondence is a bijection between $S_n$ and the set of triples $(\lambda, P, Q)$, where $\lambda \vdash n$ and $P, Q$ are two SYTs of shape $\lambda$. 

Connection with Longest Increasing Subsequences

A property of the Robinson-Schensted correspondence is that if \( \sigma \mapsto (\lambda, P, Q) \), then the first part of \( \lambda \) satisfies

\[
\lambda_1 = L(\sigma)
\]

where \( L(\sigma) \) is the length of a Longest Increasing Subsequence (LIS) of \( \sigma \).
Connection with Longest Increasing Subsequences

A property of the Robinson-Schensted correspondence is that if $\sigma \mapsto (\lambda, P, Q)$, then the first part of $\lambda$ satisfies

$$\lambda_1 = L(\sigma)$$

where $L(\sigma)$ is the length of a Longest Increasing Subsequence (LIS) of $\sigma$.

Example: for $\sigma = (3, 1, 6, 7, 2, 5, 4)$, we have $L(\sigma) = 3$. 
Connection with Longest Increasing Subsequences

A property of the Robinson-Schensted correspondence is that if \( \sigma \mapsto (\lambda, P, Q) \), then the first part of \( \lambda \) satisfies

\[
\lambda_1 = L(\sigma)
\]

where \( L(\sigma) \) is the length of a Longest Increasing Subsequence (LIS) of \( \sigma \).

Example: for \( \sigma = (3, 1, 6, 7, 2, 5, 4) \), we have \( L(\sigma) = 3 \).

There is a more general statement (Greene’s theorem) but we will not discuss it here.
Connection with Longest Increasing Subsequences

A property of the Robinson-Schensted correspondence is that if \( \sigma \mapsto (\lambda, P, Q) \), then the first part of \( \lambda \) satisfies

\[
\lambda_1 = L(\sigma)
\]

where \( L(\sigma) \) is the length of a Longest Increasing Subsequence (LIS) of \( \sigma \).

Example: for \( \sigma = (3, 1, 6, 7, 2, 5, 4) \), we have \( L(\sigma) = 3 \).

There is a more general statement (Greene’s theorem) but we will not discuss it here.

The Longest Increasing Subsequence problem consists in understanding the asymptotic behaviour as \( n \to \infty \) of \( L_n := L(\sigma_n) = \lambda_1^{(n)} \), where \( \sigma_n \) denotes a uniform random permutation in \( S_n \), and \( \lambda^{(n)} \) the random partition to which it maps via the RS correspondence, and whose law is the Plancherel measure.
Some partial history of the LIS problem

- The problem was formulated by Ulam (1961) who suggested investigating it using Monte Carlo simulations and observed that $L_n$ should be of order $\sqrt{n}$.
Some partial history of the LIS problem

- The problem was formulated by Ulam (1961) who suggested investigating it using Monte Carlo simulations and observed that $L_n$ should be of order $\sqrt{n}$.

- It was then popularized by Hammersley (1972) who introduced a nice graphical method (closely related with the RSK correspondence) and proved that $L_n/\sqrt{n}$ converges in probability to a constant $c \in [\pi/2, e]$.

Vershik-Kerov and Logan-Shepp (1977) proved independently that $c = 2$, as a consequence of a more general limit shape theorem for the Plancherel measure on partitions. (See Chapter 1 of Romik’s book.)

Baik-Deift-Johansson (1999) proved the most precise result $P(L_n - 2\sqrt{n} \leq s) = F_{\text{GUE}}(s)$, $n \to \infty$ where $F_{\text{GUE}}$ is the Tracy-Widom GUE distribution. (See Chapter 2.)

The unusual exponent $n^{1/6}$ was previously conjectured by Odlyzko-Rains and Kim based on numerical evidence and bounds.
Some partial history of the LIS problem

- The problem was formulated by Ulam (1961) who suggested investigating it using Monte Carlo simulations and observed that $L_n$ should be of order $\sqrt{n}$.
- It was then popularized by Hammersley (1972) who introduced a nice graphical method (closely related with the RSK correspondence) and proved that $L_n/\sqrt{n}$ converges in probability to a constant $c \in [\pi/2, e]$.
- Vershik-Kerov and Logan-Shepp (1977) proved independently that $c = 2$, as a consequence of a more general limit shape theorem for the Plancherel measure on partitions. (See Chapter 1 of Romik’s book.)
Limit shape

A Plancherel random partition of size 10000 (courtesy of D. Betea)
Some partial history of the LIS problem

- The problem was formulated by Ulam (1961) who suggested investigating it using Monte Carlo simulations and observed that $L_n$ should be of order $\sqrt{n}$.
- It was then popularized by Hammersley (1972) who introduced a nice graphical method (closely related with the RSK correspondence) and proved that $L_n/\sqrt{n}$ converges in probability to a constant $c \in [\pi/2, e]$.
- Vershik-Kerov and Logan-Shepp (1977) proved independently that $c = 2$, as a consequence of a more general limit shape theorem for the Plancherel measure on partitions. (See Chapter 1 of Romik’s book.)
Some partial history of the LIS problem

- The problem was formulated by Ulam (1961) who suggested investigating it using Monte Carlo simulations and observed that \( L_n \) should be of order \( \sqrt{n} \).
- It was then popularized by Hammersley (1972) who introduced a nice graphical method (closely related with the RSK correspondence) and proved that \( L_n/\sqrt{n} \) converges in probability to a constant \( c \in [\pi/2, e] \).
- Vershik-Kerov and Logan-Shepp (1977) proved independently that \( c = 2 \), as a consequence of a more general limit shape theorem for the Plancherel measure on partitions. (See Chapter 1 of Romik’s book.)
- Baik-Deift-Johansson (1999) proved the most precise result

\[
P \left( \frac{L_n - 2\sqrt{n}}{n^{1/6}} \leq s \right) = F_{GUE}(s), \quad n \to \infty
\]

where \( F_{GUE} \) is the Tracy-Widom GUE distribution. (See Chapter 2.)

