Le lien entre Michael Jordan et Catalan

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Basketball walks

Basketball walk: integer-valued walk with step-set \([-2, -1, +1, +2]\)
Theorem (Banderier & Krattenthaler & Krinik & Kruchinin & Kruchinin & Nguyen & Wallner ’16)

The generating function $G$ of basketball walks from 0 to 1 that are positive except at the origin, counted with weight $z$ per step is given by

$$G(z) = -\frac{1}{2} + \frac{1}{2} \sqrt{\frac{2 - 3z - 2\sqrt{1 - 4z}}{z}}.$$

$G := \{\text{basketball walks from 0 to 1 that are positive except at the origin}\}$

$|\mathbf{w}|$: number of steps of $\mathbf{w}$

$$G(z) := \sum_{\mathbf{w} \in G} z^{|\mathbf{w}|} \quad \quad = \sum_{n=0}^{\infty} \left| \{\mathbf{w} \in G : |\mathbf{w}| = n\} \right| z^n$$
Catalan is everywhere!

The previous authors observed that

\[ 1 + G(z) + G^2(z) = \text{Cat}(z) \]  \hspace{1cm} (1)

where \( \text{Cat} \) is the Catalan generating function.

\[ \text{Cat}(z) = \sum_{n=0}^{\infty} c_n z^n \]  where \( c_n := \frac{1}{n+1} \binom{2n}{n} \) is the \( n \)-th Catalan number

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012

\( n \)-edge rooted trees, \( n + 1 \)-leaf binary rooted trees, \( 2n \)-step Dyck walks, well-parenthesized words with \( n \) pairs of parentheses, rooted triangulations of the \( n + 2 \)-gon, noncrossing partitions of the \( n \)-set, etc.
**C-walks**

**C-walk:** basketball walk from 0 to 0 that visits 1 and is positive except at the extremities

\[ C := \{ \text{C-walks} \} \]

\[ C(z) := \sum_{w \in C} z^{|w|} \]

\[ C = ZG + ZG + G \]
**C-walks**

**C-walk**: basketball walk from 0 to 0 that visits 1 and is positive except at the extremities

\[ C := \{ C\text{-walks}\} \]

\[ C(z) := \sum_{w \in C} z^{|w|} \]

**Equation (1)** becomes \[ C(z) = z \text{Cat}(z) - z \], which is the generating function of nontrivial binary trees counted with weight \( z \) per leaf.
Introduction & motivations

Binary trees

IUBTs

Refined enumeration

even step: step starting at even height
odd step: step starting at odd height

Proposition

The number of $C$-walks with $2d \pm 1$-steps, $\ell$ odd $+2$-steps or even $-2$-steps, and $r$ odd $-2$-steps or even $+2$-steps is equal to

$$\frac{1}{d(d-1)} \binom{2d-2}{\ell+r+2d-2} \binom{\ell+r}{\ell}.$$
Matched statistics

Proposition

\[ n\text{-step C-walk} \iff n\text{-leaf binary tree} \]

- \pm 1\text{-steps} \iff \text{double leaves}

- odd +2 / even −2 \iff \text{left leaves}

- even +2 / odd −2 \iff \text{right leaves}
Increasing unary-binary tree

increasing unary-binary tree of size $n$: plane tree with $n$ vertices labeled 1, 2, $\ldots$, $n$ such that each vertex has at most 2 children, and all have larger labels.
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We associate with it the permutation obtained by reading the labels of the tree in breadth-first search order.
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**Increasing unary-binary tree** of size $n$: plane tree with $n$ vertices labeled $1$, $2$, $3$, ..., $n$ such that each vertex has at most 2 children, and all have larger labels.

We associate with it the permutation obtained by reading the labels of the tree in breadth-first search order.
Permutation avoiding 213
Permutation avoiding 213

contains 213

avoids 213
Counting IUBTs

IUBT: increasing unary-binary tree with associated permutation avoiding 213

Theorem

IUBTs are counted by $G$-walks (basketball walks from 0 to 1 that are positive except at the origin).

Proposition

For $n \geq 1$ and $0 \leq k \leq \lfloor (n - 1)/2 \rfloor$, the number of $n$-vertex IUBTs with exactly $n - 1 - 2k$ unary nodes is

$$\frac{1}{n} \binom{2n}{k} \binom{n - k}{k + 1}.$$
**Matched statistics**

**IUBT**: increasing unary-binary tree with associated permutation avoiding 213

**Proposition**

\[ n\text{-step } G\text{-walk} \leftrightarrow n\text{-vertex IUBT} \]

**staggered ±2-steps** \[\leftrightarrow\] **unary nodes**

The red and purple steps are paired.

