Combinatorics of groups via Stallings graphs

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– Mini-course content –

- **Stallings Graphs** : a unique finite representation
  - of finitely generated subgroups of a free group,
  - of quasi-convex subgroups of an automatic group.

- **Algebraic properties** via Stallings graphs.

- **Statistical properties** of **finitely generated subgroups** of a free groups.

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- **Algebraic properties** via Stallings graphs.

- **Statistical properties** of **finitely generated subgroups** of a free groups.

In this mini-course, the ambient groups are always finitely generated.

I. Free Groups

– Free groups –

- A group is a set equipped with an associative internal composition law, an identity element 1, and such that every element a has an inverse a^{-1}.
- A group F is free on a set A if every element of F can be uniquely expressed as a reduced product of elements from $A \cup A^{-1}$, where reduced means that no sub-product of the form $a \cdot a^{-1}$ occurs.

– The free group F_A on A –

- Let $A = \{a, b, \ldots\}$ be a **finite** set of letters of cardinality $r \geq 2$
- Let A^{-1} be the set of formal inverses of the letters
$$A^{-1} = \{a^{-1}, b^{-1} \ldots\}$$
- A word w on the alphabet $A \cup A^{-1}$ is reduced when it contains no pattern aa^{-1} or $a^{-1}a$:

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- If w is a word, its associated reduced word $\overline{w}$ is obtained by repeatedly removing **in any order** the aa^{-1} or $a^{-1}a$:

$$w = abbb^{-1}a^{-1}ccc^{-1}c^{-1}a^{-1} \rightarrow aba^{-1}ccc^{-1}c^{-1}a^{-1}$$
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$$\to aba^{-1}cc^{-1}a^{-1} \to aba^{-1}a^{-1} = \overline{w}$$

- The free group F_A on A is the set of all reduced words on $A \cup A^{-1}$ with the operation $u \cdot v = \overline{uv}$

– Cayley graph of a group –

Let G be a group and S a generating set of G, the Cayley graph of G is the graph

- whose vertices are the elements g of G
- and whose edges are of the form $g \to gs$ and of color c_s for $g \in G$ and $s \in S$.

In a Cayley graph, all edges can be read backward.

– The Cayley graph of F_A with $A = \{a, b\}$ –

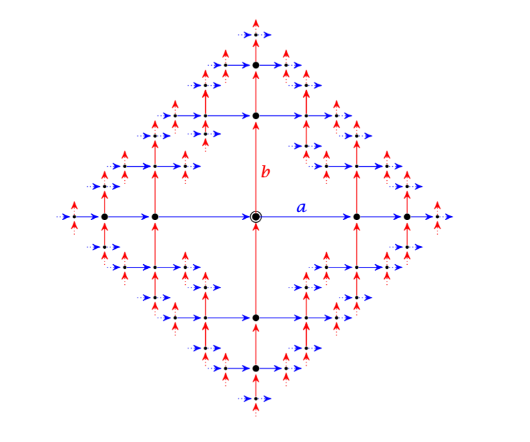

Figure: The Cayley graph of F_A with $A = \{a, b\}$

– Basis of a free group –

- The set A is a base of F_A : each element of F_A can be uniquely written as a reduced word on $A \cup A^{-1}$.
- It is not unique ($\{ab^{-1}, b\}$, $\{aba, ba\}$, $\ldots$).
- But all bases have the same cardinality.

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- But all bases have the same cardinality.

If $F_A = F_B$ then the following diagram commutes

$$\begin{array}{ccc} F_A & \xrightarrow{\phi \text{ isomorphism}} & F_B \\ \downarrow & & \downarrow \\ (\mathbb{Z}/2\mathbb{Z})^A & \stackrel{\psi}{\to} & (\mathbb{Z}/2\mathbb{Z})^B \end{array}$$

and ψ is surjective. So $\dim (\mathbb{Z}/2\mathbb{Z})^A \geq \dim (\mathbb{Z}/2\mathbb{Z})^B$ and $|A| \geq |B|$. Using the same argument with ϕ^{-1} shows that $|A| \leq |B|$. Thus $|A| = |B|$.

– Cosets and quotient group –

- Let H be a subgroup of G, for any $g \in G$, the right coset of g is

$$Hg = \{hg \mid h \in H\}.$$

- The number of right cosets (or of left cosets) of a subgroup H in a group G is its index $[G : H]$.

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- If H is normal, its set of right cosets is the quotient group with

$$Hg_1 \cdot Hg_2 = Hg_1g_2.$$

– Presentations of groups and free group –

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- Any group is isomorphic to a quotient group of some free group.
- A group G has the presentation $G = \langle S \mid R \rangle$ if G is the "freest group" generated by S subject only to the relations R.
- The group G have this presentation if it is isomorphic to the quotient of F_S by the normal closure of R.

Examples:

- The free group F_S on S has the presentation $F_S = \langle S \mid \emptyset \rangle$.
- The cyclic group of order n has the presentation $\langle a \mid a^n = 1 \rangle$.

– The modular group –

The modular group has the presentation

$$\mathsf{PSL}_2(\mathbb{Z}) = \langle a, b \mid a^2 = 1, b^3 = 1 \rangle.$$

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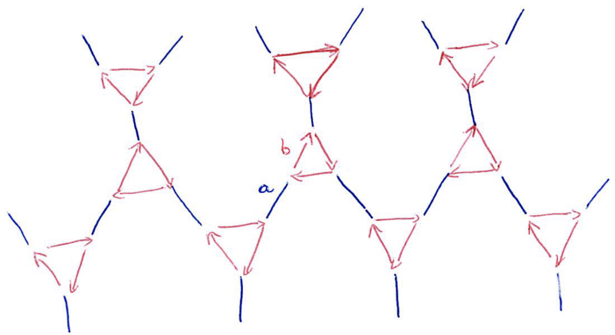

Figure: The Cayley graph of $\mathsf{PSL}_2(\mathbb{Z}) = \langle a, b \mid a^2 = 1, b^3 = 1 \rangle$.