The unusual exponent \( n^{1/6} \) was previously conjectured by Odlyzko-Rains and Kim based on numerical evidence and bounds.
Topics of the lectures

We will discuss some properties of the Plancherel measure.

1. We will show that the poissonized Plancherel measure (to be defined) is closely related with a determinantal point process (DPP) called the discrete Bessel process. Plan:
   ▶ Some general theory of DPPs
   ▶ Connection with Plancherel measure via fermions

2. We will then investigate asymptotics, in the following regimes:
   ▶ Bulk limits: the VKLS limit shape and the discrete sine process
   ▶ Edge limit: the Airy process and the Baik-Deift-Johansson theorem
Topics of the lectures

We will discuss some properties of the Plancherel measure.

1. We will show that the poissonized Plancherel measure (to be defined) is closely related with a determinantal point process (DPP) called the discrete Bessel process. Plan:
   - Some general theory of DPPs
   - Connection with Plancherel measure via fermions

2. We will then investigate asymptotics, in the following regimes:
   - Bulk limits: the VKLS limit shape and the discrete sine process
   - Edge limit: the Airy process and the Baik-Deift-Johansson theorem

These results were obtained independently in two papers by Borodin, Okounkov and Olshanski (2000) and by Johansson (2001). But we use a different approach developed later by Okounkov et al., which may be generalized to Schur measures and Schur processes. We concentrate on the Plancherel measure for simplicity.
The poissonized Plancherel measure of parameter $\theta$ is the measure

$$\text{Prob}(\lambda) = \frac{d^2_\lambda}{(|\lambda|!)^2} \theta^{|\lambda|} e^{-\theta}.$$ 

It is a mixture of the Plancherel measures of fixed size, where the size is a Poisson random variable of parameter $\theta$.

We denote by $\lambda^{(\theta)}$ a random partition distributed according to the poissonized Plancherel measure, $\lambda^{(n)}$ denoting a Plancherel random partition of size $n$. 
To a partition $\lambda$, here $(4, 2, 1)$, we associate a set $S(\lambda) \subset \mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$ by

$$S(\lambda) = \{ \lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \lambda_3 - \frac{5}{2}, \ldots \}$$

Here $S(\lambda) = \{ \frac{7}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{7}{2}, -\frac{9}{2}, \ldots \}$. Elements of $S(\lambda)$ ("particles" •) correspond to the down-steps of the blue curve.
Main result of today

**Theorem [Borodin-Okounkov-Olshanski 2000, Johansson 2001]**

The particle configuration $S(\lambda^{\langle \theta \rangle})$ associated with the poissonized Plancherel measure is a determinantal point process in the sense that, for any distinct points $\{u_1, \ldots, u_n\} \subset \mathbb{Z}'$, we have

$$
\mathbb{P} \left( \{u_1, \ldots, u_n\} \subset S(\lambda^{\langle \theta \rangle}) \right) = \det_{1 \leq i,j \leq n} J_{\theta}(u_i, u_j).
$$
Main result of today

**Theorem [Borodin-Okounkov-Olshanski 2000, Johansson 2001]**

The particle configuration $S(\lambda^{(\theta)})$ associated with the poissonized Plancherel measure is a determinantal point process in the sense that, for any distinct points $\{u_1, \ldots, u_n\} \subset \mathbb{Z}'$, we have

$$\mathbb{P}\left(\{u_1, \ldots, u_n\} \subset S(\lambda^{(\theta)})\right) = \det_{1 \leq i, j \leq n} J_\theta(u_i, u_j).$$

The correlation kernel $J_\theta$ is the discrete Bessel kernel

$$J_\theta(s, t) = \sum_{\ell \in \mathbb{Z}'_0} J_{s+\ell}(2\sqrt{\theta})J_{t+\ell}(2\sqrt{\theta}), \quad s, t \in \mathbb{Z}'$$

where $J_n$ is the Bessel function of order $n$. 
Main result of today

**Theorem [Borodin-Okounkov-Olshanski 2000, Johansson 2001]**

The particle configuration $S(\lambda^{\langle \theta \rangle})$ associated with the poissonized Plancherel measure is a determinantal point process in the sense that, for any distinct points $\{u_1, \ldots, u_n\} \subset \mathbb{Z}'$, we have

$$
\mathbb{P} \left( \{u_1, \ldots, u_n\} \subset S(\lambda^{\langle \theta \rangle}) \right) = \det_{1 \leq i, j \leq n} J_\theta(u_i, u_j).
$$

The correlation kernel $J_\theta$ is the discrete Bessel kernel

$$
J_\theta(s, t) = \sum_{\ell \in \mathbb{Z}' > 0} J_{s+\ell}(2\sqrt{\theta})J_{t+\ell}(2\sqrt{\theta}), \quad s, t \in \mathbb{Z}'
$$

where $J_n$ is the Bessel function of order $n$.

By the general theory of DPPs, knowing $J_\theta$ gives all the information on the point process.
Asymptotics of $J_\theta$, using saddle point computations. Again this is different from the original techniques of BOO/J, our approach follows Okounkov and Reshetikhin and are robust ("universality").
Part II

21 March 2019
A refresher on fermions

A **fermionic configuration** is a subset $S \subset \mathbb{Z'} := \mathbb{Z} + \frac{1}{2}$ that contains finitely many positive elements, and whose complement contains finitely many negative elements. We denote by $S$ the (countable) set of fermionic configurations.
A refresher on fermions

A fermionic configuration is a subset $S \subset \mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$ that contains finitely many positive elements, and whose complement contains finitely many negative elements. We denote by $S$ the (countable) set of fermionic configurations.

