Comment probabiliser le monoïde de trace ?

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travail commun avec **Samy Abbes** (PPS - Univ Paris Diderot)

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The algebraic approach

Alphabet: $\Sigma = \{a, b, c\}$
Free monoid: $\Sigma^* = \{1, a, b, c, aa, ab, ac, \dots\}$

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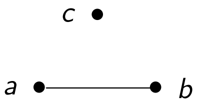
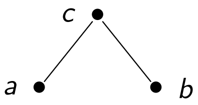
Dependence relation: $D = (\Sigma \times \Sigma) \setminus I$



Independence relation I



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Independence relation I *Dependence* relation D

Definition

The *trace monoid* associated with (Σ, I) is the finitely presented monoid

$$\mathcal{M} = \langle a, b, c \mid ab = ba \rangle .$$

The combinatorial approach

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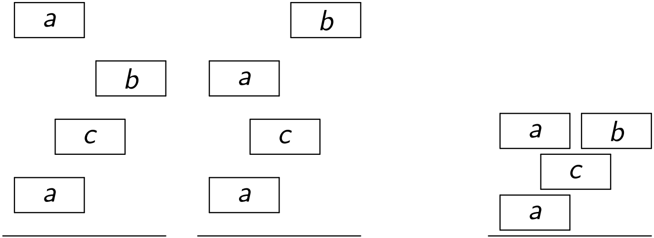
- Consider the *heaps* obtained by letting pieces fall *vertically*



word *acba* word *acab* trace $a \cdot c \cdot a \cdot b = a \cdot c \cdot b \cdot a$

Heaps identify with traces!

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Statistical model-checking: generate uniformly and randomly executions of length n

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Consider the σ-algebra $\mathfrak{F} = \langle \uparrow u \ : \ u \in \mathcal{M} \rangle$, where

$$\uparrow u = \{\omega \in \partial\mathcal{M} \ : \ u \leq \omega\} \qquad \text{the "cylinder" of base } u$$

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Definition

A probability measure $\mathbb{P}$ on $(\partial\mathcal{M}, \mathfrak{F})$ is **uniform** if:

$$\forall u, v \in \mathcal{M}, \quad |u| = |v|, \qquad \mathbb{P}(\uparrow u) = \mathbb{P}(\uparrow v)\,.$$

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Heap formed by random pieces falling one after the other

- Does *not* induce a uniform probability measure on traces.
Ex: $\mu = \{1/3, 1/3, 1/3\}$, then

$$\mathbb{P}(\uparrow ccc) = 1/27, \ \mathbb{P}(\uparrow abc) = 2/27 \, .$$

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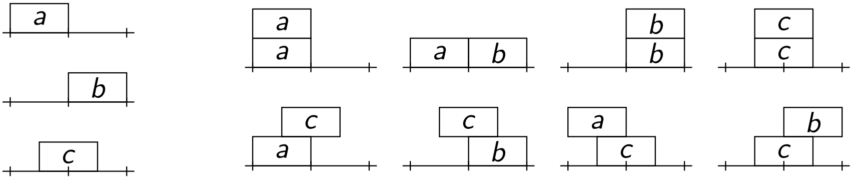

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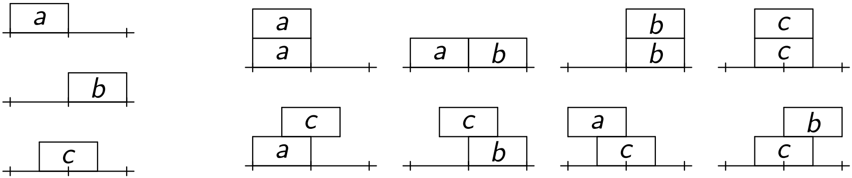
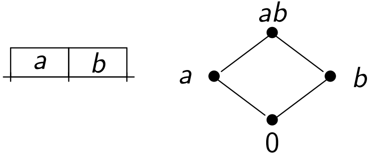


$$\frac{1}{3} = \mu_1(a) \neq \mu_2(aa) + \mu_2(ab) + \mu_2(ac) = \frac{3}{8}$$

Where does the difficulty come from ?

Concurrency interpretation

- For the concurrent events a and b, it makes no sense to say that one has occurred *before* the other.
- There is no *global clock* at the scale of the system



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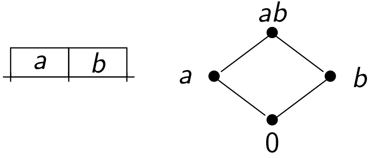

Combinatorics interpretation

- The cylinders of a given size are *not disjoint*

$$\uparrow a \cap \uparrow b = \uparrow(ab)$$

Bernoulli probability measures

Definition

A probability measure $\mathbb{P}$ on $(\mathcal{M}, \mathfrak{F})$ is

- *Bernoulli* if: $\exists (p_a, p_b, p_c) \in (0,1)^3$,

$$\forall n, \forall u_1, \ldots, u_n \in \Sigma, \qquad \mathbb{P}(\uparrow u_1 \cdots u_n) = p_{u_1} \times \cdots \times p_{u_n} \,.$$

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The converse is true.

