The splitting necklace problem

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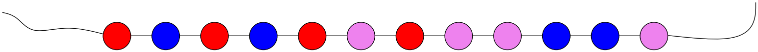
CERMICS, Optimisation et Systèmes

Two thieves and a necklace

n beads, t types of beads, a_i (even) beads of each type.

Two thieves: Alice and Bob.

Beads fixed on the string.

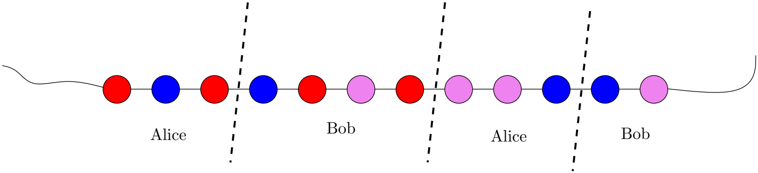


Fair splitting = each thief gets $a_i/2$ beads of type i

The splitting necklace theorem

Theorem (Alon, Goldberg, West, 1985-1986)

There is a fair splitting of the necklace with at most t cuts.



t is tight

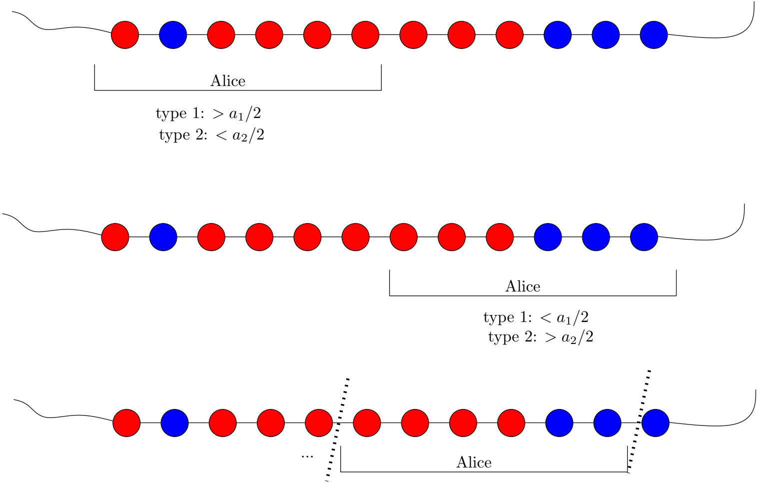
t cuts are sometimes necessary:

Plan

1. Proofs and algorithms
2. A special case: the binary necklace problem
3. Generalizations
4. Open questions

Proofs and algorithms

Easy proof when there are two types of beads



Continuous necklace theorem

Proof for any t via...

Theorem

Let $\mu_1, \ldots, \mu_t$ be continuous probability measures on $[0,1]$. Then $[0,1]$ can be partitioned into $t+1$ intervals $I_1, \ldots, I_{t+1}$ and $[t+1]$ can be partitioned into two sets A_1 and A_2 such that

$$\sum_{i \in A_r} \mu_j(I_i) = \frac{1}{2} \quad \text{for } j \in [t] \text{ and } r \in \{1,2\}.$$

$\mu_j(U) =$ fraction of type j beads in $U \subseteq [0,1]$.

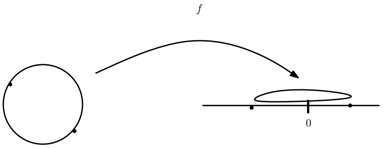
$\Rightarrow$ Splitting necklace theorem.

Proof of continuous necklace theorem

$$S^t = \left\{ (x_1, \ldots, x_{t+1}) \in \mathbb{R}^{t+1} : \sum_{i=1}^{t+1} |x_i| = 1 \right\}.$$

$$\begin{array}{rccc} f: & S^t & \longrightarrow & \mathbb{R}^t \\ & (x_1, \ldots, x_{t+1}) & \longmapsto & \displaystyle\sum_{i=1}^{t+1} \text{signe}(x_i)\mu_j[|x_i|, |x_{i+1}|). \end{array}$$

- f is antipodal $f(-x) = -f(x)$.
- f is continuous.
- Borsuk-Ulam theorem:
  $\exists x_0 : \; f(x_0) = 0$.



Combinatorial or algorithmic proofs

Proof by Borsuk-Ulam: non-constructive.