Partitions are embedded into fermionic configurations by the mapping

$$\lambda \mapsto S(\lambda) := \{\lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \lambda_3 - \frac{5}{2}, \ldots\}$$

It is not a bijection but the mapping $$(\lambda, c) \mapsto S(\lambda) + c, \text{ with } c \in \mathbb{Z},$$ is.
A refresher on fermions

A fermionic configuration is a subset $S \subset \mathbb{Z}^\prime := \mathbb{Z} + \frac{1}{2}$ that contains finitely many positive elements, and whose complement contains finitely many negative elements. We denote by $S$ the (countable) set of fermionic configurations.

Partitions are embedded into fermionic configurations by the mapping

$$\lambda \mapsto S(\lambda) := \{\lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \lambda_3 - \frac{5}{2}, \ldots\}$$

It is not a bijection but the mapping $(\lambda, c) \mapsto S(\lambda) + c$, with $c \in \mathbb{Z}$, is.

The fermionic Fock space $\mathcal{F}$ consists of columns vectors indexed by $S$. The standard basis is denoted by $(v_S)_{S \in \mathcal{S}}$ and the dual basis (of row vectors) by $(v_S^*)_{S \in \mathcal{S}}$. Operators on $\mathcal{F}$ are naively viewed as matrices with rows and columns indexed by $S$. 
A refresher on fermions

A fermionic configuration is a subset $S \subset \mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$ that contains finitely many positive elements, and whose complement contains finitely many negative elements. We denote by $S$ the (countable) set of fermionic configurations.

Partitions are embedded into fermionic configurations by the mapping

$$\lambda \mapsto S(\lambda) := \{\lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \lambda_3 - \frac{5}{2}, \ldots\}$$

It is not a bijection but the mapping $(\lambda, c) \mapsto S(\lambda) + c$, with $c \in \mathbb{Z}$, is.

The fermionic Fock space $\mathcal{F}$ consists of columns vectors indexed by $S$. The standard basis is denoted by $(v_S)_{S \in S}$ and the dual basis (of row vectors) by $(v_S^*)_{S \in S}$. Operators on $\mathcal{F}$ are naively viewed as matrices with rows and columns indexed by $S$.

We use the shorthand notation $v_\lambda := v_{S(\lambda)}$ for partitions and $v_{\emptyset} := v_{\mathbb{Z}'_{<0}}$ corresponds to the (nonzero!) vacuum vector.
A refresher on fermions

We defined the fermionic creation/annihilation operators through their action on the standard basis:

\[ \psi_k v_S := \begin{cases} 0 & \text{if } k \in S, \\ (-1)^{\#(S \cap \mathbb{Z}'_{> k})} v_{S \cup \{k\}} & \text{if } k \notin S, \end{cases} \]

\[ \psi^*_k v_S := \begin{cases} 0 & \text{if } k \notin S, \\ (-1)^{\#(S \cap \mathbb{Z}'_{> k})} v_{S \setminus \{k\}} & \text{if } k \in S. \end{cases} \]
A refresher on fermions

We defined the fermionic creation/annihilation operators through their action on the standard basis:

\[ \psi_k v_S := \begin{cases} 0 & \text{if } k \in S, \\ (-1)^{\#(S \cap \mathbb{Z}^*_k)} v_{S \cup \{k\}} & \text{if } k \notin S, \end{cases} \]

\[ \psi^*_k v_S := \begin{cases} 0 & \text{if } k \notin S, \\ (-1)^{\#(S \cap \mathbb{Z}^*_k)} v_{S \setminus \{k\}} & \text{if } k \in S. \end{cases} \]

These operators satisfy the canonical anticommutation relations (CAR)

\[ \psi_k \psi_\ell + \psi_\ell \psi_k = 0 \]
\[ \psi_k^* \psi_\ell^* + \psi_\ell^* \psi_k^* = 0 \]
\[ \psi_k \psi_\ell^* + \psi_\ell \psi_k^* = \delta_{k,\ell}. \]
A refresher on fermions

We defined the fermionic creation/annihilation operators through their action on the standard basis:

\[
\psi_k v_S := \begin{cases} 
0 & \text{if } k \in S, \\
(-1)^{\#(S \cap \mathbb{Z}'} v_{S \cup \{k\}} & \text{if } k \notin S,
\end{cases}
\]

\[
\psi_k^* v_S := \begin{cases} 
0 & \text{if } k \notin S, \\
(-1)^{\#(S \cap \mathbb{Z}'} v_{S \setminus \{k\}} & \text{if } k \in S.
\end{cases}
\]

These operators satisfy the canonical anticommutation relations (CAR)

\[
\psi_k \psi_\ell + \psi_\ell \psi_k = 0 \\
\psi_k^* \psi_\ell^* + \psi_\ell^* \psi_k^* = 0 \\
\psi_k \psi_\ell^* + \psi_\ell^* \psi_k = \delta_{k,\ell}.
\]

The diagonal operator \(N_k := \psi_k \psi_k^*\) “measures” whether there is a particle at position \(k\).
A refresher on fermions

We defined the “box” creation/annihilation operators by

\[
\alpha^* := \sum_{k \in \mathbb{Z}'} \psi_k^* \psi_{k+1}^*, \quad \alpha := \sum_{k \in \mathbb{Z}'} \psi_{k+1} \psi_k^*.
\]
A refresher on fermions

