Graph recolouring
Graph colouring

⇒

Reconfiguration graph

Solutions // Nodes. Most similar solutions // Neighbours.
Graph recolouring

\[ \Rightarrow \]

Solutions // Nodes. Most similar solutions // Neighbours.
Graph recolouring

Graphs $G$ and $G'$.
Graph recolouring
Graph recolouring \(\Rightarrow\) Reconfiguration graph

Solutions // Nodes. Most similar solutions // Neighbours.
Graph 3-recolouring: bad cases (1)
Graph 3-recolouring: bad cases (2)

\[
\begin{align*}
\alpha_w(\alpha, C_5) &= -3 \\
\beta_w(\beta, C_5) &= 3
\end{align*}
\]
Graph 3-recolouring: bad cases (2)

\[ \alpha \]

\[ w(\alpha, C_5) = -3 \]

\[ \beta \]

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Marthe Bonamy

Graph recolouring
Graph 3-recolouring: bad cases (2)

\[ \alpha \]
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\[ \alpha \]

\[ \beta \]

\[ w(\alpha, C_5) = -3 \]

\[ w(\beta, C_5) = 3 \]
Graph 3-recolouring: bad cases (2)

\[ \alpha_{w}(\alpha, C_{5}) = -3 \]

\[ \beta_{w}(\beta, C_{5}) = 3 \]
Graph 3-recolouring: bad cases (2)

\[ \alpha \left( w, C_5 \right) = -3 \]

\[ \beta \left( w, C_5 \right) = 3 \]
Graph 3-recolouring: bad cases (2)

\[ \alpha \left( w(\alpha, C_5) \right) = -3 \]

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Graph 3-recolouring: bad cases (2)

\[ w(\alpha, C_5) = -3 \quad \text{and} \quad w(\beta, C_5) = 3 \]
Theorem (Cereceda, Johnson, van den Heuvel ’11)

For any graph $G$, any two 3-colourings $\alpha, \beta$, if
- Neither $\alpha$ nor $\beta$ contain a frozen cycle, and
- $\alpha$ and $\beta$ have the same wrapping number on every cycle,
then $G$ can be recoloured from $\alpha$ to $\beta$. 
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then $G$ can be recoloured from $\alpha$ to $\beta$.

Corollary (Wrochna ’15)

All graphs with no cycle of length multiple of 3 are 3-colourable.
Conjecture (Folklore ’15)

*Every graph with no induced cycle of length multiple of 3 contains an edge whose removal does not create an induced cycle of length multiple of 3.*
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Hypothetical Corollary (Wrochna ’15)

All graphs with no induced cycle of length multiple of 3 are 3-colourable.

Theorem (Bonamy, Charbit, Thomassé ’15)

All graphs with no induced cycle of length multiple of 3 are \( O(1) \)-colourable.
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Every graph with no induced cycle of length multiple of 3 contains an edge whose removal does not create an induced cycle of length multiple of 3.

Nope! (Wrochna ’18)

Hypothetical Corollary (Wrochna ’15)

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Theorem (Bonamy, Charbit, Thomassé ’15)

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Kempe equivalence
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Kempe equivalence: Goal

\( \Delta \): Maximum degree of the graph
Kempe equivalence: Goal

Δ: Maximum degree of the graph

Theorem (Brooks ’41)

Every graph is Δ-colourable, except for cliques and odd cycles.
Kempe equivalence: Goal

\(\Delta\): Maximum degree of the graph

**Theorem (Brooks ’41)**

*Every graph is \(\Delta\)-colourable, except for cliques and odd cycles.*

**Conjecture (Mohar ’05)**

*All the \(\Delta\)-colourings of a graph are Kempe equivalent.*
Kempe equivalence: Results

The conjecture is false! (van den Heuvel '13)
Theorem (Feghali, Johnson, Paulusma '15)

True for all graphs with $\Delta \leq 3$ (other than the 3-prism).
Kempe equivalence: Results (2)

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| Theorem (B., Bousquet, Feghali, Johnson ’15) | True for all graphs (other than the 3-prism). |
Kempe equivalence: Results (2)

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Understand Glauber Dynamics (analyse Antiferromagnetic Potts Model when the temperature tends to 0)
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Understand Glauber Dynamics (analyse Antiferromagnetic Potts Model when the temperature tends to 0)

Murkier picture when fewer colours involved...
Hadwiger’s conjecture

\( K_t\)-minor: \( t \) pairwise disjoint connected subgraphs that are pairwise adjacent.

**Conjecture (Hadwiger ’43)**

*Any graph with no \( K_t\)-minor is \((t - 1)\)-colourable.*
Hadwiger’s conjecture

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Any graph with no $K_t$-minor is $(t - 1)$-colourable.

True for $t \leq 6$. 
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Marthe Bonamy  Graph recolouring
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Theorem (B., Heinrich, Legrand, Narboni ’21)

For any $\varepsilon > 0$ and large enough $t$, the $(\frac{3}{2} − \varepsilon)t$-colourings of a graph with no $K_t$-minor are not necessarily Kempe equivalent.
Theorem (Vizing '64)

For any graph, \( \Delta \leq \chi' \leq \Delta + 1 \).
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Proof through “Kempe changes”.
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Reconfiguration as a tool

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![Graph diagram](image-url)
Reconfiguration as a tool

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Proof through “Kempe changes”.

\[
\begin{align*}
\text{Diagram:} & \\
\text{Initial coloring:} & \\
\text{Intermediate coloring:} & \\
\text{Final coloring:} & 
\end{align*}
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Reconfiguration as a tool

Theorem (Vizing '64)

For any graph, $\Delta \leq \chi' \leq \Delta + 1$.

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**Theorem (Vizing ’64)**

For any graph, for any proper edge colouring $\alpha$, there is a proper $(\Delta + 1)$-edge colouring $\beta$ such that $\alpha$ and $\beta$ are Kempe-equivalent.

**Conjecture (Vizing ’65)**

For any graph, for any proper edge colouring $\alpha$, there is a proper $\chi'$-edge colouring $\beta$ such that $\alpha$ and $\beta$ are Kempe-equivalent.

**Conjecture (Mohar ’06)**

For any graph $G$, for any two $(\Delta(G) + 2)$-edge colourings $\alpha$ and $\beta$ of $G$, they are Kempe-equivalent.

**Theorem (B., Defrain, Klimová, Lagoutte, Narboni ’20)**

For any triangle-free graph, all $(\chi' + 1)$-edge-colourings are Kempe-equivalent.
## Vizing’s and Mohar’s conjectures

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Reconfiguration Graphs

Two solutions:
- In the same connected component?
- What distance between them?
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Reconfiguration graph:
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Reconfiguration Graphs

- **Two solutions:**
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  - What distance between them?

- **Reconfiguration graph:**
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  - Maximal diameter of a connected component?

Various problems, various elementary steps...
Reconfiguration graph connected $\Rightarrow$ Efficient enumeration?
Elementary steps

Reconfiguration graph connected $\Rightarrow$ Efficient enumeration?
Sampling?
Reconfiguration graph connected ⇒ Efficient enumeration?
Sampling? Approximate counting?
Elementary steps

Reconfiguration graph connected $\Rightarrow$ Efficient enumeration? Sampling? Approximate counting?

Almost every thing is PSPACE-hard.
$\Rightarrow$ Restricted graph classes, Fixed Parameter Tractability
Merci !
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