Exercises
The GRAPH MOTIF Problem

Exercise 1 (FPT in $\Delta(G)$ ? In $\mu(G)$ ?)

1. Show that unless $P = NP$, COLORFUL GRAPH MOTIF is not FPT when the considered parameter is $\Delta(G)$ (maximum degree of $G$).

2. Show that unless $P = NP$, COLORFUL GRAPH MOTIF is not FPT when the considered parameter is $\mu(G)$ (maximum number of occurrences of a given color in $G$).

3. Reconsider Questions 1. and 2. above for the COLORFUL GRAPH MOTIF problem.

Exercise 2 (CGM – motif with few colors)

The following problem is known to be $NP$-complete :

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<td>Instance : a set $X = {x_1, x_2, \ldots, x_{3q}}$, a set $S = {S_1, S_2, \ldots, S_n}$ of 3-elements subsets of $X$ s.t. $</td>
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<td>Question : Does there exist $S' \subseteq S$ such that each element of $X$ is included in exactly one subset $S_i \in S'$ ?</td>
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Show that GRAPH MOTIF is $NP$-complete even if $|M^*| = 2$, $G$ is bipartite and of maximum degree 4.

Exercise 3 (Bounded Search Tree for CGM in Trees)

Consider CGM in trees, and let $T$ be the input tree.

We suppose in the rest of the exercise that $T$ is rooted at a vertex $r$, and that $r$ must be contained in the sought occurrence of the motif $M$ (we will try several roots – always a $O(n)$ – if necessary).

1. Suppose that for every color $c$, at most one vertex is a non-leaf and is of color $c$. Show that CGM is polynomially solvable in that case.

2. Suppose that at least one color $c$ appears at least twice as a non-leaf vertex in $T$. Devise a bounded search tree algorithm that solves CGM on trees.

3. Determine the size of the search tree from Question 2.

4. Conclude that CGM in trees is FPT in $\ell$.

Exercise 4 (Find all occurrences of $M$ in $G$)

Disclaimer : I do not have clear answers to the following questions (I have not properly thought about it, and the answer may or may not be simple !).

Call ALL GRAPH MOTIF (resp. ALL COLORFUL GRAPH MOTIF) the variant of GRAPH MOTIF (resp. CGM) that asks for all the occurrences of $M$ in $G$. Suppose that $K$ such occurrences exist in the input instance.

1. Is ALL GRAPH MOTIF (resp. ALL COLORFUL GRAPH MOTIF) solvable in $O(f(k) \cdot K \cdot poly(n))$ ? In $O(f(k) \cdot poly(K) \cdot poly(n))$ ?

2. Same question with $\ell$ instead of $k$. 
Hints for Exercise 2:
1. Graph: create one vertex per $x_i$, one vertex per $S_j$, add vertices to ensure connectivity (of an occurrence of the motif) while maintaining a bipartite graph.
2. Colors: give color black to vertices representing the $S_j$s, white to the others.
3. Motif: as many whites as there are in $G$, $q$ black only.

Hints for Exercise 3:
1. Consider two cases depending on whether $c$ appears once or at least twice in $T$. For the latter, think of a greedy algorithm.
2. Consider two such vertices, and branch on two cases.