Counting planar Eulerian orientations

Claire Pennarun
Joint work with Nicolas Bonichon, Mireille Bousquet-Mélou and Paul Dorbec

LaBRI, Bordeaux

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Some definitions

We consider:

- **planar maps**, rooted in a corner
- with loops and multiple edges

\[ n: \text{number of edges} \ (= 4) \]
\[ v: \text{root-vertex} \]
\[ \Delta: \text{root-degree} \ (= 4) \]
Adding structure

Statistical physics and combinatorics: maps equipped with a structure

- proper $q$-colouring [Tutte 73-84...]
- spanning tree [Mullin 67...]
- Ising model [Kazakov 86...]
- Schnyder woods [Schnyder 89...]
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Nice bijections with other classes, good properties (lattice structure, specializations...)

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In this talk → Eulerian orientations
**Eulerian orientations (PEO)**

An oriented planar map is a **planar Eulerian orientation (PEO)** if every vertex has **in-degree and out-degree equal**.
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Decomposition of PEO

Two ways of creating a PEO:

- **merge** two PEOs $O_1, O_2$ and orient the new edge
- **split** the root-vertex at index $i$ iff the resulting map is still a PEO

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Splits at index 1 or $\Delta - 1$ are always possible; oth. we must check!
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Splits at index 1 or $\Delta - 1$ are always possible; oth. we must check!

Remember the **full orientation** around the root: no recurrence relation with a finite number of parameters
Computing the first terms

Let $o(n)$ be the number of PEO with $n$ edges.
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Let $o(n)$ be the number of PEO with $n$ edges. PEO of size $n$: results either from a merge of two PEOs of sizes summing to $n - 1$, or from a split on a PEO of size $n - 1$. 

\begin{tabular}{|c|c|}
\hline
$n$ & $o(n)$
\hline
0 & 1
\hline
1 & 6
\hline
2 & 37
\hline
3 & 370
\hline
4 & 33558
\hline
5 & 340670
\hline
6 & 3522993
\hline
7 & 3522993
\hline
8 & 340670
\hline
9 & 33558
\hline
10 & 370
\hline
11 & 37
\hline
12 & 1
\hline
\end{tabular}

Not already in the OEIS!

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**Computing the first terms**

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Approximation of the growth rate

\[ \mu = \text{growth rate of PEOs} = \lim_{n \to \infty} o(n)^{1/n} \]
Approximation of the growth rate

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Merging two PEOs with $n$ and $n'$ edges gives a PEO with $n + n'$ edges

$\rightarrow \{o(n)\}_{n \geq 0}$ is super-multiplicative, i.e. $o(n + n') \geq o(n) \cdot o(n')$.
**Approximation of the growth rate**

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Variant of Fekete’s Lemma (1923):

\[ \mu = \sup_{n \geq 1} o(n)^{1/n} \in \mathbb{R}^*_+ \]

\[ \Rightarrow \mu \geq (o(15))^{1/15} \approx 8.145525470 \]
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PEO \( \subset \) arbitrary orientations of Eulerian maps
⇒ \( 8.14 < \mu < 16 \)
**Approximation of the Growth Rate**

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PEO \( \subset \) arbitrary orientations of Eulerian maps

\[ \Rightarrow 8.14 < \mu < 16 \]

\[ \frac{o(n + 1)}{o(n)} \text{ as a function of } 1/n \rightarrow \]
Prime decomposition of maps

A map is **prime** if the root-vertex appears **exactly once** on the root-face.
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Planar map = concatenation of prime maps

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**Prime decomposition of maps**

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Prime decomposition of maps

A map is **prime** if the root-vertex appears **exactly once** on the root-face.

Planar map = concatenation of prime maps

Operations to create a prime map:
- Add a loop around any map
- Split at index $i \leq \Delta(P_\ell)$ in the last prime $P_\ell$ of any map
Subsets (and supersets) of $\mathcal{O}$

Two families of sets of orientations $\mathcal{O}^+_k$ and $\mathcal{O}^-_k$ s.t.

