Combinatoire des groupes via les graphes de Stallings

▶ **Exercice 1** [**Construction of a Stallings graph in F_A where $A = \{a, b\}$.**]
In F_A where $A = \{a, b\}$

- Build the Stallings graph of the subgroup $H = \langle aba^{-1}, a^{-1}b^{-1}a^{-1}ba^{-1}, abab^{-1}a \rangle$.
- Compute the rank of H.
- Find a base of H.

▶ **Exercice 2**[**Finite index.**]
The *index* $[G : H]$ of a subgroup H in a group G is its number of right cosets (or of left cosets).

Prove the following result and the Schreier index formula:

Theorem: A subgroup H of the a free group is of finite index if and if its Stallings graphs is finite and complete.
Moreover if H is a subgroup of free group of finite index :

$$\mathrm{rk}(H) - 1 = [F : H](\mathrm{rk}(F) - 1) \qquad \text{(Schreier index formula)}$$

▶ **Exercice 3**[**Base and rank.**]
Prove that the following procedure is correct and deduce the theorem below.

- Let H be a finitely generated subgroup of a free group F. To obtain a base of H, choose a spanning tree T of the Stallings graph of H. The elements of the base are the labels of the loops at the base vertex using e and edges in the spanning tree for each edge e that is not in T .

  Hint : Show that the labels of the edges that are not in the spanning tree are not reduced in a product unless this product is of the form ss^{-1}, where s is a generator of the base builded for H.

- *Theorem:* The rank of the finitely generated subgroup H of the free group F is computable from its Stallings graph and:

$$\mathrm{rk}(H) = E - V + 1.$$

▶ **Exercice 4**[**Intersection.**]
Show that the intersection of two finitely generated subgroups of a free group F is finetely generated.

▸ **Exercice 5**[**Quasi-convexity and finitness of Stallings graphs.**]

A subgroup $H \leq G$ is *quasi-convex* if there exists $k > 0$, such that any geodesic path in the Cayley graph of G connecting two elements of H remains within a k-neighborhood of H.

This implies that a shortest path $u = a_1 \dots a_p$ between any two elements of H does not stray too far from H itself : for $1 \leq i \leq p$, $\exists |v_i| \leq k$ such that $a_1 \dots a_i v_i \in H$.

Show the following result.

Proposition: (Kharlampovich, Miasnikov, Weil 2017) A subgroup of a finitely generated group is quasi-convex if and only if its Stallings graph is finite

▸ **Exercice 6**[**Malnormality.**]

A subgroup H is *malnormal* if, for all $g \notin H$, $H^g \cap H = 1$.

Show the following result:

Proposition: A finitely generated subgroup H of a free group F is malnormal

- if and only if every non-diagonal (*i.e.* that do not contain the vertices (v, v)) connected component of the product graph of the Stallings graph of H $\Gamma(H)$ with itself is a tree;
- if and only if there are no two loops of the same label in the Stallings graph of H.

Hint: Show that the Stallings graph of the conjugate $H^g = g^{-1}Hg$ of H is obtain by moving the base vertex **1** of the Stallings graph of H $\Gamma(H)$ to the vertex reached after reading g from the base vertex **1** in the graph $\Gamma(H)$.