A **staggered ±2-step** is a ±2-step that is not paired with any other ±2-step.
Decomposition of binary trees

\[ \mathcal{N} \]: class of nontrivial binary trees counted by number of leaves

**Goal**

Understand bijectively that \( \mathcal{C} = \mathcal{N} \)
Decomposition of binary trees

$\mathcal{N}$: class of nontrivial binary trees counted by number of leaves

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Understand bijectively that $\mathcal{C} = \mathcal{N}$
Decomposition of binary trees

$\mathcal{N}$: class of nontrivial binary trees counted by number of leaves

Goal

Understand bijectively that $\mathcal{C} = \mathcal{N}$

$$
\mathcal{N} = \mathcal{N} \times (\mathbb{Z} + \mathcal{N}) \times (\mathbb{Z} + \mathcal{N})
$$
Elementary decomposition of basketball walks
Elementary decomposition of basketball walks

A

B

C
Elementary decomposition of basketball walks

A

B

C = B

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Elementary decomposition of basketball walks

\[ A = B + C = B + A \]
Elementary decomposition of basketball walks

\[ A = B + A + B \]
Elementary decomposition of basketball walks

\[ A = \mathbb{Z} + \mathbb{Z} \]

\[ B = \mathbb{Z} + \mathbb{Z} \]

\[ C = B \cup A \cup B \]
Elementary decomposition of basketball walks

\[ A = \mathbb{Z} + \mathbb{Z} + A \]

\[ B = \mathbb{Z} + \mathbb{Z} + B \]

\[ C = B \]
Elementary decomposition of basketball walks

\[ A = \mathbb{Z} + \mathbb{Z} \]

\[ B = \mathbb{Z} \]

\[ C = B \]
Elementary decomposition of basketball walks

\[ A = 1 \]

\[ B = \mathbb{Z} + \mathbb{A} + \mathbb{B} \]

\[ C = \mathbb{B} \mathbb{A} \mathbb{B} \]
Elementary decomposition of basketball walks

\[ A = 1 + Z + A + Z \]

\[ B = Z + Z + A + B \]

\[ C = B + A + B \]
Elementary decomposition of basketball walks

\[ A = 1 + Z A Z A \]

\[ B = Z Z + Z A B \]

\[ C = B A B \]
Elementary decomposition of basketball walks

\[ A = 1 + Z A Z A + B A B A \]

\[ B = Z Z + Z A B \]

\[ C = B A B \]
The bijection

\[
+1 - 1
\]
The bijection

\[ \Phi: (+1 - 1) \rightarrow (\varepsilon, \varepsilon) \]
The bijection

\[ \Phi \quad (\varepsilon, \varepsilon) \]
The bijection

\( \Phi \quad (\varepsilon, \varepsilon) \)

\( b_1 \quad c \quad a \quad \bar{b}_2 \)

\( \Phi \quad (\bar{c}, \quad b_1 \quad a \quad \bar{b}_2) \)
The bijection

\[ a - 2 \]

\[ b_1 \quad b_2 \quad a \quad -2 \]
The bijection

\[ \Phi : (b_1, b_2, a, -2) \mapsto (\varepsilon, b_1, \overline{a}, \overline{b}_2) \]
The bijection

\[ \Phi \quad \begin{pmatrix} \varepsilon, \\ b_1, \bar{a}, \bar{b}_2 \end{pmatrix} \]

\[ \begin{align*}
\Phi : & \quad b_1, b_2, a, -2 \\
\Phi^{-1} : & \quad +2, a, \bar{b}, -1
\end{align*} \]
The bijection

\[
\begin{align*}
\Phi & \quad (\varepsilon, \ b_1 \ a \ b_2) \\
\Phi & \quad (\varepsilon, \ b \ a \ b_2)
\end{align*}
\]
The bijection

\[ \begin{align*}
\Phi & \quad \mapsto \quad (\varepsilon, \quad \overline{a}, \quad \overline{b}_2) \\
\Phi & \quad \mapsto \quad (\overline{a}, \quad a, \quad -1) \\
\Phi & \quad \mapsto \quad (\overline{b}_2, \quad a_2, \quad -2) \\
\end{align*} \]
The bijection

\[
\begin{align*}
\Phi & \mapsto (\varepsilon, (b_1, a, b_2)) \\
\Phi & \mapsto (\varepsilon, (b, a, b_1)) \\
\Phi & \mapsto (\varepsilon, (b_1, a_1, b_2, a_2, -2))
\end{align*}
\]
The bijection

\[ \Phi \]

\[ (\varepsilon, \varepsilon) \]

\[ (\bar{c}, b_1, a, \bar{b}_2) \]

\[ (\varepsilon, \bar{a}, \bar{b}_2) \]

\[ (\bar{b}, \bar{a}, -1, \varepsilon) \]

\[ (\bar{b}_1, a_1, b_2, \bar{a}_2, -2, \varepsilon) \]
The bijection
The bijection
The bijection
The bijection
The bijection
The bijection
The bijection
The bijection
The bijection
The bijection

\begin{center}
\begin{tikzpicture}

% Draw the first tree
\draw[red] (0,0) -- (1,1) -- (2,0) -- (3,1) -- (4,0);
\draw[green] (0,0) -- (1,1) -- (2,0) -- (3,1) -- (4,0);

% Draw the second tree
\draw[red] (5,0) -- (6,1) -- (7,0) -- (8,1) -- (9,0);
\draw[green] (5,0) -- (6,1) -- (7,0) -- (8,1) -- (9,0);

