Distributions of parameters in restricted classes of maps and $\lambda$-terms

Alexandros Singh (LIPN, Paris 13)
Olivier Bodini (LIPN, Paris 13)
Noam Zeilberger (LIX, Polytechnique)

$\mathcal{N}\left(\left(\frac{2n}{3}\right)^{1/3}, \left(\frac{2n}{3}\right)^{1/3}\right)$

$\mathcal{N}\left(\left(\frac{2n}{3}\right)^{2/3}, \left(\frac{2n}{3}\right)^{2/3}\right)$

Poisson(1)

ALEA, 14 March 2021

Olivier Bodini (LIPN, Paris 13)
Alexandros Singh (LIPN, Paris 13)
Noam Zeilberger (LIX, Polytechnique)
What is the $\lambda$-calculus?
What is the \( \lambda \)-calculus?

- A **universal** system of computation
What is the λ-calculus?

- A **universal** system of computation
- Its terms are formed using the following grammar

\[
\begin{align*}
\text{x} & \mid \lambda \text{x}. \text{t} \mid (\text{s} \text{ t}) \\
\end{align*}
\]
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\[ \chi \mid \lambda \chi . t \mid (s \ t) \]

- **variable**
- **abstraction** represents an *anonymous* function
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$$ \chi \mid \lambda \chi.t \mid (s\ t) $$

- **variable**
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- **application** feeding an argument $t$ to a function $s$
What is the λ-calculus?

- A **universal** system of computation
- Its terms are formed using the following grammar

\[ \text{variable} \mid \lambda x. t \mid (s \, t) \]

- variable
- abstraction represents an *anonymous* function
- application feeding an argument \( t \) to a function \( s \)

- We’re interested in terms up to \( \alpha \)-equivalence:

\[
(\lambda x.xx)(\lambda x.xx) \overset{\alpha}{=} (\lambda y.yy)(\lambda x.xx) \neq (\lambda y.ya)(\lambda x.xx)
\]
Subfamilies of $\lambda$-terms

General terms: no restrictions on variable use

\[ \lambda x.\lambda y. x \]

\[ \lambda x.\lambda y. x \ (y \ a) \]

\[ (\lambda x.x x)(\lambda y.y y) \]
Subfamilies of $\lambda$-terms

General terms: no restrictions on variable use

\[ \lambda x.\lambda y.x \]
free variable

\[ \lambda x.\lambda y.x \]
unused abstraction

\[ (\lambda x.xx)(\lambda y.yy) \]
var. used twice

\[ \lambda x.\lambda y.x (y \ a) \]
Subfamilies of $\lambda$-terms

General terms: no restrictions on variable use

- $\lambda x.\lambda y. x$ (free variable)
- $\lambda x.\lambda y. x$ (unused abstraction)
- $(\lambda x.xx)(\lambda y.yy)$ (variable used twice)

Affine Terms: bound variables occur at most once

- $(\lambda x.\lambda y.a)(\lambda x.x)$
- $\lambda x.\lambda y.\lambda z.(x a) y$
- $\lambda x.\lambda y.y$

Subfamilies of $\lambda$-terms

General terms: no restrictions on variable use

- $\lambda x.\lambda y. x \ (y \ a)$
- $\lambda x.\lambda y. \lambda z. (x \ a) \ y$
- $\lambda x.\lambda y. (y \ x) a$
- $\lambda x.\lambda y. (y \ a) (b \ x)$

Free variable

Unused abstraction

Variable used twice

Affine Terms: bound variables occur at most once

- $(\lambda x.\lambda y. a) (\lambda x. x)$
- $\lambda x.\lambda y. y$
- $\lambda x.\lambda y. (y \ x) a$
- $\lambda x. a (\lambda z. (\lambda y. y (x \ z)))$

Linear Terms: bound variables occur exactly once

- $(\lambda x.\lambda y. x) (\lambda y. y)$
- $\lambda x.\lambda y. (y \ x) a$
- $\lambda x.\lambda y. (y \ a) (b \ x)$
- $\lambda x. a (\lambda z. (\lambda y. y (x \ z)))$
What are maps?
What are maps?
What are maps?