We defined the “box” creation/annihilation operators by

\[ \alpha^* := \sum_{k \in \mathbb{Z}'} \psi_k \psi^*_{k+1}, \quad \alpha := \sum_{k \in \mathbb{Z}'} \psi_{k+1} \psi^*_k. \]

In terms of partitions their action read

\[ \nu_\lambda^* \alpha^* = \sum_{\mu : \lambda \nearrow \mu} \nu^*_\mu, \quad \alpha \nu_\lambda = \sum_{\mu : \lambda \nearrow \mu} \nu_\mu \]

where \( \nearrow \) means “adding a box”.
A refresher on fermions

We defined the “box” creation/annihilation operators by

\[ \alpha^* := \sum_{k \in \mathbb{Z}'} \psi_k \psi^*_k, \quad \alpha := \sum_{k \in \mathbb{Z}'} \psi_{k+1} \psi^*_k. \]

In terms of partitions their action read

\[ v^*_\lambda \alpha^* = \sum_{\mu: \lambda \to \mu} v^*_\mu, \quad \alpha v_\lambda = \sum_{\mu: \lambda \to \mu} v_\mu \]

where \( \to \) means “adding a box”.

By iterating we get

\[ v^*_\emptyset (\alpha^*)^n = \sum_{\lambda \vdash n} d_\lambda v^*_\lambda, \quad \alpha^n v_\emptyset = \sum_{\lambda \vdash n} d_\lambda v_\lambda \]

or, equivalently,

\[ v^*_\emptyset e^{x \alpha^*} = \sum_{\lambda} \frac{d_\lambda x^{\mid \lambda \mid}}{\mid \lambda \mid !} v^*_\lambda, \quad e^{x \alpha} v_\emptyset = \sum_{\lambda} \frac{d_\lambda x^{\mid \lambda \mid}}{\mid \lambda \mid !} v_\lambda. \]
A refresher on fermions

Final result of yesterday

The correlation function $\rho(U)$ of the poissonized Plancherel measure admit the fermionic expression (with $\theta = x^2$)

$$\rho(U) := \mathbb{P} \left( \{ u_1, \ldots, u_n \} \subset S(\lambda^{\langle \theta \rangle}) \right) = \frac{v_\emptyset^* e^{x\alpha^*} N_{u_1} \cdots N_{u_n} e^{x\alpha} v_\emptyset}{e^{x^2}}$$

where we recall that $N_u = \psi_u \psi_u^*$ “indicates” if there is a particle at $u$. 

A refresher on fermions

Final result of yesterday

The correlation function \( \rho(U) \) of the poissonized Plancherel measure admit
the fermionic expression (with \( \theta = x^2 \))

\[
\rho(U) := \mathbb{P} \left( \{ u_1, \ldots, u_n \} \subset S(\lambda^{\langle \theta \rangle}) \right) = \frac{v^*_\emptyset e^{x\alpha^*} N_{u_1} \cdots N_{u_n} e^{x\alpha} v_\emptyset}{e^{x^2}}
\]

where we recall that \( N_u = \psi_u \psi^*_u \) “indicates” if there is a particle at \( u \).

It remains to identify the rhs as a determinant with Bessel kernel entries. There are two main steps, which both exploit the CAR algebra structure:
A refresher on fermions

Final result of yesterday

The correlation function $\rho(U)$ of the poissonized Plancherel measure admit the fermionic expression (with $\theta = x^2$)

$$\rho(U) := \mathbb{P} \left( \{u_1, \ldots, u_n\} \subset S(\lambda^{\langle \theta \rangle}) \right) = \frac{v_\emptyset^* e^{x\alpha^*} N_{u_1} \cdots N_{u_n} e^{x\alpha} v_\emptyset}{e^{x^2}}$$

where we recall that $N_u = \psi_u \psi_u^*$ “indicates” if there is a particle at $u$.

It remains to identify the rhs as a determinant with Bessel kernel entries. There are two main steps, which both exploit the CAR algebra structure:

- eliminate the $\alpha$’s to rewrite

$$\rho(U) = v_\emptyset^* \hat{\psi}_{u_1} \hat{\psi}_{u_1}^* \cdots \hat{\psi}_{u_n} \hat{\psi}_{u_n}^* v_\emptyset,$$

  with $\hat{\psi}_u := \sum_{\ell \in \mathbb{Z}} J_\ell(2x) \psi_{u-\ell}$
A refresher on fermions

Final result of yesterday

The correlation function $\rho(U)$ of the poissonized Plancherel measure admit the fermionic expression (with $\theta = x^2$)

$$
\rho(U) := \mathbb{P} \left( \{u_1, \ldots, u_n\} \subset S(\lambda^\langle \theta \rangle) \right) = v_0^* e^{x\alpha^*} N_{u_1} \cdots N_{u_n} e^{x\alpha} v_0 e^{x^2}
$$

where we recall that $N_u = \psi_u \psi_u^*$ "indicates" if there is a particle at $u$.

It remains to identify the rhs as a determinant with Bessel kernel entries. There are two main steps, which both exploit the CAR algebra structure:

- **eliminate** the $\alpha$’s to rewrite

  $$
  \rho(U) = v_0^* \hat{\psi}_{u_1} \hat{\psi}_{u_1}^* \cdots \hat{\psi}_{u_n} \hat{\psi}_{u_n}^* v_0, \quad \text{with} \quad \hat{\psi}_u := \sum_{\ell \in \mathbb{Z}} J_\ell(2x) \psi_{u-\ell}
  $$

- **apply Wick’s lemma** to get

  $$
  \rho(U) = \det_{1 \leq i,j \leq n} v_0^* \hat{\psi}_{u_i} \hat{\psi}_{u_j}^* v_0 = \det_{1 \leq i,j \leq n} J_\theta(u_i, u_j).
  $$
Eliminating the $\alpha$’s

From the CAR we deduce the commutation relations
\[
\left[ \alpha^*, \psi_k \right] = \psi_k - 1,
\left[ \psi_k, \alpha \right] = -\psi_k + 1
\]
and their duals.

