Counting perfect matchings

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CNRS, Laboratoire G-SCOP, Grenoble, France

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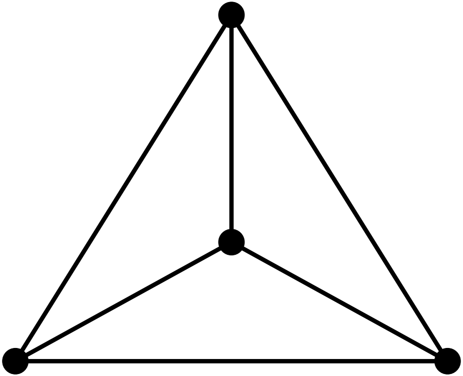
March 16, 2015

Cubic graphs and perfect matchings



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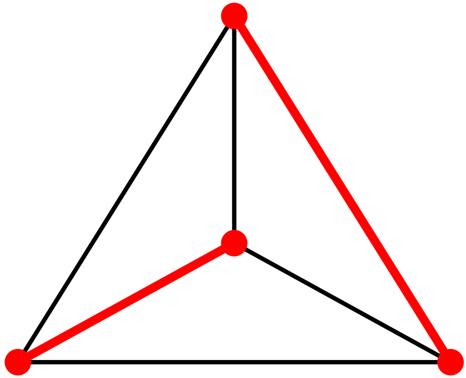

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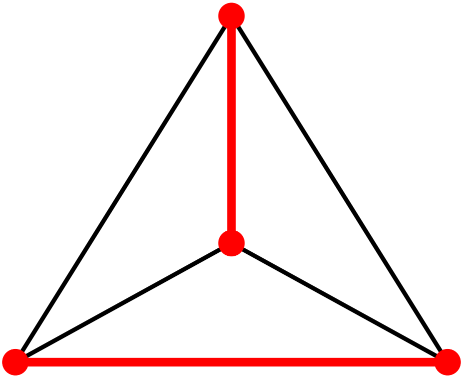

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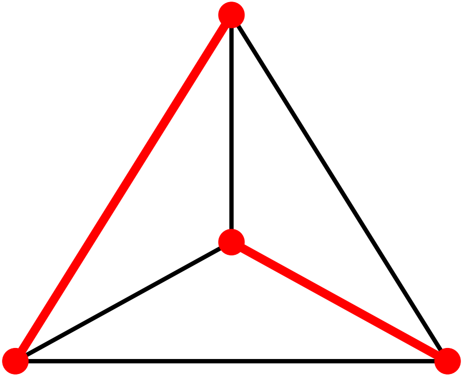

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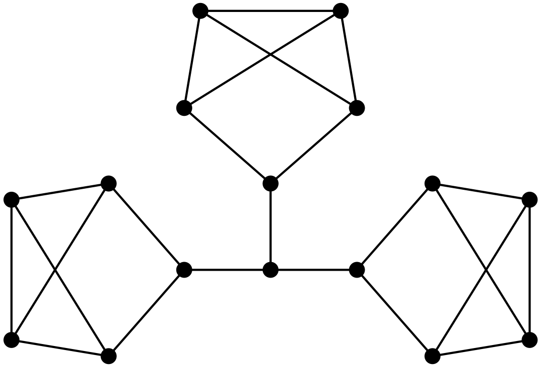

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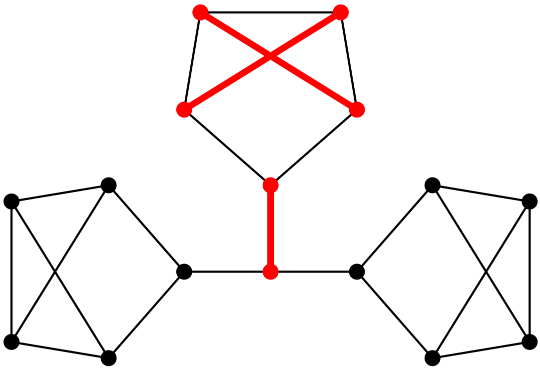

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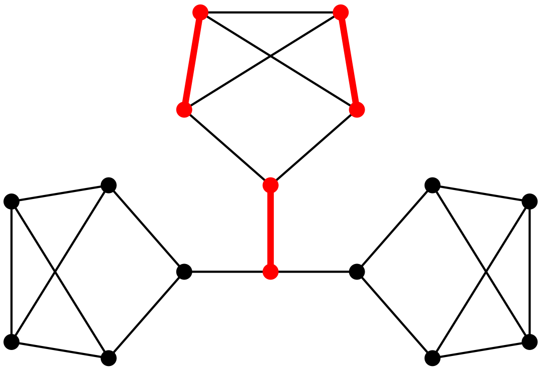