We’re interested in unrestricted genus, restricted vertex degrees
Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16]

\[ \bullet = \chi \\
\chi = \lambda x. t \\
(\circ) = (s \ t) \]
Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16] \((\lambda y.\lambda z.(y \lambda w.w)z))(\lambda u.\lambda v.a \ u)\)

\[\bullet = \chi\]

\[\chi = \lambda x. t\]

\[s \ t = (s \ t)\]

(order matters!)
Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16] \((\lambda y.\lambda z.(y \lambda w.w)z))(\lambda u.\lambda v.a\ u)\)

- Free var \(\leftrightarrow\) unary vertex

\[\bullet = \chi\]

\[
t\xrightarrow{\chi} = \lambda x. t
\]

\[
s \quad t \xrightarrow{\text{order matters!}} = (s \ t)
\]
Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16] \((\lambda y.\lambda z.(y\ \lambda w.w)z))\(\lambda u.\lambda v.\alpha \ u)\)

\[\bullet = x\]

\[\begin{array}{c}
\text{order matters!} \\
\end{array}\]

\[\begin{array}{c}
\lambda x.t \\
(t\ x) \\
s \ t \\
\end{array}\]

\[= (s\ t)\]

**Dictionary**

- Free var ↔ unary vertex
- Unused \(\lambda \leftrightarrow \) binary vertex
Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16] \((\lambda y. \lambda z. (y \lambda w.w)z)) (\lambda u. \lambda v. a\ u)\)

\[\bullet = x\]

\[t \xrightarrow{x} \lambda x.t\]

\[s \quad t\]

\[= (s\ t)\]

**Dictionary**

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Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16] 

\[
(\lambda y.\lambda z.(y \lambda w.w)z)(\lambda u.\lambda v.au)
\]

\[
\bullet = \chi
\]

\[
x = \lambda x.t
\]

\[
(s \circ t) = (s \cdot t)
\]

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• Free var ⇔ unary vertex
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• \# subterms ⇔ \# edges
Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16]

\[
\lambda y.\lambda z.(y \lambda w.w)z)(\lambda u.\lambda v.a u)
\]

\[\ldots\]

Dictionary

- Free var $\leftrightarrow$ unary vertex
- Unused $\lambda$ $\leftrightarrow$ binary vertex
- Identity-subterm $\leftrightarrow$ loop
- Closed subterm $\leftrightarrow$ bridge
- $\#\text{ subterms} \leftrightarrow \#\text{ edges}$

Closed linear terms $\leftrightarrow$ trivalent maps
Closed affine terms $\leftrightarrow$ (2,3)-valent maps
Established in [BGJ13, BGGJ13]
Why should you, a combinatorialist, be interested in λ-terms?
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Decomposing rooted $(1,3)$-valent maps $(+ \bullet)$ à la Tutte [AB00]
Why should you, a combinatorialist, be interested in $\lambda$-terms?

Decomposing rooted (1,3)-valent maps $(+ \bullet)$ à la Tutte [AB00]

$$\text{OT}(z, u) = z$$

**Diagram:**
- Edges
- Unary vertices
Why should you, a combinatorialist, be interested in $\lambda$-terms?

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$$OT(z, u) = z + zOT(z)^2$$
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Decomposing rooted (1,3)-valent maps ($+ \bullet$) à la Tutte [AB00]

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Decomposing rooted \((1,3)\)-valent maps \((+ \bullet) \) à la Tutte [AB00]

\[
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\]

\[
\text{lin. term} = \chi
\]
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\end{array}
\]

\[
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(s \ t) \\
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Why should you, a combinatorialist, be interested in $\lambda$-terms?