Equivalently, in terms of
\[
\psi(z) := \sum_{k \in \mathbb{Z}'} \psi_k z^k,
\]
\[
\left[ \alpha^*, \psi(z) \right] = z \psi(z),
\left[ \psi(z), \alpha \right] = -z - 1 \psi(z).
\]
By “exponentiating” we get
\[
\text{Ad}_{e^{x \alpha^*}}(\psi(z)) = e^{xz} \psi(z),
\text{Ad}_{e^{-x \alpha}}(\psi(z)) = e^{-xz-1} \psi(z).
\]
Equivalently, 
\[
\text{Ad}_{e^{x \alpha^*}}(\psi_k) = \sum_{\ell \in \mathbb{Z}} x^\ell \ell! \psi_k - \ell,
\text{Ad}_{e^{-x \alpha}}(\psi_k) = \sum_{\ell \in \mathbb{Z}} (-x)^\ell \ell! \psi_k + \ell.
\]
Eliminating the \( \alpha \)'s

From the CAR we deduce the commutation relations

\[
[\alpha^*, \psi_k] = \psi_{k-1}, \quad [\psi_k, \alpha] = -\psi_{k+1}
\]

and their duals.
Eliminating the $\alpha$’s

From the CAR we deduce the commutation relations

$$[\alpha^*, \psi_k] = \psi_{k-1}, \quad [\psi_k, \alpha] = -\psi_{k+1}$$

and their duals. Equivalently, in terms of $\psi(z) := \sum_{k \in \mathbb{Z}'} \psi_k z^k$,

$$[\alpha^*, \psi(z)] = z \psi(z), \quad [\psi(z), \alpha] = -z^{-1} \psi(z).$$
Eliminating the $\alpha$’s

From the CAR we deduce the commutation relations

$$[\alpha^*, \psi_k] = \psi_{k-1}, \quad [\psi_k, \alpha] = -\psi_{k+1}$$

and their duals. Equivalently, in terms of $\psi(z) := \sum_{k \in \mathbb{Z}'} \psi_k z^k$,

$$[\alpha^*, \psi(z)] = z\psi(z), \quad [\psi(z), \alpha] = -z^{-1}\psi(z).$$

By “exponentiating” we get

$$\text{Ad}_{e^{x\alpha^*}} (\psi(z)) = e^{xz}\psi(z), \quad \text{Ad}_{e^{-x\alpha}} (\psi(z)) = e^{-xz^{-1}}\psi(z)$$

where $\text{Ad}_g(Y) := ghg^{-1}$. 
Eliminating the $\alpha$’s

From the CAR we deduce the commutation relations

$$[\alpha^*, \psi_k] = \psi_{k-1}, \quad [\psi_k, \alpha] = -\psi_{k+1}$$

and their duals. Equivalently, in terms of $\psi(z) := \sum_{k \in \mathbb{Z}'} \psi_k z^k$,

$$[\alpha^*, \psi(z)] = z \psi(z), \quad [\psi(z), \alpha] = -z^{-1} \psi(z).$$

By “exponentiating” we get

$$\text{Ad}_{e^{x \alpha^*}} (\psi(z)) = e^{xz} \psi(z), \quad \text{Ad}_{e^{-x \alpha}} (\psi(z)) = e^{-xz^{-1}} \psi(z)$$

where $\text{Ad}_g(Y) := ghg^{-1}$. Equivalently,

$$\text{Ad}_{e^{x \alpha^*}} (\psi_k) = \sum_{\ell \in \mathbb{Z}} \frac{x^\ell}{\ell!} \psi_{k-\ell}, \quad \text{Ad}_{e^{-x \alpha}} (\psi_k) = \sum_{\ell \in \mathbb{Z}} \frac{(-x)^\ell}{\ell!} \psi_{k+\ell}$$
Eliminating the $\alpha$’s

Also, we have

$$[\alpha^*, \alpha] = 1$$

which implies, by the Baker-Campbell-Hausdorff formula,

$$e^{x\alpha^*} e^{x\alpha} = e^{x^2} e^{x\alpha} e^{x\alpha^*}.$$
Eliminating the $\alpha$’s

Also, we have

$$[\alpha^*, \alpha] = 1$$

which implies, by the Baker-Campbell-Hausdorff formula,

$$e^{x\alpha^*} e^{x\alpha} = e^{x^2} e^{x\alpha} e^{x\alpha^*}.$$

Note that

$$\text{Ad}_{e^{x\alpha^*} e^{-x\alpha}} (\psi(z)) = e^{x(z-z^{-1})} \psi(z)$$
Eliminating the $\alpha$’s

Also, we have

$$[\alpha^*, \alpha] = 1$$

which implies, by the Baker-Campbell-Hausdorff formula,

$$e^{x^* \alpha^*} e^{x \alpha} = e^{x^2} e^{x \alpha} e^{x^* \alpha^*}.$$

Note that

$$\text{Ad}_{e^{x \alpha^*} e^{-x \alpha}} (\psi(z)) = e^{x(z-z^{-1})} \psi(z)$$

and

$$\text{Ad}_{e^{x \alpha^*} e^{-x \alpha}} (\psi_k) = \sum_{\ell \in \mathbb{Z}} J_\ell (2x) \psi_{k-\ell} =: \hat{\psi}_k$$
Eliminating the $\alpha$'s