% Draw the third tree
\draw[red] (10,0) -- (11,1) -- (12,0) -- (13,1) -- (14,0);
\draw[green] (10,0) -- (11,1) -- (12,0) -- (13,1) -- (14,0);

% Draw the fourth tree
\draw[red] (15,0) -- (16,1) -- (17,0) -- (18,1) -- (19,0);
\draw[green] (15,0) -- (16,1) -- (17,0) -- (18,1) -- (19,0);

\end{tikzpicture}
\end{center}
The bijection
The bijection
The bijection

\[ \text{Diagram showing two binary trees and their bijection.} \]
The bijection
The bijection
The bijection
Statistics
Statistics
Statistics
Valid permutations on a unary-binary tree

Lemma

A permutation is valid for a tree if and only if it avoids 213 and, the value taken at each node is a right-to-left minimum.
Valid permutations on a unary-binary tree

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**Lemma**

A permutation is valid for a tree if and only if it avoids 213 and, the value taken at each node is a right-to-left minimum.
Encoding by decorated Motzkin paths
Encoding by decorated Motzkin paths

\begin{tikzpicture}
\t\node (1) at (0,0) {1};
\t\node (2) at (-1,-1) {2};
\t\node (3) at (1,-1) {3};
\t\node (4) at (-2,-2) {4};
\t\node (5) at (0,-2) {5};
\t\node (6) at (2,-2) {6};
\t\node (7) at (-1,-3) {7};
\t\node (8) at (0,-3) {8};
\t\node (9) at (1,-3) {9};
\t\node (10) at (-2,-4) {10};
\t\node (11) at (-1,-4) {11};
\t\node (12) at (0,-4) {12};
\t\node (13) at (1,-4) {13};
\t\node (14) at (2,-4) {14};
\t\node (15) at (1,-5) {15};
\t\node (16) at (2,-6) {16};
\t\node (17) at (1,-6) {17};
\t\node (18) at (0,-6) {18};
\t\node (19) at (-1,-6) {19};
\end{tikzpicture}
Encoding by decorated Motzkin paths
Encoding by decorated Motzkin paths
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Introduction & motivations

Binary trees

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March 20, 2017
Encoding by decorated Motzkin paths
Encoding by decorated Motzkin paths
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Encoding by decorated Motzkin paths
How to reconstruct the permutation
How to reconstruct the permutation
How to reconstruct the permutation
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How to reconstruct the permutation
How to reconstruct the permutation
Counting decorated Motzkin walks

\( \mathcal{T} \): decorated Motzkin walks

**Aim**

We want to show that \( \mathcal{T} = \mathcal{G} \).

\[ \mathcal{G} \]
Counting decorated Motzkin walks

\( \mathcal{T} \): decorated Motzkin walks

**Aim**

We want to show that \( \mathcal{T} = \mathcal{G} \).

\[
\mathcal{G} = \mathcal{Z} \cdot \mathcal{A} + \mathcal{Z} \cdot \mathcal{G} \cdot \mathcal{A}
\]
Counting decorated Motzkin walks

\( T \): decorated Motzkin walks

**Aim**

We want to show that \( T = G \).

\[
G = (ZA) + (ZA)^2 + (ZA)^3 + \ldots = \text{Seq}_{\geq 1}(ZA)
\]
Step 1: $\mathcal{T} = \text{Seq}_{\geq 1}(\mathcal{M})$

$\mathcal{M}$: decorated Motzkin walks whose last 0-step or +1-step is a +1-step.

Claim

$\mathcal{T} = \text{Seq}_{\geq 1}(\mathcal{M})$
Step 1: $\mathcal{T} = \text{Seq}_{\geq 1}(\mathcal{M})$

$\mathcal{M}$: decorated Motzkin walks whose last 0-step or +1-step is a +1-step

Claim

$\mathcal{T} = \text{Seq}_{\geq 1}(\mathcal{M})$

$\mathcal{T} - \mathcal{M} = \mathcal{M} \times \mathcal{T}$
Step 2: $\mathcal{M} = ZA$

We observed that $B = z/(1 - zA)$ and $A = 1 + (zA)^2 + (BA)^2$. Thus,

$$zA = z \left( 1 + (zA)^2 + \frac{(zA)^2}{(1 - zA)^2} \right)$$
Step 2: $\mathcal{M} = ZA$

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We observed that $B = z/(1 - zA)$ and $A = 1 + (zA)^2 + (BA)^2$. Thus,

$$zA = z \left( 1 + (zA)^2 + \frac{(zA)^2}{(1 - zA)^2} \right)$$
Step 2: $\mathcal{M} =ZA$

We observed that $B = z/(1 - zA)$ and $A = 1 + (zA)^2 + (BA)^2$. Thus,

$$ZA = z \left(1 + (zA)^2 + \frac{(zA)^2}{(1 - zA)^2}\right)$$

$$M = z (1 + T - M)^2 = z \left(1 - M + \frac{M}{1 - M}\right)^2 = z \left(1 + M^2 + \frac{M^2}{(1 - M)^2}\right)$$
Merci !