Two combinatorial and constructive proofs:

1. M. 2008: via Ky Fan's cubical lemma (combinatorial version of Borsuk-Ulam for cubical complexes).

2. Pálvölgyi 2009: via combinatorial Tucker's lemma
   Lemma. Let
   $\lambda : \{+,-,0\}^n \setminus \{(0,\ldots,0)\} \to \{-(n-1),-(n-2),\ldots,-1,0,+1,\ldots,+(n-2),+(n-1)\}$
   satisfy simultaneously the following two properties:

   $$\lambda(-x) = -\lambda(x) \text{ for all } x \qquad \text{and} \qquad \lambda(x)+\lambda(y) \neq 0 \text{ for all } x \preceq y.$$

   Then there exists x_0 such that $\lambda(x_0) = 0$.

Topological ideas still inside, but provide algorithms (with unknown complexity).

Elementary proofs and algorithms?

All known proofs rely on the Borsuk-Ulam theorem

Is there a direct/elementary proof?

Direct proofs for $t = 2$ and $t = 3$ (M. 2008)

Open question (Papadimitriou 1994)

Is there a polynomial algorithm computing a fair splitting of the necklace with at most t cuts?

- Naïve $O(n^t)$; less naïve $O(n^{t-1})$ (proof using Ham-sandwhich and moment curve)
- Minimizing the number of cuts: NP-hard.

Binary necklace problem

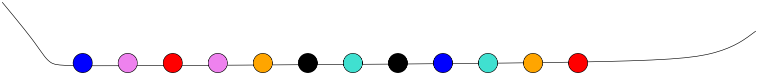
Definition

Binary Necklace Problem [Epping, Hochstättler, Oertel 2001]

Input. Necklace with t types of beads, 2 beads per type.

i.e. $n = 2t$ and $a_i = 2$ for all i.

Output. Fair splitting minimizing the number of cuts.



Defined in an operations research context as the paintshop problem (automotive industry)

Minimizing the number of cuts

Splitting necklace theorem: OPT $\leq t$, but obvious: greedy algorithm.

Challenge here: optimization.

Proposition (Epping, Hochstättler 2006)

The binary necklace problem is NP*-hard.*

Proof by MAX-CUT in 4-regular graphs.

M., Sebö 2009

- APX-hard

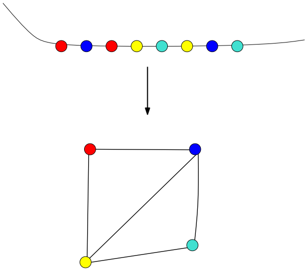
Gupta et al. 2013

- No polytime fixed-ratio approximation (assuming Unique Games Conjecture)

Positive results

M., Sebö 2009

OPT $\leq \frac{3}{4}t + \frac{1}{4}\beta$, *where* $\beta =$ *"forced cuts"* *(and). Can be found in polytime.*



Let $G = ([t], E)$, with $ij \in E$ if types i and j adjacent on necklace.

G planar $\Rightarrow$ polytime.

Greedy algorithm

Two questions.

- Expected number of cuts when applying the greedy algorithm? (fixed-size input drawn uniformly at random)

- When is the greedy algorithm optimal?

Expected number of cuts

g: number of cuts computed by the greedy algorithm (fixed-size input drawn uniformly at random)

Theorem (Andres, Hochstättler 2010)

$$\lim_{t \to +\infty} \frac{1}{t} \mathbb{E}_t(g) = \frac{1}{2}.$$

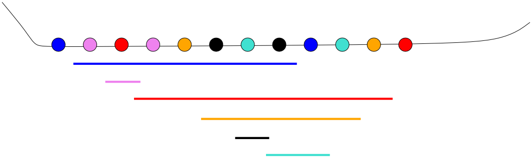
Proof.

$$\mathbb{E}_t(g) = \mathbb{E}_{t-1}(g) + \frac{2(t-1)^2 - 1}{4(t-1)^2 - 1}.$$

□

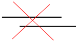
Equivalent formulation of the binary necklace problem

Input. Collection $\mathcal{I}$ of intervals of $\mathbb{R}$.
Output. $X \subseteq \mathbb{R}$ such that $|X \cap I|$ is odd for all $I \in \mathcal{I}$ and $|X|$ minimum.



Lower bound and greedy algorithm

$\mathcal{I}$ is evenly laminar if

- no crossings: 
- each interval properly contains an even number of intervals

Lemma

If $\mathcal{I}$ is evenly laminar, then $|\mathcal{I}| = \mathrm{OPT}$ and the greedy algorithm finds the optimal solution.