$\mathcal{O}^-_k \subset \mathcal{O}^-_{k+1} \subset \mathcal{O} \subset \mathcal{O}^+_k \subset \mathcal{O}^+_{k+1} \subset \mathcal{O}^+_k$

Definition

A map of $\mathcal{O}^-_k$ is obtained by either:
Subsets (and supersets) of $\mathcal{O}$

Two families of sets of orientations $\mathcal{O}^{-}_k$ and $\mathcal{O}^{+}_k$ s.t.

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Definition

A map of $\mathcal{O}^{-}_k$ is obtained by either:

- a concatenation of prime maps of $\mathcal{O}^{-}_k$, 

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**Subsets (and supersets) of $O$**

Two families of sets of orientations $O_k^-$ and $O_k^+$ s.t.

$O_k^- \subset O_{k+1}^- \subset O \subset O_{k+1}^+ \subset O_k^+$

**Definition**

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- adding a loop around a map $O \in \mathcal{O}_k^-$ and orienting it,
- a split on the last prime component $P_\ell$ of a map $P_1 \ldots P_\ell \in \mathcal{O}_k^-$ at index $i < 2k$ or $i = \Delta(P_\ell) - 1$. 

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The atomic map (one vertex, no edges) is in $\mathcal{O}_k^-$. Fewer splits allowed $\rightarrow$ the number of orientations necessary to look at form now a word of finite length, which we can use as a parameter.
Algebraic system for $\mathcal{O}^{(k)}_- \equiv \mathcal{O}^-$

The root-word $w(O)$ of a map $O$ is the binary word formed as follows in counterclockwise order around the root-vertex:

- 1 if there is an out-edge,
- 0 if there is an in-edge.
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A word $w$ is **balanced** iff $|w|_0 - |w|_1 = 0$. 

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1110000101
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A word $w$ is **balanced** iff $|w|_0 - |w|_1 = 0$.

$F_w(t)$ : g.f. of the set $\{O \in \mathcal{O}^- | w(O) = w\}$

$L_w(t)$ : g.f. of the set $\{O \in \mathcal{O}^- | w(O) = uw \text{ for some } u\}$

$F'_w(t), L'_w(t)$ : their counterparts for prime maps of $\mathcal{O}^-$. 
An example: equation for $F'_w(t)$
Prime oriented maps of $\mathcal{O}^-$ with root-word $w$. 
An example: equation for $F'_w(t)$

Prime oriented maps of $\partial^-$ with root-word $w$.

$w_s$: maximal proper suffix of $w$, $w_c$: central factor of $w$ ($w = \alpha w_c \bar{\alpha}$)
An example: equation for $F'_{w}(t)$

Prime oriented maps of $\mathcal{O}^{-}$ with root-word $w$.

$w_{s}$: maximal proper suffix of $w$, $w_{c}$: central factor of $w$ ($w = \alpha w_{c} \bar{\alpha}$)

For $w$ balanced, $2 \leq |w| \leq 2k$: 

\[ F'_{w}(t) = tF_{w_{c}} + tL_{\epsilon} L'_{w_{s}} \]
An example: equation for $F'_w(t)$

Prime oriented maps of $\mathcal{O}^-$ with root-word $w$.

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For $w$ balanced, $2 \leq |w| \leq 2k$:

$$F'_w = tF_{w_c} + tL_\varepsilon L'_{w_s}$$

$w_s$ is a suffix of $w(P_\ell)$
Algebraic system for $O^{(k)}$–

\[
\begin{align*}
F_w &= \sum_{w=uv} F_u F'_v \\
L_w &= \begin{cases} 
L_\varepsilon L'_w + \sum_{w=uv,u\neq \varepsilon} L_u F'_v \\
1 + L_\varepsilon L'_\varepsilon
\end{cases} \\
F'_w &= tF_w + tL_\varepsilon L'_w \\
L'_w &= \begin{cases} 
tL_\varepsilon + \sum_{u=vw} (L'_u - F_u) + tL_\varepsilon (L'_w - F'_w) \\
2tL_\varepsilon + tL_\varepsilon L'_\varepsilon
\end{cases}
\end{align*}
\]

$|w| \leq 2k - 2$

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$|w| \leq 2k$

$w = \varepsilon$

$w = \varepsilon$ \Rightarrow $F_w = 1, F'_w = 0$

$w$ non-balanced \Rightarrow $F_w = F'_w = 0$. 