Decomposing rooted (1,3)-valent maps (+ •) à la Tutte [AB00]
and open linear terms! [Z16]

$$\text{OT}(z, u) = z + z\text{OT}(z)^2 + z\partial_u\text{OT}(z)$$

$$\text{lin.term} = \chi (s \ t) \lambda x. t$$
Recap: $\lambda$-terms and maps

- Syntactic diagrams of families of $\lambda$-terms yield maps
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- Syntactic diagrams of families of $\lambda$-terms yield maps
- $\lambda$-terms as invariants of maps encoding decomposition data
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Our results: limit distributions
Trivalent maps $\leftrightarrow$ closed linear terms

$$\lambda x.\lambda y. (y \lambda w. w)x$$
Our results: limit distributions

Trivalent maps $\leftrightarrow$ closed linear terms

$\# \text{ loops} = \# \text{ id-subterms}$

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Our results: limit distributions

Trivalent maps $\leftrightarrow$ closed linear terms

$\#\text{ loops} = \#\text{ id-subterms}$

$\lambda x.\lambda y. (y \lambda w.w) x$

$X_n^{id} \xrightarrow{D} \text{Poisson}(1)$
Our results: limit distributions

Trivalent maps $\leftrightarrow$ closed linear terms

$$\lambda x.\lambda y. (y \lambda z.\lambda w. zw)x$$
Our results: limit distributions

Trivalent maps $\leftrightarrow$ closed linear terms

$\# \text{ bridges} = \# \text{ closed subterms}$

\[
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\]
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$\# \text{ bridges} = \# \text{ closed subterms}$

$\lambda x.\lambda y. (y \lambda z.\lambda w. zw)x$
Our results: limit distributions

Trivalent maps $\leftrightarrow$ closed linear terms

\[ \lambda x.\lambda y.(y \lambda z.\lambda w.z w)x \]

$\# \text{bridges} = \# \text{closed subterms}$

bad news for remote villages in rooted trivalent maps...

one bridge $\leftrightarrow$ no bridge

$X^\text{sub}_n \xrightarrow{D} \text{Poisson}(1)$
Our results: limit distributions

$(1,3)$-valent maps $\leftrightarrow$ open linear terms

\[(a \ (\lambda x. \lambda y. (y \ b)(c \ x)))\]
Our results: limit distributions
(1,3)-valent maps $\leftrightarrow$ open linear terms

$\#\text{ unary vertices} = \#\text{ free vars}$

\[
(a \ (\lambda x. \lambda y. (y \ b) (c \ x)))
\]
Our results: limit distributions

(1,3)-valent maps $\leftrightarrow$ open linear terms

$\#$ unary vertices $= \#$ free vars

\[ X_{n} \text{ free} - \mu_{n} \sqrt{\frac{\sigma^{2}_{n}}{n}} \overset{D}{\to} \mathcal{N}(0,1) \]

for $\mu = \sigma^{2} = (2n)^{1/3}$
Our results: limit distributions

(2,3)-valent maps ↔ closed affine terms

\((\lambda x.\lambda y.(\lambda z.x)y)(\lambda w.\lambda v.\lambda u.u)\)
Our results: limit distributions

(2,3)-valent maps ↔ closed affine terms

# binary vertices = # unused \( \lambda \)

\[(\lambda x. \lambda y. (\lambda z. x)y)(\lambda w. \lambda v. \lambda u. u)\]
Our results: limit distributions

(2,3)-valent maps ↔ closed affine terms

# binary vertices = # unused λ

\(\lambda x.\lambda y. (\lambda z.x)y)(\lambda w.\lambda v.\lambda u. u)\)

\[
\frac{X_n^\lambda - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} N(0, 1)
\]

for \(\mu = \sigma^2 = \frac{2n}{2}^{2/3}\)
Our workflow:
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we have a lot of ’em, but only some are tractable!

1) Establish good bijections to obtain specifications for the bivariate OGFs
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OGFs are purely formal, which makes them difficult to analyse!