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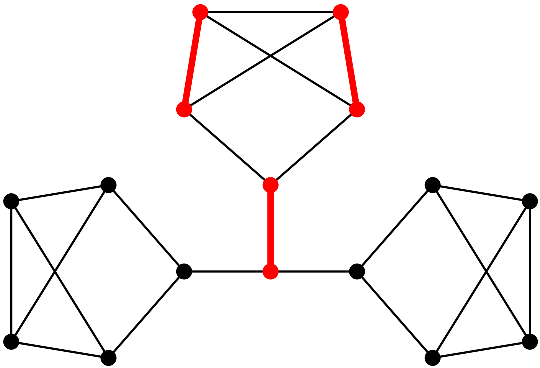
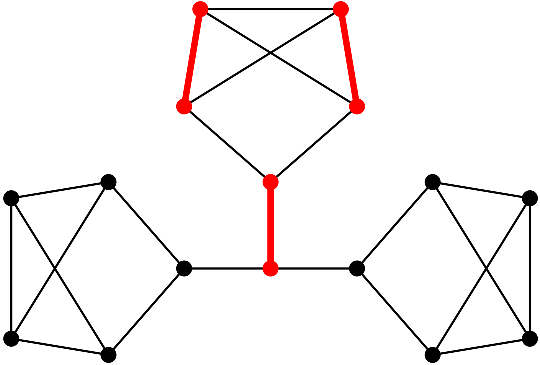


Theorem (Petersen 1891)

Every cubic bridgeless graph contains a perfect matching.

The Lovász-Plummer conjecture



Conjecture (Lovász & Plummer 70's)

There exists a constant $c > 0$, such that any n-vertex cubic bridgeless graph contains at least 2^{cn} perfect matchings.

Bipartite graphs

Theorem (Voorhoeve 1979)

Every cubic bipartite graph with n vertices contains at least $6 \cdot (4/3)^{n/2-3}$ perfect matchings.

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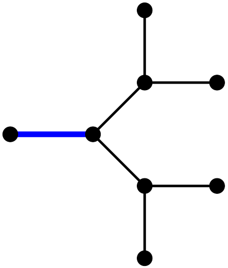

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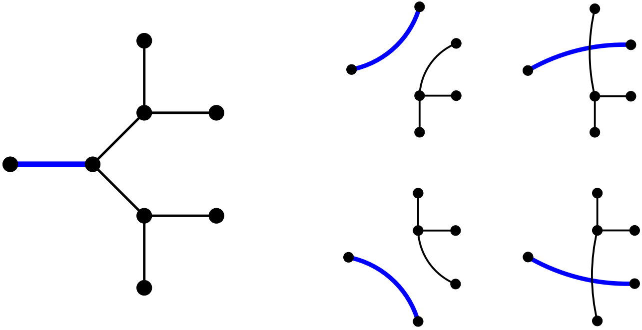

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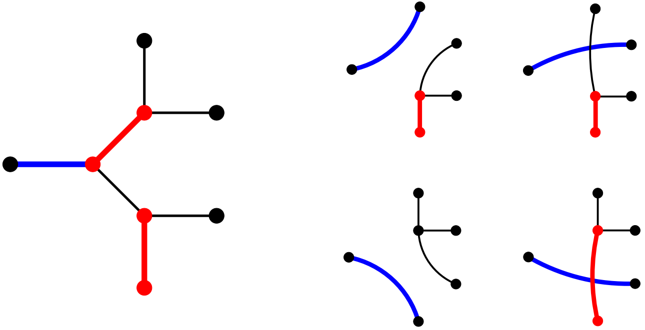

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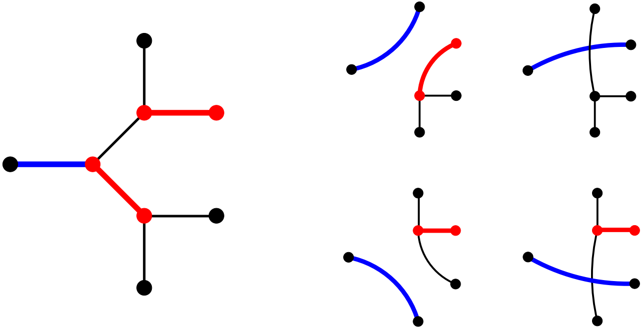

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Theorem (Chudnovsky & Seymour 2008)

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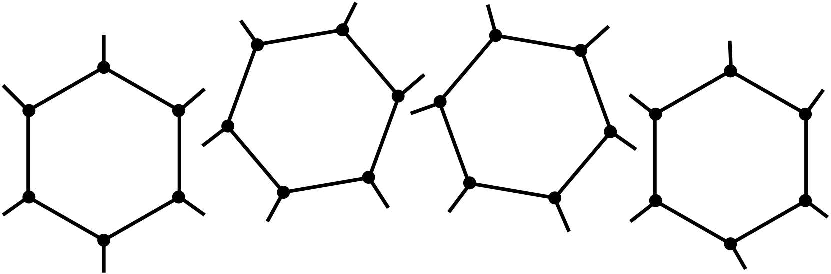

