On the asymptotic number of $m$-ary weakly increasing trees

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Introduction to increasing tree labellings
Increasing trees

- There are different kinds of trees (labelled, plane, rooted, restricted degrees, ...).

- Analysis of permutations and data structures like binary search trees using increasing binary trees.
  
  \cite{Drm09, Maj92, FGM06}

- Trees of epidemic spreading and manuscript reconstruction are also increasing trees.

- Study the number of executions of a parallel process and their synchronisations. This leads to repeated labellings. \cite{BGP16, BGR17}
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Some related works on increasing trees

- Increasing trees [BFS92].
  No label repetitions and labellings along branches are strictly increasing.

- Monotone functions on trees [PU83].
  The maximum label is fixed and does not depend on the size of the tree.
  Labellings along branches are weakly increasing and some labels may be skipped.

- Strictly monotonic binary [BGGW20].
  Specific case of strictly monotonic trees with arity 2.

- Ranked Schröder trees [BGN19].
  Two increasing labellings where all arities are allowed.

- Families of monotonic trees [BGNS20].
  A general asymptotic for cases where the number of repetitions allowed is not bounded.
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Increasing labellings on trees

- It is possible to define several *increasing labellings on tree structures*.
- We always consider labellings *without gaps* that is if $m$ is the maximum label appearing in the tree all labels between 1 and $m$ also appear.
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This labellings can be defined on any tree structures but in this talk we will focus only on plane $d$-ary trees.
Specification of increasing labellings on binary trees
Evolution process for increasing binary trees

Start at step 0 with a leaf; at each step $i \geq 1$ do:

1. Choose a leaf of the so-far built tree.
2. Replace it with an internal node labelled $i$ and attach to it 2 new leaves.

Symbolic method with ordinary generating functions

$$IB(z) = z + z^2 IB'(z)$$

The first coefficients are:

\[(IB_n)_{n \geq 0} = 0, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...\]

referenced under EIS A000142.
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(IBM)_{n=6} &= 720, \\
(IBM)_{n=7} &= 5040, \\
(IBM)_{n=8} &= 40320, \\
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![Tree Diagram]

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Evolution process for strict monotonic binary trees

Start at step 0 with a leaf; at each step $i \geq 1$ do:

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Symbolic method with ordinary generating functions

$$SB(z) = z + SB(z + z^2) - SB(z)$$

The first coefficients are:

$$(SB_n)_{n \geq 0} = 0, 1, 1, 2, 7, 34, 214, 1652, 15121, 160110, 1925442, \ldots$$

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We covered two labellings where the paths along branches are strictly increasing.
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**Idea to get weakly increasing labellings along branches**

Instead of replacing leaves with new binary nodes, we can replace them by entire tree shapes.
Evolution process monotonic binary trees

Start at step 0 with a leaf; at each step $i \geq 1$ do:

1. Choose a non-empty subset $L$ of leaves of the so-far built tree such that $|L| \in r$.
2. For each $\ell \in L$ choose an unlabelled binary tree shape.
3. Replace $\ell$ with the chosen unlabelled tree, labelling all its internal nodes $i$.  

Symbolic method with ordinary generating functions

$$MB(z) = z + MB\left(\text{cat}(z)\right) - MB(z)$$

where $\text{cat}(z) = z + z^2 + 2z^3 + 5z^4 + 14z^5 + \ldots$

The first coefficients are:

$$(MB_n)_{n \geq 0} = 0, 1, 1, 4, 22, 152, 1264, 12304, 137332, 1729584, 24265584, \ldots$$
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Asymptotics of increasing labellings on binary trees
Methodology of the proof

- All labellings can be specified in the world of OGF with simple evolution processes.
- From these specifications we get divergent series.

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A method was described in [BGGW20] to obtain the asymptotic enumeration of strict monotonic binary trees. The ideas are the following:
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where \( cat(z) = z + z^2 + 2z^3 + 5z^4 + 14z^5 + \ldots \)

A method was described in [BGGW20] to obtain the asymptotic enumeration of strict monotonic binary trees. The ideas are the following:

- From the specification we get:

\[
B_1 = 1, \\
B_n = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k}{k} B_{n-k}.
\]
Methodology of the proof

- All labellings can be specified in the world of OGF with simple evolution processes.
- From these specifications we get divergent series.