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- Schema based on ODEs, yielding Poisson limit law:
  \[ \partial^k_u F(z, u) \]
  Only certain terms contribute

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• Schema based on ODEs, yielding Poisson limit law:
  \[ \partial^k_u F(z, u) \]
  Only certain terms contribute

• Schema based on compositions (see also [B75,FS93,B18,P19,BKW21]):
  \[ F(z, u, G(z, u)) \]
  \[ G(z, u) \]
  inherits the limit law of

we have a lot of 'em, but only some are tractable!
Proof sketch for loops/id-subterms:
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\[ T^{id}(z, u) = (u - 1)z^2 + zT^{id}(z, u)^2 + \partial_u T^{id}(z, u) \]
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Pumping \( T^{id}(z, u) \)
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Pumping \( T^{id}(z, u) \)

\[
[z^n] \left. \partial_u T^{id} \right|_{v=1} = T^{id} - (u - 1)z^2 - z(T^{id})^2 \quad \sim \quad [z^n] \partial_u T^{id}(z, 1)
\]
Proof sketch for loops/id-subterms:

\[ T^{\text{id}}(z, u) = (u - 1)z^2 + zT^{\text{id}}(z, u)^2 + \partial_u T^{\text{id}}(z, u) \]

Pumping \( T^{\text{id}}(z, u) \)

\[
\begin{align*}
[z^n] \quad \partial_u T^{\text{id}} \big|_{v=1} &= T^{\text{id}} - (u - 1)z^2 - z(T^{\text{id}})^2 \\
[z^n] \quad \partial_u^2 T^{\text{id}} \big|_{v=1} &= \partial_u T^{\text{id}} - z^2 + 2zT^{\text{id}} - 2zT^{\text{id}} \partial_u T^{\text{id}}
\end{align*}
\]
Proof sketch for loops/id-subterms:

\[ T^{id}(z, u) = (u - 1)z^2 + zT^{id}(z, u)^2 + \partial_u T^{id}(z, u) \]

<table>
<thead>
<tr>
<th>Proof sketch for loops/id-subterms:</th>
</tr>
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Pumping $T^{id}(z, u)$

\[
\begin{align*}
[z^n] \left. \partial_u T^{id} \right|_{v=1} &= T^{id} - (u - 1)z^2 - z(T^{id})^2 \\
&\sim [z^n] \partial_u T^{id}(z, 1) \\
[z^n] \left. \partial^2_u T^{id} \right|_{v=1} &= \partial_u T^{id} - z^2 + 2zT^{id} - 2zT^{id}\partial_u T^{id} \\
&= T^{id} - 2u^2z^5 - 8uz^4(T^{id})^2 - \ldots \sim [z^n] \partial_u T^{id}(z, 1)
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\[ [z^n] \partial_u^2 T^{id} \Big|_{v=1} = \partial_u T^{id} - z^2 + 2zT^{id} - 2zT^{id}\partial_u T^{id} \]

\[ = T^{id} - 2u^2z^5 - 8uz^4T^{id} - \ldots \sim [z^n] \partial_u T^{id}(z, 1) \]

\[ [z^n] \partial_u^{k+1} T^{id} \Big|_{v=1} = \partial_u^{k} T^{id} - S - 2z T^{id} \partial_u^{k} T^{id} \sim [z^n] \partial_u^{k} \sim T^{id}(z, 1) \]
Proof sketch for loops/id-subterms:

$$T^{id}(z, u) = (u - 1)z^2 + zT^{id}(z, u)^2 + \partial_u T^{id}(z, u)$$

Pumping $T^{id}(z, u)$

$$[z^n] \partial_u T^{id}_{v=1} = T^{id} - (u - 1)z^2 - z(T^{id})^2 \sim [z^n] \partial_u T^{id}(z, 1)$$