Also, we have

$$[\alpha^*, \alpha] = 1$$

which implies, by the Baker-Campbell-Hausdorff formula,

$$e^{x\alpha^*} e^{x\alpha} = e^{x^2} e^{x\alpha} e^{x\alpha^*}.$$

Note that

$$\text{Ad}_{e^{x\alpha^*} e^{-x\alpha}} (\psi(z)) = e^{x(z-z^{-1})} \psi(z)$$

$$\text{Ad}_{e^{x\alpha^*} e^{-x\alpha}} (\psi_k) = \sum_{\ell \in \mathbb{Z}} J_\ell (2x) \psi_{k-\ell} =: \hat{\psi}_k$$

$$v_0^* e^{x\alpha} = v_0^* \quad e^{x\alpha^*} v_0 = v_0$$
Eliminating the $\alpha$’s

Also, we have

$$[\alpha^*, \alpha] = 1$$

which implies, by the Baker-Campbell-Hausdorff formula,

$$e^{\alpha^*} e^{x \alpha} = e^{x^2} e^{x \alpha} e^{x \alpha^*}.$$

Note that

$$\text{Ad}_{e^{x \alpha^*} e^{-x \alpha}} (\psi(z)) = e^{x (z - z^{-1})} \psi(z)$$

$$\text{Ad}_{e^{x \alpha^*} e^{-x \alpha}} (\psi_k) = \sum_{\ell \in \mathbb{Z}} J_\ell(2x) \psi_{k-\ell} =: \hat{\psi}_k$$

$$\nu^*_0 e^{x \alpha} = \nu^*_0 \quad e^{x \alpha^*} \nu_0 = \nu_0$$

and combining everything

$$\rho(U) = \frac{\nu^*_0 e^{x \alpha^*} \psi_{u_1} \psi^*_{u_1} \cdots \psi_{u_n} \psi^*_{u_n} e^{x \alpha} \nu_0}{e^{x^2}} = \nu^*_0 \hat{\psi}_{u_1} \hat{\psi}^*_{u_1} \cdots \hat{\psi}_{u_n} \hat{\psi}^*_{u_n} \nu_0.$$
Wick’s lemma (fermionic version)

Let \( \langle O \rangle := \nu_\emptyset^* O \nu_\emptyset \) denote the *vacuum expectation value* of an operator \( O \).
Wick’s lemma (fermionic version)

Let \( \langle \mathcal{O} \rangle := \nu_0^* \mathcal{O} \nu_0 \) denote the vacuum expectation value of an operator \( \mathcal{O} \).

Wick’s lemma (see Gaudin 1960 for a simple proof using CAR)

Let \( \varphi_1, \varphi_3, \ldots, \varphi_{2n-1} \) denote linear combinations of the \( \psi_k \)'s and \( \varphi_2^*, \varphi_4^*, \ldots, \varphi_{2n}^* \) denote linear combinations of the \( \psi_k^* \)'s. Then we have

\[
\langle \varphi_1 \varphi_2^* \varphi_3 \varphi_4^* \cdots \varphi_{2n-1} \varphi_{2n}^* \rangle = \det_{1 \leq i, j \leq n} C_{i,j}
\]

where \( C_{i,j} = \begin{cases} 
\langle \varphi_{2i-1} \varphi_{2j}^* \rangle & \text{if } i \leq j \\
-\langle \varphi_{2j}^* \varphi_{2i-1} \rangle & \text{if } i > j
\end{cases} \) \( (\text{“time-ordered correlator”}) \).
Wick’s lemma (fermionic version)
Let $\langle O \rangle := \nu_0^* O \nu_0$ denote the vacuum expectation value of an operator $O$.

Wick’s lemma (see Gaudin 1960 for a simple proof using CAR)
Let $\varphi_1, \varphi_3, \ldots, \varphi_{2n-1}$ denote linear combinations of the $\psi_k$’s and $\varphi_2^*, \varphi_4^*, \ldots, \varphi_{2n}^*$ denote linear combinations of the $\psi_k^*$’s. Then we have

$$
\langle \varphi_1 \varphi_2^* \varphi_3 \varphi_4^* \cdots \varphi_{2n-1} \varphi_{2n}^* \rangle = \det_{1 \leq i, j \leq n} C_{i,j}
$$

where $C_{i,j} = \begin{cases} 
\langle \varphi_{2i-1} \varphi_{2j}^* \rangle & \text{if } i \leq j \\
-\langle \varphi_{2j}^* \varphi_{2i-1} \rangle & \text{if } i > j
\end{cases}$ (“time-ordered correlator”).

Example
For $n = 2$ we have

$$
\langle \varphi_1 \varphi_2^* \varphi_3 \varphi_4^* \rangle = \begin{vmatrix}
\langle \varphi_1 \varphi_2^* \rangle & \langle \varphi_1 \varphi_4^* \rangle \\
-\langle \varphi_2^* \varphi_3 \rangle & \langle \varphi_3 \varphi_4^* \rangle
\end{vmatrix} = \langle \varphi_1 \varphi_2^* \rangle \cdot \langle \varphi_3 \varphi_4^* \rangle + \langle \varphi_1 \varphi_4^* \rangle \cdot \langle \varphi_2^* \varphi_3 \rangle.
$$
Applying Wick’s lemma

We deduce that

\[ \rho(U) = \langle \hat{\psi}_{u_1} \hat{\psi}_{u_1}^* \cdots \hat{\psi}_{u_n} \hat{\psi}_{u_n}^* \rangle = \det \langle \hat{\psi}_{u_i} \hat{\psi}_{u_j}^* \rangle_{1 \leq i, j \leq n} \]

(“time-ordering” does not matter here as \(-\hat{\psi}_u \hat{\psi}_v = \hat{\psi}_v \hat{\psi}_u^*\) for \(u \neq v\)).
Applying Wick’s lemma

We deduce that

\[ \rho(U) = \langle \hat{\psi}_{u_1} \hat{\psi}^*_{u_1} \cdots \hat{\psi}_{u_n} \hat{\psi}^*_{u_n} \rangle = \det_{1 \leq i, j \leq n} \langle \hat{\psi}_{u_i} \hat{\psi}^*_{u_j} \rangle \]

("time-ordering" does not matter here as \( -\hat{\psi}^* \hat{\psi}_v = \hat{\psi}_v \hat{\psi}^*_u \) for \( u \neq v \)).