II. Stallings graph of a subgroup

– Schreier coset graph of a subgroup –

Let H be a subgroup of G, S a generating set of G, the Schreier coset graph $Sch(H)$ is the graph

- whose vertices are the right cosets $Hg = \{hg : h \in H\}$ for $g \in G$
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Properties:

- In a Schreier graph, all edges can be read backward : $Hg \xrightarrow{s} Hgs$ and $Hg \xleftarrow{s^{-1}} Hgs$
- A Schreier graph is connected, deterministic, co-deterministic and complete.
- The Cayley graph of the group G itself is the Schreier coset graph for $H = \{1_G\}$.

– Example in the free group –

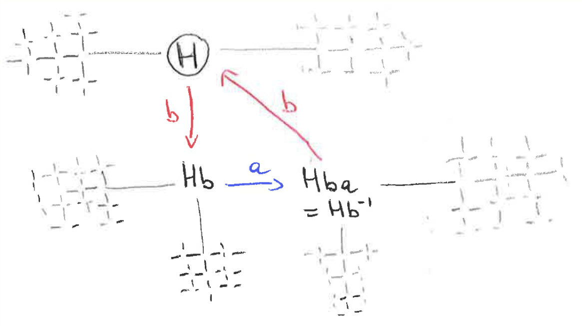

Figure: The Schreier graph of $H = \langle bab \rangle$ in F_A with $A = \{a, b\}$.

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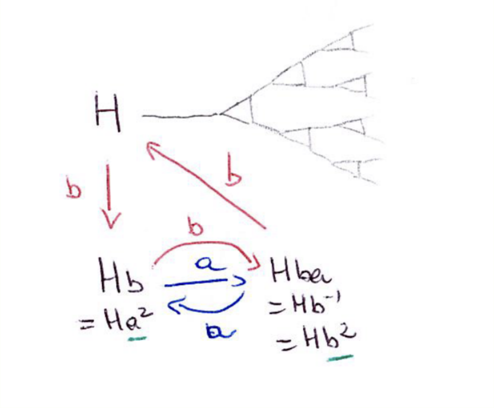

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– Stallings graph of a subgroup –

Let H a subgroup of G, S a generating set G, the Stallings graph of is the unique subgraph of $Sch(H)$

- rooted at the vertex corresponding to the subgroup H
- and spanned by all loops (closed paths) originating and ending at H labeled by a geodesic (shortest path) representation of an element in H in the Schreier graph of H.

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- The vertex H is called the base vertex.
- This construction provides a graphical representation of the subgroup facilitating the study of its structural properties.

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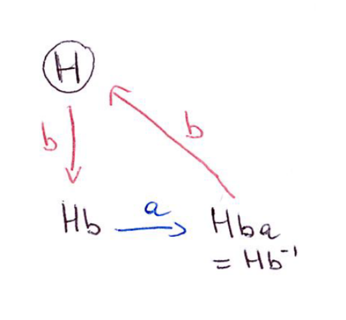

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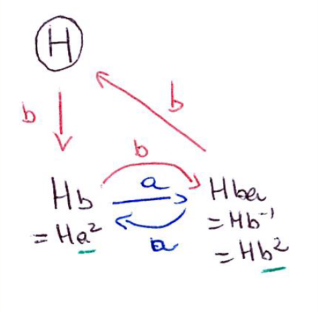

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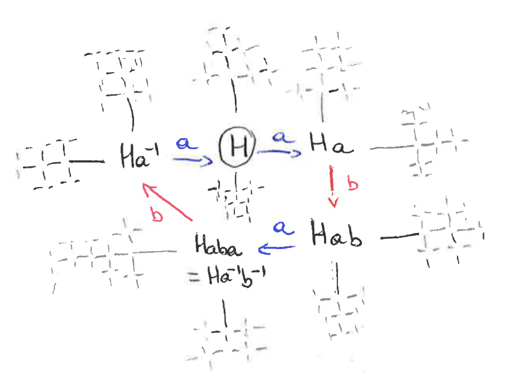

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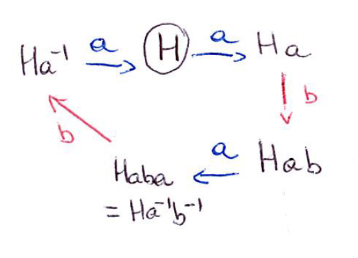

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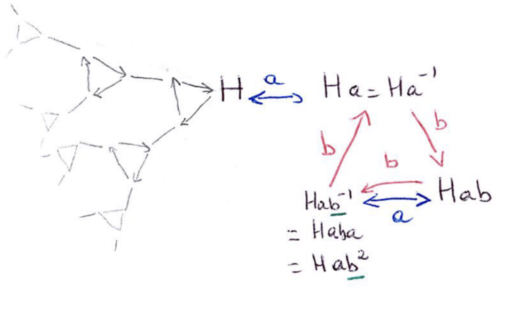

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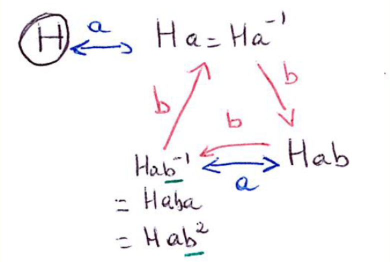

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– Stallings graphs and their generalizations –

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- Delgado and Ventura (2013) : a decorated graph structure to represent subgroups of direct products of free and free abelian groups.