In general:

$|\mathcal{J}| \leq \mathrm{OPT}$ for $\mathcal{J}$ evenly laminar $\subseteq \mathcal{I}$.

Optimality of the greedy algorithm

Necklace = word w on a alphabet of size t.

Theorem (M., Sebö, 2009; Rautenbach, Szigeti 2012)

If w contains none of

abaccb *abbcddad* *abbcdcad*

as a subword, then the greedy algorithm is optimal.

Proof.

Not containing *abaccb*, *abbcddad*, *abbcdcad*

$$\Longleftrightarrow$$

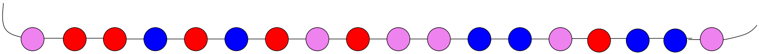
The intervals are evenly laminar.

$\square$

Generalizations

q thieves and a necklace

n beads, t types of beads, a_i (multiple of q) beads of each type.
q thieves: Alice, Bob, Charlie,...

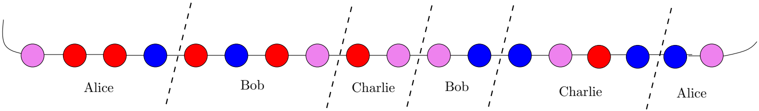


A generalization

Fair splitting = each thief gets a_i/q beads of type i

Theorem (Alon 1987)

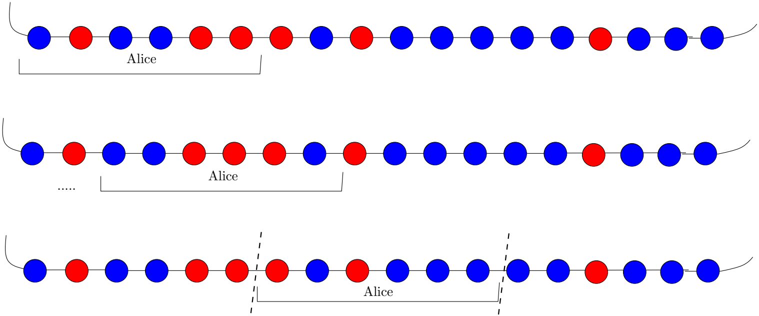
There is a fair splitting of the necklace with at most $(q-1)t$ cuts.



$(q-1)t$ is tight

$(q-1)t$ cuts are sometimes necessary:

An easy case: two types of beads (again!)



... and induction on the number of thieves.

Generalized continuous necklace theorem

Proof for any t via...

Theorem (Alon 1987)

Let $\mu_1, \ldots, \mu_t$ be continuous probability measures on $[0, 1]$. Then $[0, 1]$ can be partitioned into $(q-1)t+1$ intervals $I_1, \ldots, I_{(q-1)t+1}$ and $[(q-1)t+1]$ can be partitioned into $A_1, \ldots, A_q$

$$\sum_{i \in A_r} \mu_j(I_i) = \frac{1}{q} \quad \text{for } j \in [t] \text{ and } r \in [q].$$

$\mu_j(U) =$ fraction of type j beads in $U \subseteq [0, 1]$.

$\Rightarrow$ Generalized splitting necklace theorem.

Proof of the generalized continuous necklace theorem: prime case

Assume q prime.
Z_q = qth roots of unity.

$$\Sigma^{(q-1)t} = \left\{ (\omega_i, x_i)_{i \in [(q-1)t+1]} \in (Z_q \times \mathbb{R}_+)^{(q-1)t+1} : \sum_{i=1}^{(q-1)t+1} x_i = 1 \right\}.$$

$$f: \quad \Sigma^{(q-1)t} \quad \longrightarrow \quad \prod_{\omega \in Z_q} \mathbb{R}^t$$

- f is equivariant $f(\omega x) = \omega \cdot f(x)$.
- f is continuous.
- Dold theorem: $\exists x_0, \forall \omega \in Z_q, \; f(x_0) = \omega \cdot f(x_0)$.

Proof of the generalized continuous necklace theorem: nonprime case

Theorem.
Let $\mu_1, \ldots, \mu_t$ be continuous probability measures on $[0,1]$. Then $[0,1]$ can be partitioned into $(q-1)t+1$ intervals $I_1, \ldots, I_{t+1}$ and $[(q-1)t+1]$ can be partitioned into $A_1, \ldots, A_q$

$$\sum_{i \in A_r} \mu_j(I_i) = \frac{1}{q} \quad \text{for } j \in [t] \text{ and } r \in [q].$$

Proposition

If the generalized continuous necklace theorem is true for q_1 and q_2 (whatever are the other parameters), then it is true for $q_1 q_2$.