<table>
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- From the specification we get:
  
  $B_1 = 1,$

  $B_n = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k}{k} B_{n-k}.$

- Define a new recurrence $b_n$, that normalises the coefficients of $B_n$.

  $b_n = \frac{B_n}{(n-1)! (\ln 2)^n}.$
Methodology of the proof II

- Write a new recurrence on $b_n$:

$$b_n = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(\ln 2)^k}{k!} \frac{(n-k)(n-k-1)\ldots(n-2k+1)}{(n-1)(n-2)\ldots(n-k)} b_{n-k} \cdot \gamma_{n,k}$$
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Find bounds on $\gamma_{n,k}$:

For $1 \leq k \leq n$ we can show that:

$$1 - \frac{k(k-1)}{n} - \frac{k(k-1)^2}{n^2} \leq \gamma_{n,k} \leq 1 - \frac{k(k-1)}{n} + \frac{k(k-1)^3}{2n^2}.$$
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- Deduce that:
  \[ b_n \underset{n \to \infty}{=} c \left(\frac{1}{\ln 2}\right)^n n^{-\ln 2} \quad \text{where } c \text{ is a constant that involves the evaluation of } a'(z) \]
In summary

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- By noticing that the substitution can be written as an unbounded sum.
- All 4 specifications can be represented as parameters of the following general specification.

$$B(z) = z + \sum_{i \in r} \frac{1}{i!} B^{(i)}(z) \cdot \phi(z)^i.$$  

$\frac{B^{(i)}(z)}{i!}$ corresponds to erasing $i$ leaves.

<table>
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<th>$r$</th>
</tr>
</thead>
<tbody>
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<td>${1}$</td>
</tr>
<tr>
<td>Connected monotonic</td>
<td>$\text{cat}(z) - z$</td>
<td>${1}$</td>
</tr>
<tr>
<td>Strict monotonic</td>
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<td>$\mathbb{N}^*$</td>
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\( \phi_i = [z^i] \phi(z) \)

### Condition

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Let \( B_n^{\phi,r} \) be the number of trees of size \( n \) built via the evolution process.
A general theorem

The following theorem generalises the method described in [BGGW20]. Let 
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Let \( B_{n}^{\phi, r} \) be the number of trees of size \( n \) built via the evolution process

**Theorem**

Let \( \phi(z) \) be a coloured degree function as in the condition, and \( r \subset \mathbb{N}^* \), with \( r \neq \emptyset \). Let \( \min(r) = 1 \), then as \( n \) tends to infinity

\[ B_{n}^{\phi, r} \xrightarrow{n \to \infty} \kappa n! \left( \frac{\phi_2}{\rho} \right)^n n^{-1 + \frac{\rho \phi_3}{\phi_2^2} - \frac{\rho f''(\rho)}{f'(\rho)}} \],

where \( \kappa \) is a constant that depends on \( \phi(z) \) and \( r \). Let \( f(z) = \sum_{i \in r} \frac{z^i}{i!} \), then \( \rho \) is the smallest positive real of the equation \( f(z) - 1 = 0 \).
Comparison of binary trees increasing labellings

<table>
<thead>
<tr>
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<th>$\phi(z)$</th>
<th>Asymptotics</th>
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<tr>
<td>1</td>
<td>$z^2$</td>
<td></td>
<td>$(n - 1)!$</td>
<td>[FS09], Theorem I</td>
</tr>
<tr>
<td>Connected monotonic</td>
<td>${1}$</td>
<td>$(\text{cat}(z) - z)$</td>
<td>$c_3 \cdot n! \cdot n$</td>
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</tr>
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<td>Strict monotonic</td>
<td>$\mathbb{N}^*$</td>
<td>$z^2$</td>
<td>$c_4 \cdot (n - 1)! \cdot \left(\frac{1}{\ln 2}\right)^n \cdot n^{-\ln 2}$</td>
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</tr>
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<td>$\mathbb{N}^*$</td>
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**Table 1:** Comparison of the asymptotic behaviour of labelled binary trees under different labelling models.