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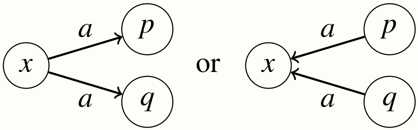
- The rank of H is the cardinal of a minimal set of generators of H.
- A free group with finite rank contains subgroups with any countable rank.
- $\langle b^i ab^{-i} \mid 0 \leq i < k \rangle$ has rank k.
- $\langle b^i ab^{-i} \mid i \in \mathbb{N} \rangle$ has an infinite rank.

– Action of the group generators in the Stalling graph –

A Stallings graph is deterministic and co-deterministic. Hence the two following configurations never occur for $p \neq q$:



Every generator of the ambiant group acts as a partial injection on the set of vertices of the Stallings graph.

– Stallings graph of a finitely generated subgroup of a free group –

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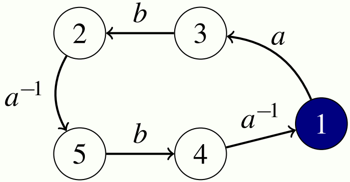

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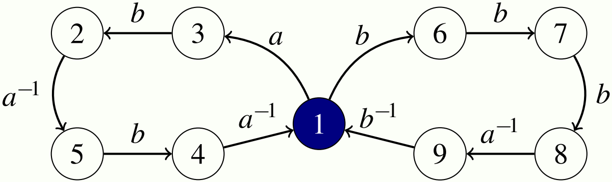

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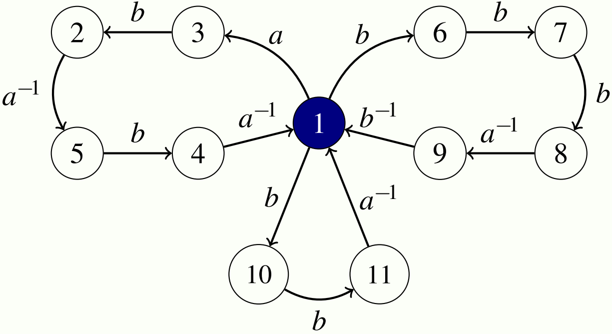
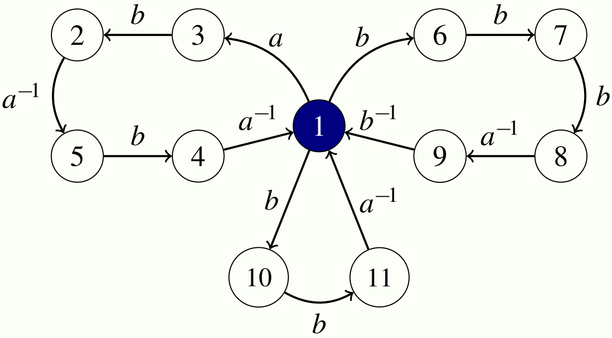
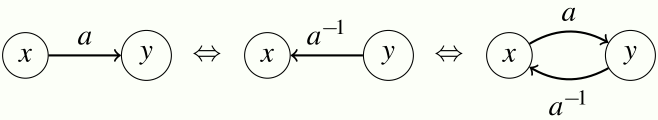
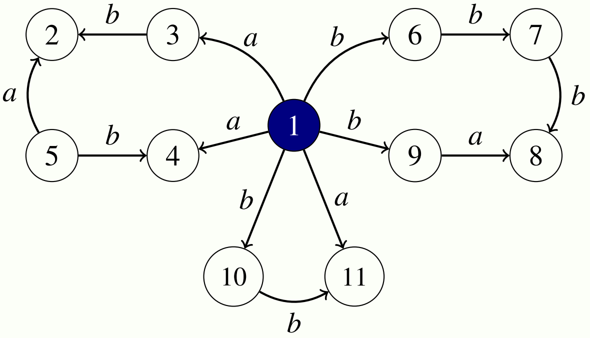




Edges can be used backward:



The flower graph with positive labels for
$Y = \{aba^{-1}ba^{-1}, bbba^{-1}b^{-1}, bba^{-1}\}$:



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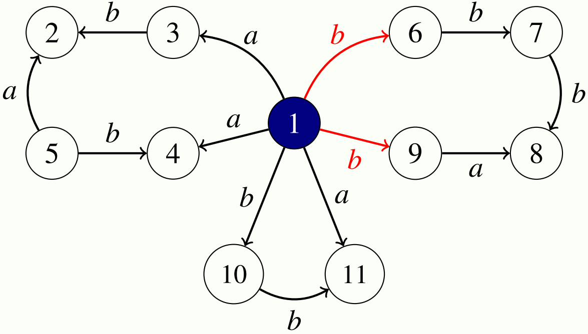

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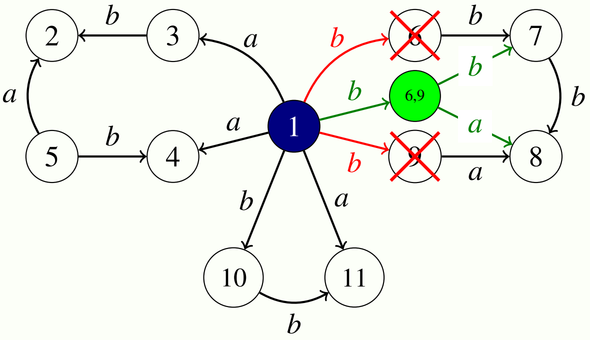