Proof.

A super-thief = q_2 thieves.

Make a first splitting among q_1 super-thieves: $(q_1 - 1)t$ cuts.

For each super-thieves: $(q_2 - 1)t$ cuts.

In total: $q_1(q_2 - 1)t + (q_1 - 1)t = (q_1 q_2 - 1)t$ cuts. □

Results

- Complexity of finding a fair splitting of at most $(q-1)t$ cuts: unknown.

- Optimization version: NP-hard.

- Combinatorial and constructive proof (M., 2014): via a Z_q-version of Ky Fan's cubical lemma (combinatorial version of Dold's theorem for simplotopal complexes)

- No known algorithmic proof.

Yet another generalization for q thieves

n beads, t types of beads, a_i beads of each type, q thieves.

Fair splitting = each thief gets $\lfloor a_i/q \rfloor$ or $\lceil a_i/q \rceil$ beads of type i, for all i.

Theorem (Alon, Moshkovitz, Safra 2006)

There is a fair splitting of the necklace with at most $(q-1)t$ cuts.

Is also a consequence of the combinatorial proof.

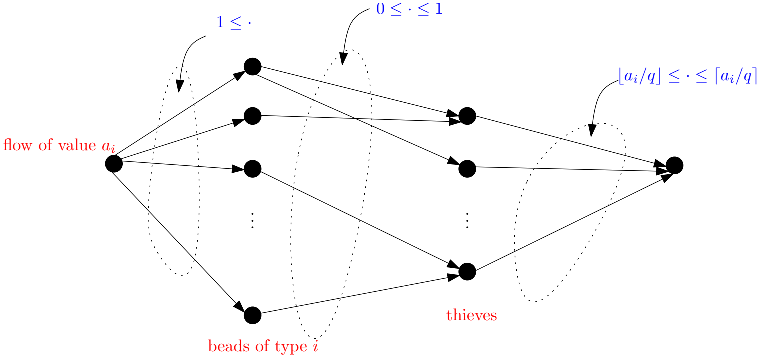
Direct proofs

- for $t = 2$
- $1 \leq a_i \leq q$ for all i.

Yet another generalization: proof

Generalized continuous necklace theorem: fair splitting with at most $(q-1)t$ cuts, but may be non-integral.

For each type i, build a directed graph D_i



Existence of a flow $\Rightarrow$ existence of an integral flow (Polytime "rounding" procedure).

Yet another generalization for q thieves?

Conjecture (M. 2008, Pálvölgyi 2009)

There is a fair splitting of the necklace with at most $(q-1)t$ cuts such that for each type i, we can decide which thieves receive $\lfloor a_i/q \rfloor$ and which receive $\lceil a_i/q \rceil$.

True in any of the following cases

- $q = 2$
- $t = 2$
- $1 \leq a_i \leq q$ for all i.

A generalization for two thieves

n beads, t types, a_i (even) beads per type i, two thieves Alice and Bob.

Theorem (Simonyi 2008)

If there is no fair splittings with $t-1$ cuts, then whatever are $A, B \subseteq [t]$, disjoint and no both empty, there is a splitting with at most $t-1$ cuts such that

- *Alice is advantaged for beads in A*
- *Bob is advantaged for beads in B*
- *the splitting is fair for beads in $[t] \setminus (A \cup B)$.*

Multidimensional continuous necklaces

Theorem (de Longueville, Živaljević 2008)

Let $\mu_1, \ldots, \mu_t$ be continuous probability measures on $[0,1]^d$. Let $m_1, \ldots, m_d$ be positive integers such that $m_1 + \cdots + m_d = (q-1)t$. Then there exists a fair division of $[0,1]^d$ determined by m_i hyperplanes parallel to the ith coordinate hyperplane.

The discrete version is not true (Lasoń 2015).

Open questions (summary)

Open questions

- Complexity of computing a fair splitting with at most t cuts when there are two thieves.
- Complexity of computing a fair splitting with at most $(q-1)t$ cuts when there are q thieves.
- Existence of a fair splitting with choice of the advantaged thieves.
- Elementary proof of the splitting necklace theorem (any version).
- Simonyi's generalization for q thieves?

- Expected value of the optimum for the binary necklace problem.
- Extend results of the binary necklace problem to general necklaces (at least for two thieves).

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Thank you