The final step is to observe that

\[ \langle \psi_k \psi^*_\ell \rangle = \delta_{k, \ell} \mathbb{1}_{k < 0} \]

and therefore

\[ \langle \hat{\psi}_s \hat{\psi}^*_t \rangle = \sum_{u \in \mathbb{Z}_>^l} J_{s+u}(2x) J_{t+u}(2x) =: J_\theta(s, t). \]
Main result of “yesterday”

**Theorem [Borodin-Okounkov-Olshanski 2000, Johansson 2001]**

The particle configuration $S(\lambda^{\langle \theta \rangle})$ associated with the poissonized Plancherel measure is a determinantal point process in the sense that, for any distinct points $\{u_1, \ldots, u_n\} \subset \mathbb{Z}'$, we have

$$
\mathbb{P} \left( \{u_1, \ldots, u_n\} \subset S(\lambda^{\langle \theta \rangle}) \right) = \det_{1 \leq i, j \leq n} J_{\theta}(u_i, u_j).
$$

The correlation kernel $J_{\theta}$ is the discrete Bessel kernel

$$
J_{\theta}(s, t) = \sum_{\ell \in \mathbb{Z}'_{>0}} J_{s+\ell}(2\sqrt{\theta}) J_{t+\ell}(2\sqrt{\theta}), \quad s, t \in \mathbb{Z}'
$$

where $J_n$ is the Bessel function of order $n$. 
“Today”: asymptotics

Basically all we need to do is to understand the asymptotics of $J_{\theta}$. 
"Today": asymptotics

Basically all we need to do is to understand the asymptotics of $J_\theta$. We will use contour integral representations:

$$J_n(2x) = \frac{1}{2i\pi} \oint_{|z|=r} e^{x(z-z^{-1})} \frac{dz}{z^{n+1}}$$

$$J_\theta(s, t) = \frac{1}{(2i\pi)^2} \iint_{|z|>|w|>0} \frac{e^{x(z-z^{-1})}}{e^{x(w-w^{-1})}} \cdot \frac{dz \cdot dw}{(z - w)z^{s+\frac{1}{2}}w^{-t+\frac{1}{2}}}.$$
Theorem 1 \[BOO/J\]

Fix $A \in \mathbb{R}$ and consider the asymptotic regime $\theta = x^2 \to \infty$ with $s, t \sim Ax$ and $s - t$ fixed. Then we have

$$J_{\theta}(s, t) \to K \sin(s - t; \chi) := \begin{cases} \chi \pi & \text{if } s = t, \\ \sin \chi(s - t) \pi(s - t) & \text{if } s \neq t, \end{cases}$$

where

$$\chi := \begin{cases} \arccos(A/2) & \text{if } |A| \leq 2, \\ 0 & \text{if } A > 2, \\ \pi & \text{if } A < -2. \end{cases}$$

We deduce immediately that, if $u_1, \ldots, u_n$ are such that $u_i \sim Ax$ and $u_i - u_j$ remains fixed for all $i, j$, then

$$P(\{u_1, \ldots, u_n\} \subset \mathcal{S}(\lambda \langle \theta \rangle)) \to \det 1 \leq i, j \leq n K \sin(u_i - u_j; \chi).$$
**Theorem 1 [BOO/J]**

Fix $A \in \mathbb{R}$ and consider the asymptotic regime $\theta = x^2 \to \infty$ with $s, t \sim Ax$ and $s - t$ fixed. Then we have

$$J_\theta(s, t) \to K_{\sin}(s - t; \chi) := \begin{cases} \frac{\chi}{\pi} & \text{if } s = t, \\ \frac{\sin \chi(s-t)}{\pi(s-t)} & \text{if } s \neq t, \end{cases}$$

where

$$\chi := \begin{cases} \arccos(A/2) & \text{if } |A| \leq 2, \\ 0 & \text{if } A > 2, \\ \pi & \text{if } A < -2. \end{cases}$$
Bulk limit: discrete sine kernel

**Theorem 1 [BOO/J]**

Fix $A \in \mathbb{R}$ and consider the asymptotic regime $\theta = x^2 \to \infty$ with $s, t \sim Ax$ and $s - t$ fixed. Then we have

$$J_\theta(s, t) \to K_{\text{sin}}(s - t; \chi) := \begin{cases} \frac{\chi}{\pi} & \text{if } s = t, \\ \frac{\sin \chi(s-t)}{\pi(s-t)} & \text{if } s \neq t, \end{cases}$$

where

$$\chi := \begin{cases} \arccos(A/2) & \text{if } |A| \leq 2, \\ 0 & \text{if } A > 2, \\ \pi & \text{if } A < -2. \end{cases}$$

We deduce immediately that, if $u_1, \ldots, u_n$ are such that $u_i \sim Ax$ and $u_i - u_j$ remains fixed for all $i, j$, then

$$\mathbb{P}\left(\{u_1, \ldots, u_n\} \subset S(\lambda^{(\theta)})\right) \to \det_{1 \leq i, j \leq n} K_{\text{sin}}(u_i - u_j; \chi).$$
Connection with Vershik-Kerov-Logan-Shepp

In particular, for $s = t$ we obtain the one-point function (particle density).