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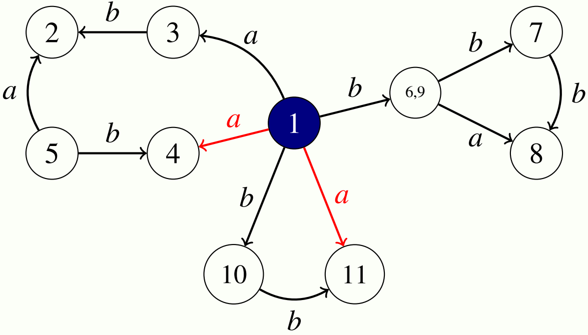

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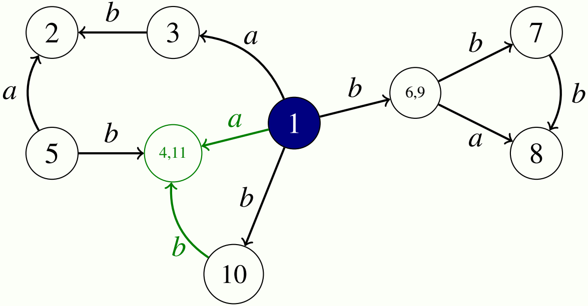

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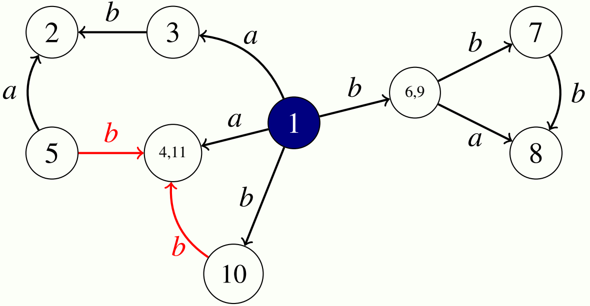
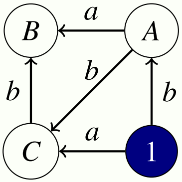


There is a problem: The graph is not co-deterministic.

– Result after foldings –



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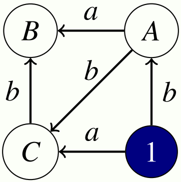

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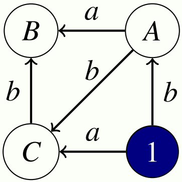

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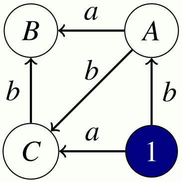

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The Stallings graph can be computed in $\mathcal{O}(m \log^* n)$, m being the number of foldings and n the sum of the length of the generators, using Union-Find (Touikan 2006).

– Characterization of Stallings graphs of subgroups of free groups –

Theorem: A positively labeled finite graph is the Stallings graph of a finitely generated subgroup of a free group if and only if

- the action of each letter is a partial injection;
- the graph is weakly connected (connected as an undirected graph);
- every vertex, but possibly the **base vertex**, has at least two incident edges (counting both ingoing and outgoing edges).

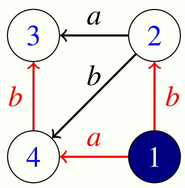
– Properties of the Stallings graph –

Theorem (membership): A reduced word u is in H if and only if it labels a loop begining and ending at the base vertex.

Theorem (rank and bases): The rank of finitely generated subgroup H of a free group is computable from its Stallings graph :

$$\mathsf{rk}(H) = E - V + 1.$$

To obtain a base, choose a spanning tree T of the Stallings graph. The elements of the base are the labels of the loops at the base vertex using e and edges in the spanning tree for each edge e that is not in T .



$H = < aba^{-1}ba^{-1}, bbba^{-1}b^{-1}, bba^{-1} >$ is of rank 2, and $\{bab^{-1}a^{-1}, bba^{-1}\}$ is a base of H.

– Properties of the Stallings graph : finite index –

The index $[G : H]$ of a subgroup H in a group G is its number of right cosets (or of left cosets).

Theorem (finite index): A subgroup $H \leq F$ of the free group F is of finite index if and if its Stallings graphs is finite and complete.
Then each letter acts like a permutation on the set of vertices and

$$\mathsf{rk}(H) - 1 = [F : H](\mathsf{rk}(F) - 1) \qquad \text{(Schreier index formula)}$$

– Properties of the Stallings graph : purity –

A subgroup $H \leq F$ of the free group F is pure if for any $g \in F$ and $n \geq 1$, $g^n \in H \Rightarrow g \in H$.

A deterministic automaton is aperiodic if : for any $g \in F$, any vertex q and any $n \geq 2$, if g^n labels a loop at q then g also labels a loop at q.

Theorem (purity): (Birget, Margolis, Meakin, Weil, 2000)

- A finitely generated subgroup of a free group is pure if and only if its Stallings graph is aperiodic.
- Testing if a Stallings graph is aperiodic is PSPACE-complete.

Testing if a finite automaton is aperiodic is PSPACE-complete (Cho, Huynh, 1991).

– The intersection of finitely generated subgroups of free groups –

Theorem (intersection): The intersection of two finitely generated subgroups can be computed in time and space $\mathcal{O}(n_1 \cdot n_2)$ where n_1 (resp. n_2) is the size (here the number of vertices) of the first (resp. second) Stallings graph.

Rank of the intersection of finitely generated subgroups :

- The intersection of two finitely generated subgroups is finitely generated (Howson, 1954).
- $\mathsf{rk}(H \cap K) - 1 \leq 2(\mathsf{rk}(H) - 1)(\mathsf{rk}(K) - 1)$ (H. Neumann, 1956).
- Hanna Neumann's conjecture / Mineyev theorem (2012):

$$\mathsf{rk}(H \cap K) - 1 \leq (\mathsf{rk}(H) - 1)(\mathsf{rk}(K) - 1).$$

– Normality and malnormality –

Conjugacy

To obtain the Stallings graph of the conjugate $H^g = g^{-1}Hg$ of H, replace the base vertex of the Stallings graph of H by the vertex reached after reading g from the base vertex v in the Schreier graph.