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Small example: subsets, $k = 1$

0/1 symmetry $\rightarrow$ divide the number of equations by 2
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0/1 symmetry $\rightarrow$ divide the number of equations by 2

\[
\begin{align*}
F'_{10} &= t + t L_\varepsilon L'_0, \\
L_\varepsilon &= 1 + L_\varepsilon L'_\varepsilon, \\
L'_\varepsilon &= 2t L_\varepsilon + t L_\varepsilon (L'_\varepsilon + 2L'_0 - 2F'_{10}), \\
L'_0 &= t L_\varepsilon + t L_\varepsilon (L'_0 + L'_0 - F'_{10}).
\end{align*}
\]
Small example: subsets, $k = 1$

0/1 symmetry $\rightarrow$ divide the number of equations by 2

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F'_{10} &= t + tL_\varepsilon L'_0, \\
L_\varepsilon &= 1 + L_\varepsilon L'_\varepsilon, \\
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L'_0 &= tL_\varepsilon + tL_\varepsilon (L'_0 + L'_0 - F'_{10}).
\end{align*}
$$

Eliminating all series but $L_\varepsilon$: cubic equation for $L_\varepsilon$:

$$t^2L_\varepsilon^3 + t(t-4)L_\varepsilon^2 + (2t+1)L_\varepsilon - 1 = 0$$
Small example: subsets, $k = 2$

\[
\begin{align*}
F_{01} &= F_{10} = F'_{01}, \\
F'_{10} &= F'_{01} = t + tL_\varepsilon L'_1,
\end{align*}
\]
\[
\begin{align*}
F'_{1100} &= tF_{10} + tL_\varepsilon L'_{100}, \\
F'_{1010} &= tF_{01} + tL_\varepsilon L'_{010}, \\
F'_{0110} &= tL_\varepsilon L'_1, \\
L_\varepsilon &= 1 + L_\varepsilon L'_\varepsilon, \\
L_0 &= L_1 = L_\varepsilon L'_0, \\
L_{00} &= L_{11} = L_\varepsilon L'_0, \\
L_{01} &= L_{10} = L_\varepsilon L'_0, \\
L'_\varepsilon &= 2tL_\varepsilon + tL_\varepsilon (L'_\varepsilon + 2(L'_0 - F'_{10} + L'_{100} - F'_{1100} + L'_{010} - F'_{1010} + L'_{110} - F'_{0110})), \\
L'_0 &= L'_1 = tL_0 + tL_\varepsilon (L'_0 + L'_0 - F'_{10} + L'_{100} - F'_{1100} + L'_{010} - F'_{1010} + L'_{110} - F'_{0110}), \\
L'_{00} &= tL_0 + tL_\varepsilon (L'_{00} + L'_{100} - F'_{1100}), \\
L'_{10} &= L'_{01} = tL'_{11} + t + tL_\varepsilon (L'_1 + L'_1 - F'_{01} + L'_{010} - F'_{1010} + L'_{110} - F'_{0110}), \\
L'_{100} &= tL_{10} + tL_\varepsilon (L'_{100} + L'_1 - F'_{1100}), \\
L'_{010} &= tL_{01} + tL_\varepsilon (L'_{010} + L'_0 - F'_{1010}), \\
L'_{110} &= tL_{11} + tL_\varepsilon (L'_{110} + L'_1 - F'_{0110}).
\end{align*}
\]