It is consistent with the VKLS limit shape.
Connection with Vershik-Kerov-Logan-Shepp

In particular, for $s = t$ we obtain the one-point function (particle density).

It is consistent with the VKLS limit shape.

We do not quite recover their theorem: here we do a first moment calculation, we should also do second moment to prove concentration, and depoissonize.
Theorem 2 \cite{BOO/J}

Fix $\sigma, \tau \in \mathbb{R}$ and consider the asymptotic regime $\theta = x^2 \to \infty$ with $s = 2x + \sigma x^{1/3} + o(x^{1/3})$, $t = 2x + \tau x^{1/3} + o(x^{1/3})$.

Then we have

$$x^{1/3} J_{\theta}(s, t) \to A(\sigma, \tau) := \hat{\infty}_0 \text{Ai}(\sigma + \nu) \text{Ai}(\tau + \nu) d\nu.$$ 

For the LIS problem we are interested in the gap probability

$$P(\lambda_{\langle \theta \rangle}^{1} < t) = \det(I - J_{\theta})\{t, t+1, ...\} \to \det(I - A) L_2(\tau, \infty) = F_2(\tau).$$

The Baik-Deift-Johansson theorem follows by a depoissonization argument!
Theorem 2 [BOO/J]

Fix $\sigma, \tau \in \mathbb{R}$ and consider the asymptotic regime $\theta = x^2 \to \infty$ with

$$s = 2x + \sigma x^{1/3} + o(x^{1/3}), \quad t = 2x + \tau x^{1/3} + o(x^{1/3}).$$

Then we have

$$x^{1/3} J_\theta(s, t) \to A(\sigma, \tau) := \int_0^\infty \text{Ai}(\sigma + \nu) \text{Ai}(\tau + \nu) d\nu$$

where $\text{Ai}$ is the Airy function given by $\text{Ai}(y) = \frac{1}{2i\pi} \int_{\Re(\zeta)=1} e^{\frac{\zeta^3}{3} - y\zeta} d\zeta$. 
**Edge limit \((A = 2)\): Airy kernel**

**Theorem 2 [BOO/J]**

Fix \(\sigma, \tau \in \mathbb{R}\) and consider the asymptotic regime \(\theta = x^2 \to \infty\) with

\[
\begin{align*}
    s &= 2x + \sigma x^{1/3} + o(x^{1/3}), \\
    t &= 2x + \tau x^{1/3} + o(x^{1/3}).
\end{align*}
\]

Then we have

\[
x^{1/3} J_\theta(s, t) \to A(\sigma, \tau) := \int_0^\infty \text{Ai}(\sigma + \nu) \text{Ai}(\tau + \nu) d\nu
\]

where \(\text{Ai}\) is the Airy function given by

\[
\text{Ai}(y) = \frac{1}{2i\pi} \int_{\Re(\zeta) = 1} e^{\frac{\zeta^3}{3} - y\zeta} d\zeta.
\]

For the LIS problem we are interested in the gap probability

\[
P(\lambda^{(\theta)}_1 < t) = \det(I - J_\theta)\{t,t+1,...\}
\]
Theorem 2 [BOO/J]

Fix $\sigma, \tau \in \mathbb{R}$ and consider the asymptotic regime $\theta = x^2 \to \infty$ with

$$s = 2x + \sigma x^{1/3} + o(x^{1/3}), \quad t = 2x + \tau x^{1/3} + o(x^{1/3}).$$

Then we have

$$x^{1/3} J_\theta(s, t) \to A(\sigma, \tau) := \int_0^\infty \text{Ai}(\sigma + \nu) \text{Ai}(\tau + \nu) d\nu$$

where $\text{Ai}$ is the Airy function given by $\text{Ai}(y) = \frac{1}{2i\pi} \int_{\Re(\zeta)=1} e^{\zeta^3/3 - y\zeta} d\zeta$.

For the LIS problem we are interested in the gap probability

$$\mathbb{P}(\lambda_1^{(\theta)} < t) = \det(I - J_\theta)_{\{t, t+1, \ldots\}} \to \det(I - A)_{L^2(\tau, \infty)} = F_2(\tau).$$
Edge limit ($A = 2$): Airy kernel

**Theorem 2 [BOO/J]**

Fix $\sigma, \tau \in \mathbb{R}$ and consider the asymptotic regime $\theta = x^2 \to \infty$ with

$$s = 2x + \sigma x^{1/3} + o(x^{1/3}), \quad t = 2x + \tau x^{1/3} + o(x^{1/3}).$$

Then we have

$$x^{1/3} \mathbf{J}_\theta(s, t) \to \mathbf{A}(\sigma, \tau) := \int_0^\infty \text{Ai}(\sigma + \nu) \text{Ai}(\tau + \nu) d\nu$$

where $\text{Ai}$ is the Airy function given by $\text{Ai}(y) = \frac{1}{2i\pi} \int_{\Re(\zeta)=1} e^{\frac{\zeta^3}{3} - y\zeta} d\zeta$.

For the LIS problem we are interested in the gap probability

$$\mathbb{P}(\lambda_{1}^{\langle \theta \rangle} < t) = \det(\mathbf{I} - \mathbf{J}_\theta)^{\{t,t+1,\ldots\}} \to \det(\mathbf{I} - \mathbf{A})_{L^2(\tau, \infty)} = F_2(\tau).$$

The Baik-Deift-Johansson theorem follows by a depoissonization argument!