Normality

- A subgroup $H \leq G$ is normal if, for all $g \in G$, $H^g = H$.
- Proposition: If a finitely generated subgroup of a free group is normal, then it is of finite index.

Malnormality

- A subgroup H is malnormal if, for all $g \notin H$, $H^g \cap H = 1$.
- Proposition: A finitely generated subgroup H of a frez group is malnormal
  - if and only if every non-diagonal connected component of the product graph of the Stallings graph of H with itself is a tree;
  - if and only if there are no two loops of the same label in the Stallings graph of H.

– Geodesically automatic group –

Let $G = \langle A \mid R \rangle$ be a group, G is geodesically automatic if there exist:

- an automaton $\mathcal{A}_G$ that accepts all the *geodesic* representatives of the elements of G;
- and, for each $a \in A \cup A^{-1} \cup \{1\}$, an automaton that accepts a pair (w_1, w_2), for all words w_i accepted by $\mathcal{A}_G$, exactly when $w_1 a = w_2$ in G.

Examples: Hyperbolic groups, RAAG.

– Quasi-convex subgroup –

- A subgroup $H \leq G$ is quasi-convex if there exists $k > 0$, such that any geodesic path in the Cayley graph of G connecting two elements of H remains within a k-neighborhood of H.

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- Proposition: A subgroup of a finitely generated group is quasi-convex if and only if its Stallings graph is finite (Kharlampovich, Miasnikov, Weil 2017).
- Theorem: The quasi-convexity of a subgroup is an undecidable property (Kapovich, 1996).

– Stallings graphs for quasi-convex subgroups –

Theorem (Kharlampovich, Miasnikov, Weil, 2017):
Let $G = \langle A \mid R \rangle$ be a finitely presented (A and R are finite) geodesically automatic group.

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Let $G = \langle A \mid R \rangle$ be a finitely presented (A and R are finite) geodesically automatic group.
Let H be quasi-convex subgroup of G.
Then the Stallings graph of H is finite and effectively computable by a partial algorithm.

– Stallings graphs for quasi-convex subgroups –

A partial algorithm computing the Stallings graph $\Gamma(H)$

1. Compute the Stallings graph of the **free subgroup** generated by the generators of H.
2. For each relator, add at each vertex of the graph a loop (closed path) labeled by this relator.
3. Fold the edges of the graph.
4. Iterate Step 2 and 3 until all the geodesic representations of the elements of H can be read in $\Gamma(H)$.
5. Remove the vertices that do not belong to a geodesic path.

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There is no bound on the time complexity of this algorithm; otherwise, it would be possible to determine whether a set of generators generates a quasi-convex subgroup.

– In the modular group $\mathsf{PSL}_2(\mathbb{Z}) = \langle a, b \mid a^2 = 1, b^3 = 1 \rangle$ –

- The geodesic words consist of alternations of $a^{\pm 1}$ and $b^{\pm 1}$.
- To obtain a geodesic representative for the element represented by a word u
  - first freely reduce u,
  - delete every occurrence of a^2, b^3 and their inverses,
  - replace every occurrence of b^2 (resp. b^{-2}) by b^{-1} (resp. b).
- $\mathsf{PSL}_2(\mathbb{Z})$ is geodesically automatic.
- Every subgroup is quasi-convex.
- Combinatorial et algebraic study of properties of the Stallings graphs of subgroups (Bassino, Nicaud, Weil, 2021, 2024, 2025+).

– Example : $H = \langle abab^{-1}, babab \rangle$ in $\mathsf{PSL}_2(\mathbb{Z}) = \langle a, b \mid a^2 = 1, b^3 = 1 \rangle$ –

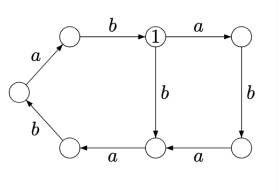

Figure: The Stallings graph of $\langle abab^{-1}, babab \rangle$ in F_A with $A = \{a, b\}$

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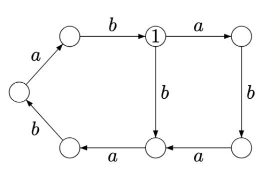

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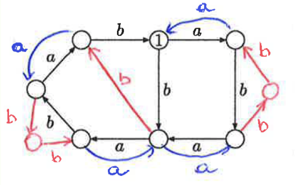

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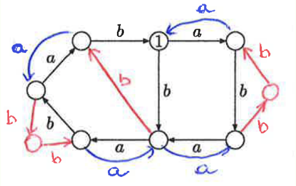

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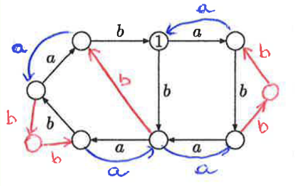

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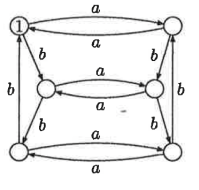

Figure: The Stallings graph of $\langle abab^{-1}, babab \rangle$ in $\mathsf{PSL}_2(\mathbb{Z})$.

– Properties of the Stallings graph –

- Membership can be tested.
- The finitness of the subgroup can be tested.
- A generating set can be computed.
- There is method to determine the rank of the subgroup (the size of minimal set of generators).
- The intersection can be computed.
- Finite index : If the Stalling graph is finite and complete, the subgroup is of finite index. But the converse is not proven.

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III. Graph-Based Distribution

– Size of a subgroup : the number of vertices –

Define the size $|H|$ of a finitely generated subgroup H as the number of vertices of its Stallings graph Γ_H.