Figure: Simulation for $n \in \{1, 100\}$ of binary trees with different increasing labellings divided by their expected asymptotic first order.
Two specifications of increasing labellings on $d$-ary trees
Specification by the number of leaves

All 4 specifications can be represented as parameters of the same functional equation

\[ B(z) = z + \sum_{i \in r} \frac{1}{i!} B(i)(z) \cdot \phi(z)^i. \]

\( B(i)(z) \) corresponds to erasing \( i \) leaves.

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<td>Connected monotonic</td>
<td>( C(z) = z + (T_d(z) - z) \cdot C'(z) )</td>
<td>( T_d(z) - z )</td>
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Where \( T_d(z) \) is the generating function of the class of plane \( d \)-ary trees counted by their number of leaves.

\[ T_d(z) = z + T_d(z)^d \]
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<th>( d )</th>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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<tr>
<td>2</td>
<td></td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>22</td>
<td>152</td>
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<td>1729584</td>
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<td>9336</td>
</tr>
<tr>
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<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>100</td>
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Table 2: First values of monotonic \( d \)-ary trees.
Known results on $d$-ary increasing trees

- 2 out of the 4 increasing labellings have asymptotic enumeration formulae that are known.

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- The classical increasing asymptotics can be solved in the world of EGF and explicit generating functions can be obtained.
- Moreover, the recurrences are D-finite.
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- The classical increasing asymptotics can be solved in the world of EGF and explicit generating functions can be obtained.
- Moreover, the recurrences are D-finite.
- The strict monotonic specifications are formal and their renormalisation does not satisfy known differential equations.
• Theorem 1 does not cover cases where there are no binary nodes.
• The sequences appear with periodicities since the size is the number of leaves.
• In fact it is possible to specify counting the number of internal nodes.

They key idea is to notice that a $d$-ary tree with $n$ internal nodes has $(d - 1)n + 1$ leaves which represent possible new nodes positions.
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$$ML_d(z) = z + ML_d(z + zd) - ML_d(z)$$

The specification by the number of internal nodes gives:

$$MT_d(z) = z + (1 + z) MT_d(z(1 + z)^{d-1}) - MT_d(z)$$
Theorem 1 does not cover cases where there are no binary nodes.
- The sequences appear with periodicities since the size is the number of leaves.
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# Specifications by the number of internal nodes

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<tr>
<th>Specification</th>
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Where \( T_d(z) \) is the generating function of the class of plane \( d \)-ary trees counted by their number of internal nodes.

\[
T_d(z) = z (1 + T_d(z))^d
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- These specifications do not involve periodicities.
- The same method presented before can be applied to obtain asymptotic enumeration formulae.
Complete asymptotic framework
We can now complete the asymptotic picture on $d$-ary trees.

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<tr>
<td>Increasing</td>
<td>$c_d \ n! \ (d-1)^n \ n^{-\frac{d-2}{d-1}}$</td>
<td>[FS09], [BFS92]</td>
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<td>Connected monotonic</td>
<td>$\alpha_d \ n! \ (d-1)^n \ n^{\frac{2}{d-1}}$</td>
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<td>$\kappa_d \ n! \ (\frac{d-1}{ln\ 2})^n \ n^{-\frac{d\ ln\ 2}{2(d-1)} - \frac{d-2}{d-1}}$</td>
<td>[BGGW20]</td>
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Conclusion and future work

- We saw that it is possible to specify label repetitions and weakly increasing labellings on $d$-ary trees in the world of OGF.
- The same method can be applied to specify plane oriented recursive trees (PORTs) and $d$-bundled trees.
- How difficult it is to have a general specification of these labellings on any simple trees class?
- What can be said about non-plane trees?
Thank you for listening
References


[FGM+06] Philippe Flajolet, Xavier Gourdon, Conrado Martinez, Philippe Flajolet, Xavier Gourdon, Conrado Martinez, Random Binary, and Search Trees. Patterns in Random Binary Search Trees To cite this version : HAL Id : inria-00073700 apport de recherche. 2006.