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$$[z^n] \partial_u^{k+1} T^{id}_{v=1} = \partial_u^{k} T^{id} - S - 2z T^{id} \partial_u^{k} T^{id} \sim [z^n] \partial_u^{k} \sim T^{id}(z, 1)$$

Schema then yields Poisson(1) limit law
Proof sketch for bridges/closed subterms:

spanning tree def’d by term
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spanning tree def’d by term

No bridges along the path
Proof sketch for bridges/closed subterms:

No bridges along the path

spanning tree def’d by term

Dist. of param. in restricted classes of maps and λ-terms - Bodini, Singh, Zeilberger ALEA 2021
Proof sketch for bridges/closed subterms:

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Dist. of param. in restricted classes of maps and $\lambda$-terms - Bodini, Singh, Zeilberger ALEA 2021
Proof sketch for bridges/closed subterms:

\[ \frac{\partial}{\partial \nu} T_{\text{sub}}(z, \nu) = - \frac{\nu^2 z T_{\text{sub}}^3(z, \nu) + z^2 T_{\text{sub}}(z, \nu) - T_{\text{sub}}^2(z, \nu)}{(\nu^3 - \nu^2) z T_{\text{sub}}(z, \nu)^2 + \nu z^2 - (\nu - 1) T_{\text{sub}}(z, \nu)} \]

spanning tree def’d by term

No bridges along the path

May be pumped using our schema
Proof sketch for vertices of given degree:
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Specifications based on exponential Hadamard products

\[ OT(z, u) = uz^2 + z^4 + z^5 \frac{\partial}{\partial z} \left( \ln \left( \exp(z^2/2) \circ \exp(z^3/3 + uz) \right) \right) \]
Proof sketch for vertices of given degree:

Specifications based on exponential Hadamard products

\[ OT(z, u) = uz^2 + z^4 + z^5 \frac{\partial}{\partial z} \left( \ln \left( \exp\left(\frac{z^2}{2}\right) \odot \exp\left(\frac{z^3}{3} + uz\right) \right) \right) \]
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Specifications based on exponential Hadamard products

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\]

\[
TT(z, u) = z \frac{\partial}{\partial z} \left( \ln \left( \exp\left(\frac{z^2}{2}\right) \odot \exp\left(\frac{z^3}{3} + uz^2\right)\right) \right)
\]

\[
A(z, u) = \frac{z^2 + z^2 TT(z^{1/2}, u)}{1 - z}
\]

(2,3)-valent maps

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Compositions for fast-growing series:

\[ F(z, u, G(z, u)) \]
Compositions for fast-growing series:

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for \( u = 1 \), analytic at 0
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\[ [z^{n-1}] G(z, 1) = o([z^n] G(z, 1)) \]

If \( F \) is the g.f of \( F \), \( G \) the one of \( G \):
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If \( F \) is the g.f of \( \mathcal{F} \), \( G \) the one of \( \mathcal{G} \):

“To build a big \( \mathcal{F}(\mathcal{G}) \) structure, pick a small \( \mathcal{F} \) one and replace one of its atoms with a big \( \mathcal{G} \)-structure”

\[ \text{dictates behavior of parameter} \]
Compositions for fast-growing series:

\[ F(z, u, G(z, u)) \]

for \( u = 1 \), analytic at 0

If \( F \) is the g.f of \( \mathcal{F} \), \( G \) the one of \( \mathcal{G} \):

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If \( F \) is the logarithm:

\[ [z^{n-1}] G(z, 1) = o([z^n] G(z, 1)) \]
Compositions for fast-growing series:

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If \( F \) is the g.f of \( F \), \( G \) the one of \( G \):

“To build a big \( F(G) \) structure, pick a small \( F \) one and replace one of its atoms with a big \( G \)-structure”

If \( F \) is the logarithm:

Asymptotically, almost all not-necessarily-connected \( G \)-structures are connected, so the distribution of params. is the same for connected and not-necessarily-so structures!
Proof sketch for bridges/closed subterms (contd.) :

\[ OT(z, u) = uz^2 + z^4 + z^5 \frac{\partial}{\partial z} \left( \ln \left( \exp \left( \frac{z^2}{2} \right) \circ \exp \left( \frac{z^3}{3} + uz \right) \right) \right) \]

\[ TT(z, u) = z \frac{\partial}{\partial z} \left( \ln \left( \exp \left( \frac{z^2}{2} \right) \circ \exp \left( \frac{z^3}{3} + \frac{uz^2}{2} \right) \right) \right) \]

\[ A(z, u) = \frac{z^2 + z^2 TT(z^{\frac{1}{2}}, u)}{1 - uz} \]
Proof sketch for bridges/closed subterms (contd.):

\[ \text{OT}(z, u) = uz^2 + z^4 + z^5 \frac{\partial}{\partial z} \left( \ln \left( \exp\left(\frac{z^2}{2}\right) \odot \exp\left(\frac{z^3}{3} + uz\right) \right) \right) \]

\[ \text{TT}(z, u) = z \frac{\partial}{\partial z} \left( \ln \left( \exp\left(\frac{z^2}{2}\right) \odot \exp\left(\frac{z^3}{3} + \frac{uz^2}{2}\right) \right) \right) \]

\[ A(z, u) = \frac{z^2 + z^2 \text{TT}(z^{\frac{1}{2}}, u)}{1 - uz} \]

Ammenable to saddle-point analysis!

Both yield Gaussian limit laws
Proof sketch for bridges/closed subterms (contd.):

\[
\begin{align*}
\text{OT}(z, u) &= uz^2 + z^4 + z^5 \frac{\partial}{\partial z} (\ln (\exp(z^2/2) \odot \exp(z^3/3 + uz))) \\
\text{TT}(z, u) &= z \frac{\partial}{\partial z} (\ln (\exp(z^2/2) \odot \exp(z^3/3 + uz^2))) \\
A(z, u) &= \left(\frac{z^2 + z^2 \text{TT}(z^{1/2}, u)}{1 - uz^2}\right)
\end{align*}
\]

Ammenable to saddle-point analysis!

Both yield Gaussian limit laws

Use schema for compositions to show that the results carry over!
Whats next?
Whats next?

- More parameters:

  Mean path length

  Profile
Whats next?

- More parameters:
  - Mean path length
  - Profile

- More map/term families: planar, bridgeless...
Whats next?

- More parameters:
  - Mean path length
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Thank you!
Bibliography


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Bibliography

An asymptotic expansion for the coefficients of some formal power series.

General combinatorial schemas: Gaussian limit distributions and exponential tails.
Discrete Mathematics, 114(1-3), 159-180.

Generating Asymptotics for Factorially Divergent Sequences.
The Electronic Journal of Combinatorics, P4-1.

Analytic Combinatorics of Composition schemes and phase transitions
mixed Poisson distributions.
Bibliography

Analytic combinatorics of connected graphs.

Asymptotics and random sampling for BCI and BCK lambda terms
Theoretical Computer Science, 502, 227-238.
Our results: limit distributions

Trivalent maps $\leftrightarrow$ closed linear terms

$\begin{align*}
\# \text{ loops} &= \# \text{id-subterms} \\
\# \text{ bridges} &= \# \text{ closed subt.}
\end{align*}$

$\{ \text{ Poisson}(1) \}$

$(1,3)$-maps $\leftrightarrow$ open linear terms

$\# \text{ unary vertices} = \# \text{ free vars}$

$\mathcal{N}(\mu, \sigma^2)$ with $\mu = \sigma^2 = (2n)^{2/3}$

$(2,3)$-maps $\leftrightarrow$ closed affine terms

$\# \text{ unary vertices} = \# \text{ free vars}$

$\mathcal{N}(\mu, \sigma^2)$ with $\mu = \sigma^2 = (2n)^{1/3}$