Let G_n denote the set of all Stallings graphs with n vertices and whose set of vertices is $[n]$, the vertex 1 being the base vertex.

Some questions:

- What can we say about the cardinality of G_n?
- Can we design an algorithm to draw an element of G_n uniformly at random?
- What are the typical algebraic properties (rank, ...) of a random element of G_n, for the uniform distribution?

[Bassino, Nicaud, Weil 2008, 2016] and [Bassino, Martino, Nicaud, Ventura, Weil 2013].

– Reminder : Characterization of Stallings graphs –

Theorem: a positively labelled graph on $[n]$ is the Stallings graph of a finitely generated subgroup if and only if

- the action of each letter is a partial injection;
- the graph is weakly connected (connected as an undirect graph);
- every vertex, but possibly **1**, has at least two edges (counting both ingoing and outgoing edges).

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A given size-n subgroup H has exactly $(n-1)!$ associated Stallings graphs (by relabeling all the vertices, but vertex 1): the uniform distribution on G_n induces the uniform distribution on size-n finitely generated subgroups.

– An Example –

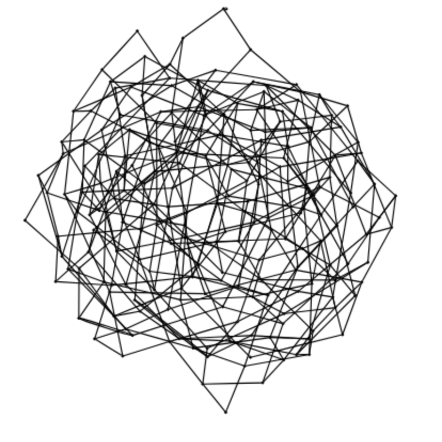
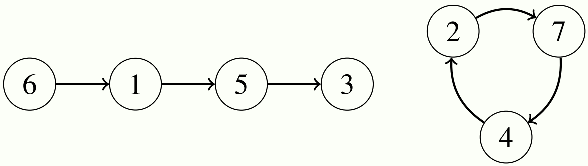


Figure: A random Stallings graph with 200 vertices.

– Partial injections –



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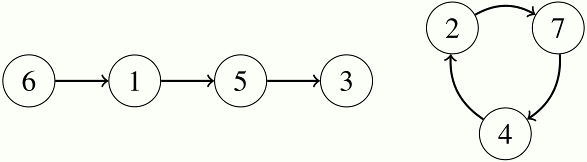
– Partial injections –



- A partial injection is a set of cycles and non-empty sequences of labeled ”atoms”.
- If $\mathcal{I}$ denote the set of all partial injection, we have

$$\mathcal{I} = \mathrm{Set}\left(\mathrm{Seq}_{\geq 1}(\mathcal{Z}) \,\dot{\cup}\, \mathrm{Cyc}(\mathcal{Z})\right),$$

- The goal is to analyze the exponential generating series of $\mathcal{I}$

$$I(z) = \sum_{n\geq 0} \frac{I_n}{n!} z^n.$$

where I_n denote the number of partial injections on a size-n set.

– Symbolic method (Flajolet, Sedgewick, 2008) –

- "sets of cycles and non-empty sequences"

Sets	$\longleftrightarrow$	$\exp(\bullet)$
"of"	$\longleftrightarrow$	composition
Cycles	$\longleftrightarrow$	$\log(\frac{1}{1-\bullet})$
"and"	$\longleftrightarrow$	$+$
Non-empty sequences	$\longleftrightarrow$	$\frac{\bullet}{1-\bullet}$
Atom	$\longleftrightarrow$	z

- This yields directly

$$I(z) = \exp\left(\log\left(\frac{1}{1-z}\right) + \frac{z}{1-z}\right) = \frac{1}{1-z}\exp\left(\frac{z}{1-z}\right)$$

– Number of partial injections –

$$I(z) = \frac{1}{1-z} \exp\left(\frac{z}{1-z}\right)$$

satisfies the conditions of the saddle point theorem, and therefore if I_n denote the number of partial injection from $[n]$ to $[n]$ we have:

$$\frac{I_n}{n!} \sim \frac{e^{-1/2}}{2\sqrt{\pi}} n^{-1/4} e^{2\sqrt{n}},$$

the saddle point is $\zeta = 1 - \frac{1}{\sqrt{n}} + \mathcal{O}(\frac{1}{n})$.

– Weakly connected? –

Proposition: For $r \geq 2$, r uniform random partial injections on a size-n set forms a weakly connected graph with probability $p_n = 1 - \frac{2^r}{n^{r-1}} + o(\frac{1}{n^{r-1}})$.

The number of pairs of partial injections is I_n^2 and the radius of convergence of $\sum \frac{1}{n!} I_n^2 z^n$ is zero!.
We cannot use analytic techniques.

The proof use a theorem of Bender.

– Proof for bijections –

Proposition: The size-n graph obtained when taking two random **permutations** uniformly at random is weakly connected with high probability.

Proof: If the graph is not connected, the set of vertices $[n]$ can be split in two subsets X and Y that are stable under the action of the two permutations.
Thus, summing over the size k of X, the probability of having such configurations is bounded from above by

$$\frac{1}{n!^2} \sum_{k=1}^{n-1} \binom{n}{k} k!^2 (n-k)!^2 = \sum_{k=1}^{n-1} \frac{1}{\binom{n}{k}},$$

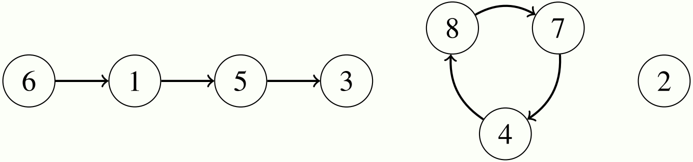
which is $\mathcal{O}(\frac{1}{n})$.