Eliminating all series but $L'_\varepsilon$ gives an equation of degree 6 for $L'_\varepsilon$.

\[2tL_5 + t^4(8 + t) + t^3(16 - 3t^2) + t^2(-5 - 3t^2 + 3t^4) + t(-17 - 7t - 7t^2) + (5t + 1) = 0.\]
Finding $L_\varepsilon$

Generate the systems automatically then **eliminate** the variables with Maple (keeping $L_\varepsilon$)

$k \geq 4$: find the first terms using the **Newton GF package**
Finding $L_\varepsilon$

Generate the systems automatically then **eliminate** the variables with Maple (keeping $L_\varepsilon$)

$k \geq 4$: find the first terms using the **Newton GF package**

<table>
<thead>
<tr>
<th>nature</th>
<th>$k$</th>
<th>degree</th>
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(*) not proven, use of quadratic approximants
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For each $k > 0$, $o_k(n) \sim \gamma n^{-3/2} \rho^{-n}$ ($\rho$ and $\gamma$ depend on $k$).
Finding $L_\varepsilon$

Generate the systems automatically then eliminate the variables with Maple (keeping $L_\varepsilon$)

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For each $k > 0$, $o_k(n) \sim \gamma n^{-3/2} \rho^{-n}$ ($\rho$ and $\gamma$ depend on $k$).

Let $\mu_k^-$ be the growth rate of the set $\mathcal{O}_k^-$. Then $\mu_k^- \rightarrow_{k \to \infty} \mu$. 
Supersets of PEO

General idea: allowing splits at indices $i \geq \Delta(P_\ell)$, creating non Eulerian orientations
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General idea: allowing splits at indices \( i \geq \Delta(P_\ell) \), creating non Eulerian orientations

One catalytic variable \( x \) (for the half-degree of the root)

Same kind of systems, but with divided differences!
Supersets of PEO

General idea: allowing splits at indices $i \geq \Delta(P_\ell)$, creating non Eulerian orientations

One catalytic variable $x$ (for the half-degree of the root)

Same kind of systems, but with divided differences!

For $k = 1$:

\[
\begin{align*}
L_\varepsilon(t, x) &= 1 + L_\varepsilon(t, x)L'_\varepsilon(t, x), \\
L'_\varepsilon(t, x) &= 2txL_\varepsilon(t, x) + tL_\varepsilon(t, 1) \left(2xL'_0(t, 1) + \frac{x}{x^x - 1} (L'_\varepsilon(t, x) - xL'_\varepsilon(t, 1))\right), \\
L'_0(t, x) &= txL_\varepsilon(t, x) + tL_\varepsilon(t, 1) \left(xL'_0(t, 1) + \frac{x}{x - 1} (L'_0(t, x) - xL'_0(t, 1))\right).
\end{align*}
\]
The supersets of PEO have **algebraic** generating functions.
Supersets of PEO

The supersets of PEO have **algebraic** generating functions.

Conjecture

For each $k > 0$, $o_k(n) \sim \gamma n^{-5/2} \rho^{-n}$ ($\rho$ and $\gamma$ depend on $k$).
The supersets of PEO have \textit{algebraic} generating functions.

\textbf{Conjecture}

For each \( k > 0 \), \( o_k(n) \sim \gamma n^{-5/2} \rho^{-n} \) (\( \rho \) and \( \gamma \) depend on \( k \)).

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<td>3</td>
<td>–</td>
<td>13.031(*)</td>
</tr>
<tr>
<td>sup</td>
<td>2</td>
<td>28</td>
<td>13.047</td>
</tr>
<tr>
<td>sup</td>
<td>1</td>
<td>3</td>
<td>13.065</td>
</tr>
</tbody>
</table>

(*) not proven, use of quadratic approximants
Finally...

- What is the **nature** of the generating function of PEOs?
- What if we restrict the **vertex degrees**? (4-regular, [Kostov 00])
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Thank you!