The results - I

Theorem 1

There exists a unique uniform measure $\mathbb{P}_0$ on $\partial\mathcal{M}$, and it satisfies:

$$\forall u \in \mathcal{M}, \qquad \mathbb{P}_0(\uparrow u) = p_0^{|u|}, \qquad p_0 = \left(3 - \sqrt{5}\right)/2 = 0.382\cdots .$$

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Sketch of the proof.

Let $\mathbb{P}$ be Bernoulli. By Poincaré inclusion-exclusion formula,
$\forall u \in \mathcal{M}$,

$$\begin{aligned} \mathbb{P}(\uparrow u) &= \mathbb{P}(\uparrow ua) + \mathbb{P}(\uparrow ub) + \mathbb{P}(\uparrow uc) - \mathbb{P}(\uparrow uab) \\ &= \mathbb{P}(\uparrow u)\big(p_a + p_b + p_c - p_a p_b\big) . \end{aligned}$$

$\square$

Cartier-Foata normal form

Idea: an infinite heap can be described, slice by slice.

Set $\mathscr{C} = \{a, b, c, ab\}$. Let $C_n : \partial\mathcal{M} \to \mathscr{C}$ be the "n-th slice"
Infinite heap "$=$" $(C_n)_n$.

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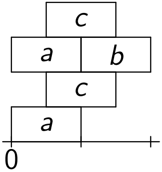
Infinite heap
$(a; c; ab; c; \cdots)$



The results - II

Theorem 2

Let $\mathbb{P}_0$ be the uniform measure on $\partial\mathcal{M}$. Set $p_0 = (3 - \sqrt{5})/2$. Then $(C_n)_n$ is a realization of the Markov chain on $\mathscr{C} = \{a, b, c, ab\}$ with initial probability measure h given by:

$$h(a) = p_0 - p_0^2, \quad h(b) = p_0 - p_0^2, \quad h(c) = p_0, \quad h(ab) = p_0^2\,,$$

and transition matrix

$$P = \begin{array}{c} a \\ b \\ c \\ ab \end{array} \begin{pmatrix} p_0 & 0 & 1-p_0 & 0 \\ 0 & p_0 & 1-p_0 & 0 \\ p_0 - p_0^2 & p_0 - p_0^2 & p_0 & p_0^2 \\ p_0 - p_0^2 & p_0 - p_0^2 & p_0 & p_0^2 \end{pmatrix}$$

The results - II

Sketch of proof.

Recall that $C_1(\omega)$ is the first layer of the infinite heap $\omega \in \partial\mathcal{M}$.
We have:

$$\{C_1 = ab\} = \uparrow(ab) \qquad \mathbb{P}(C_1 = ab) = p_0^2$$
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Consider the generating series of $\partial\mathcal{M}$:

$$S(z) = \sum_{u \in \mathcal{M}} z^{|u|} = \sum_{k \geq 0} c_k z^k \,, \qquad c_k = \#\{u \in \mathcal{M} \ : \ |u| = k\} \,.$$

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Let p_0 be the radius of convergence of $S(z)$. For $r \in (0, p_0)$, set:

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When $r \to p_0$, consider any weak limit $\mathbb{P}$ of μ_r. Then $\mathbb{P}$ satisfies:

$$\mathbb{P}(\mathcal{M}) = 0, \qquad \mathbb{P}(\Uparrow u) = \mathbb{P}(\uparrow u) = p_0^{|u|}$$

So $\mathbb{P}$ is **uniform** on ∂M.

Connection with the Patterson-Sullivan measure

Back to combinatorics.

Define the *Möbius polynomial* of $\mathcal{M}$ by:

$$P_{\mathcal{M}}(z) = \sum_{c \in \mathscr{C}} (-1)^{|c|} z^{|c|} = 1 - 3z + z^2 \,.$$

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Strongly connected and deterministic finite automaton (alphabet Σ). Let $L^\infty \subset \Sigma^\mathbb{Z}$ be the set of bi-infinite paths in the automaton (*sofic subshift*)

Parry measure $\mathbb{P}$: "uniform" measure on $L^\infty \leftrightarrow$ measure of maximal entropy on L^∞.

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Let $A \in \{0,1\}^{n\times n}$ be the incidence matrix of the automaton. By Perron-Frobenius Theorem:

$$\exists! \rho > 0,\ y \in (0,1)^n,\ \sum_i y_i = 1, \qquad Ay = \rho y\ .$$

Define:

$$Q \in \mathbb{R}_+^{n\times n}, \qquad Q_{ij} = \frac{1}{\rho} A_{ij} \frac{y_j}{y_i}$$

Connection with the Parry measure - II

Theorem (classical)

The matrix Q is stochastic. The probability measure $\mathbb{P}$ is the shift-invariant Markovian measure of transition matrix Q.

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Here:

$$Q = \begin{array}{c} a \\ b \\ c \\ ab \end{array} \begin{pmatrix} p_0 & 0 & 1-p_0 & 0 \\ 0 & p_0 & 1-p_0 & 0 \\ p_0-p_0^2 & p_0-p_0^2 & p_0 & p_0^2 \\ p_0-p_0^2 & p_0-p_0^2 & p_0 & p_0^2 \end{pmatrix}$$

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- Braid monoids

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- Uniform random sampling ...
- Braid monoids
- Trace groups, braid groups, ...
- Regular language of a trace monoid (Petri net)

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