– Last condition –

Theorem: a positively labelled graph on $[n]$ is the Srallings graph of a finitely generated subgroup if and only if

- the action of each letter is a partial injection;
- the graph is weakly connected (connected as an undirect graph)
- every vertex, but possibly **1**,has at least two incident edges (counting both ingoing and outgoing edges).

– Vertices with zero or one outgoing or ingoing edge –



- If x is a vertex with 0 edge, then x must be isolated for all injections.
- If x is a vertex with 1 edge, then x must be isolated for one injection and an endpoint for the other injection when $r = 2$.

The probability it is isolated for one injection is $\frac{I_{n-1}}{I_n}$, which is smaller than $\frac{1}{n}$.

– Computing the number of sequences –

If $A(z)$ is a power series, let $[z^n]A(z)$ denote its coefficient a_n.

Let $I_{n,k}$ be the number of size-n injections having k sequences, and let $I(z,u)$ be the bivariate generating function defined by:

$$I(z,u) = \sum_{n\geq 0}\sum_{k\geq 0} \frac{I_{n,k}}{n!} z^n u^k,$$

Observe that $I(z,1) = I(z)$ and that

$$\frac{[z^n]\,\frac{d}{du}I(z,u)\Big|_{u=1}}{[z^n]I(z)} = \frac{\sum_{k\geq 0} k\, I_{n,k}}{I_n}$$

is the expected number of sequences in a size-n injection.

Using the second derivative, we also get an expression of the variance of the number of sequences in a size-n injection.

– Using marks –

$$\mathcal{I} = \text{Set}\left(\bullet\text{Seq}_{\geq 1}(\mathcal{Z}) \,\dot{\cup}\, \text{Cyc}(\mathcal{Z})\right)$$

There is one blue mark for each non-empty sequence.
The bivariate EGS of $\mathcal{I}$ is

$$I(z,u) = \exp\left(\frac{zu}{1-z} + \log\left(\frac{1}{1-z}\right)\right) = \frac{1}{1-z}\exp\left(\frac{zu}{1-z}\right)$$

Appling the saddle point theorem on
$I(z) = \frac{1}{(1-z)^p}\exp\left(\frac{z}{1-z}\right)$ $(p = 1, 2)$, we obtain that

- the expected number of sequences is $\sqrt{n}$ with standard deviation $o(\sqrt{n})$
- the probability that a given vertex is an endpoint is in $\mathcal{O}(\frac{1}{\sqrt{n}})$
- the probability that it has 0 or 1 edge with probability $\mathcal{O}(n^{-3/2})$: there is such a vertex with probability $\mathcal{O}(n^{-1/2})$: with high probability the graph has no such vertex.

– Stallings graph with high probability –

Theorem: For $r \geq 2$, a r-tuple of partial injections of $[n]$ generically (*i.e.* with a probability that tends to 1 when n tends to ∞) forms a Stallings graph.

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Corollary: The number of finitely generated subgroups of size n is equivalent to

$$\frac{I_n^r}{(n-1)!} \sim \frac{(2e)^{-r/2}}{\sqrt{2\pi}} e^{-(r-1)n+2r\sqrt{n}} n^{(r-1)n+\frac{r+2}{4}}.$$

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Proposition: For $r \geq 2$, the average rank of a subgroup H of the free group with a size-n Stallings graph is

$$rk(H) = (r-1)n - r\sqrt{n} + o(\sqrt{n})$$

– Malnormality –

Definition: A subgroup H of G is malnormal when for every $g \notin H$, $g^{-1}Hg \cap H = \{1\}$.

Theorem: A finitely generated free subgroup H is not malnormal if and only if there are two loops with the same label in the Stallings graph of H.

Sufficient conditions:
• If there is a a-cycle of length at least 2, the subgroup is not malnormal.
• If there are at least two a-cycles of length 1, the subgroup is not marnomal.

– Malnormality –

Partial injections with no cycles:

$$\mathcal{J} = \text{Set}\left(\text{Seq}_{\geq 1}(\mathcal{Z})\right) \Longrightarrow J = \exp\left(\frac{z}{1-z}\right)$$

Partial injections with only 1 cycle which is of length 1:

$$\mathcal{K} = \mathcal{Z} \star \text{Set}\left(\text{Seq}_{\geq 1}(\mathcal{Z})\right) \Longrightarrow J = z\exp\left(\frac{z}{1-z}\right)$$

By the saddle point theorem:

Theorem:

- The probability that a size n partial injection has no cycle of length greater than 2 is asymptotically equivalent to $\frac{e}{\sqrt{n}}$.
- The probabibility that a random finitely generated subgroup of a free group is malnormal is $\mathcal{O}(n^{-r/2})$.

– Random generation of a finitely generated subgroup –

Recall that

- The action of each letter is a partial injection:

  $\longrightarrow$ Generate as many partial injections as the size of the alphabet

- The graph is weakly connected and every vertex, but possibly **1**, has at least two edges

  $\longrightarrow$ Reject if it is not true, and try again

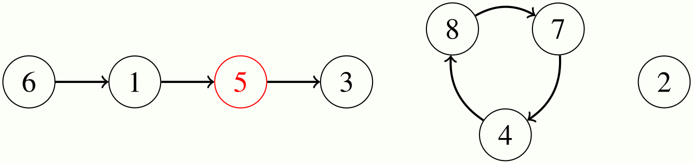
– Random generation of a partial injection –

A partial injection is a set of disjoint *components*, that are either cycles or non-empty sequences.
To recursively generate a uniform partial injection:

- choose the size k of a component according to the distribution of the sizes of components in a random size n partial injection;
- choose whether that the size k component is a cycle or a sequence – according to the distribution of these two types among size k components;
- and choose a size $n - k$ partial injection.

– Pointing –

Let $\mathcal{C} = \text{Set}(\mathcal{A})$. If we mark an atom uniformly at random, we also designate the element of $\mathcal{A}$ which contains this atom.



If $\Theta\mathcal{C}$ denotes the set of all elements of $\mathcal{C}$ with one marked atom, we have the bijection

$$\Theta\mathcal{C} \equiv \Theta\mathcal{A} \star \mathcal{C}$$

– Pointing –

The generating function of $\Theta\mathcal{A}$ is

$$\sum_{n\geq 0} n\,\frac{a_n}{n!}\,z^n = z\frac{d}{dz}A(z)$$

$\Theta\mathcal{C} \equiv \Theta\mathcal{A} \star \mathcal{C}$ is a combinatorial interpretation of

$$z\frac{d}{dz}\exp\left(A(z)\right) = z\frac{d}{dz}A(z) \times \exp\left(A(z)\right)$$

If we select one atom, the probability it is in a component of size k is

$$\frac{\frac{ka_k}{k!}\,\frac{c_{n-k}}{(n-k)!}}{\frac{nc_n}{n!}} = \binom{n}{k}\frac{ka_k c_{n-k}}{nc_n}$$

– Size of a component –

In our settings, an element of $\mathcal{A}$ is either a non-empty sequence or a cycle, hence $a_k = k! + (k-1)!$. The probability of pointing a size-k component is therefore

$$\binom{n}{k}\frac{k a_k I_{n-k}}{n I_n}$$

It can easily be computed if the I_m have been preprocessed.
If we compute the derivative of $I(z)$ we obtain that

$$I'(z) = \frac{2-z}{(1-z)^3}\exp\left(\frac{z}{1-z}\right) = \frac{2-z}{(1-z)^2}I(z)$$

And therefore

$$I_n = 2nI_{n-1} - (n-1)^2 I_{n-2},$$

$\Longrightarrow$ we can compute the I_n efficiently.

– Random generation –

A direct computation shows that a given size-k component is a cycle with probability $\frac{1}{k+1}$

We can therefore build the random injection component by component.

Theorem: There exists an algorithm to generate size-n random Stallings graphs whose average complexity is linear.

– Bibliography –

- F. Bassino, A. Martino, C. Nicaud, E. Ventura and P. Weil. Statistical properties of subgroups of free groups. Random Struct. Algorithms, 42(3):349-373, 2013.
- F. Bassino, C. Nicaud and P. Weil. Random generation of finitely generated subgroups of a free group. Int. Journ. of Algebra Comput. 18(2) 375-405, 2008
- F. Bassino, C. Nicaud and P. Weil. Generic properties of subgroups of free groups and finite presentations. In Algebra and computer science, volume 677 of Contemp. Math., pages 1-43. Amer. Math. Soc., Providence, RI, 2016.
- F. Bassino, C. Nicaud and P. Weil. Statistics of subgroups of the modular group. Int. J. Algebra Comput., 31(8):1691-1751, 2021.
- F. Bassino, C. Nicaud and P. Weil. Random generation of subgroups of the modular group with a fixed isomorphism type. Electron. J. Combin., 31(4) P4.7, 2024.
- J.-C. Birget, S. Margolis, J. Meakin, and P. Weil. PSPACE-complete problems for subgroups of free groups and inverse finite automata. Theoret. Comput. Sci 242(1):247-281, 2000.
- S. Cho, D. Huynh, Finite-automaton aperiodicity is PSPACE-complete, Theoret. Comput. Sci. 88: 99-116, 1991.
- J. Delgado and E. Ventura. A Algorithmic problems for free-abelian times free groups. J. Algebra, 391:256-283, 2013.
- J. Delgado, E. Ventura. Stallings automata. Preprint arXiv:2403.10757 (survey)
- Ph. Flajolet and R. Sedgewick. Analytic Combinatorics. Cambridge University Press 2009
- A. G. Howson. On the intersection of finitely generated free groups. Journal of the London Mathematical Society s1-29.4, p. 428-434, 1954.
- T. Jitsukawa. Malnormal subgroups of free groups. In Computational and statistical group theory, volume 298 of Contemp. Math., pages 83-95. Amer. Math. Soc., Providence, RI, 2002
- I. Kapovich. Detecting quasiconvexity: algorithmic aspects. Geometric and Computational Perspectives on Infinite Groups, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 25, , p. 91-99 Amer. Math. Soc. (1996).
- I. Kapovich, A. Myasnikov. Stallings Foldings and Subgroups of Free Groups, J. Algebra, 248 (2): 608-668. 2002. (survey)
- L. Markus-Epstein. Stallings foldings and subgroups of amalgams of finite groups. Internat. J. Algebra Comput. 17(8):1493-1535, 2007.
- I. Mineyev. ASubmultiplicativity and the Hanna Neumann Conjecture. Ann. Math. 175.1, p. 393-414, 2012.
- H. Neumann. AOn the intersection of finitely generated free groups. Publicationes Mathematicae 4: 186-189, 1956.
- P. V. Silva, X. Soler-Escrivà, and E. Ventura. Finite automata for Schreier graphs of virtually free groups. J. Group Theory, 19(1):25-54, 2016.
- J.R. Stallings. Topology of finite graphs. Inventiones Math. 71 (1983) 551-565
- N. W. M. Touikan. A Fast Algorithm for Stallings' Folding Process. Int. J. Algebra Comput. 16(6): 1031-1046, 2006.

That’s all, thanks!