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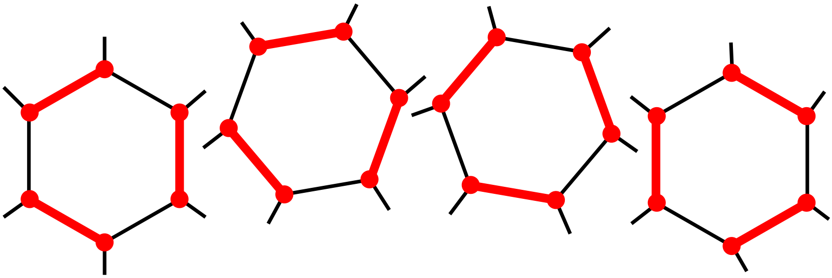

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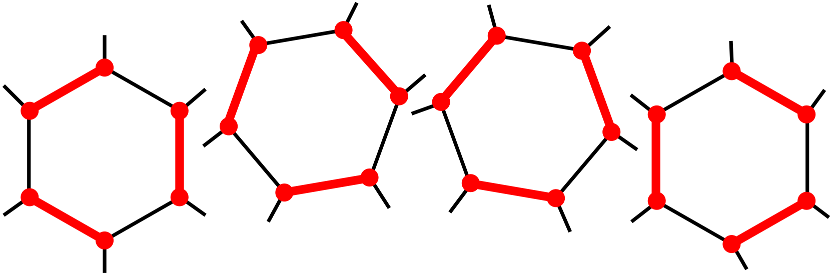

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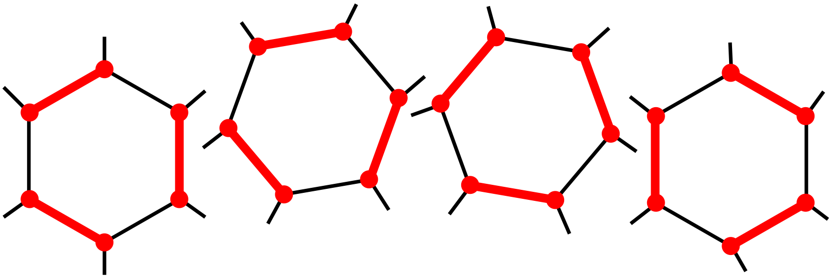

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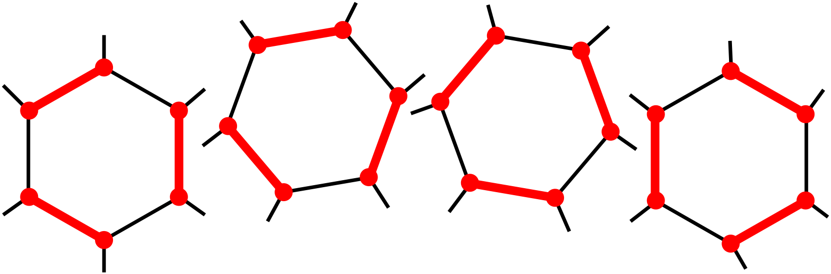

k disjoint alternating cycles $\Rightarrow 2^k$ perfect matchings

General case

Theorem (E., Kardoš & Král' 2009)

For any $a > 0$ there exists a constant b such that every cubic bridgeless graph with n vertices contains at least $an - b$ perfect matchings.

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Every cubic bridgeless graph on n vertices has at least $2^{n/3656}$ perfect matchings.

Theorem

If G is a cubic bridgeless graph on n vertices, then

1. each edge is contained in at least $2^{n/3656}$ perfect matchings, or
2. G has a perfect matching with at least $n/3656$ disjoint alternating cycles.

Sketch of the proof

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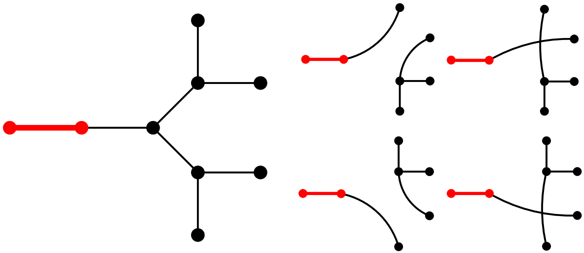

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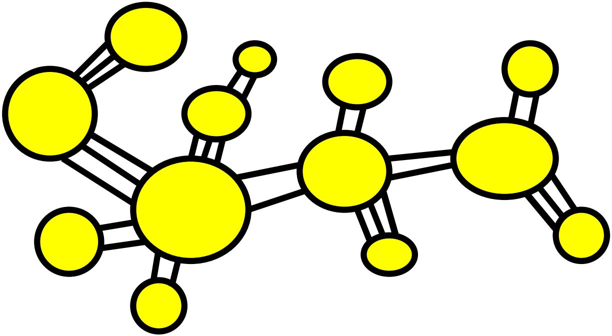

The perfect matching polytope

The perfect matching polytope of G is the convex hull of the characteristic vectors of the perfect matchings of G.

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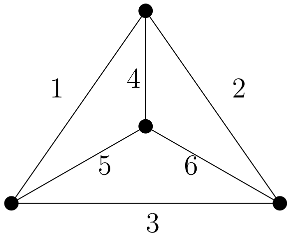

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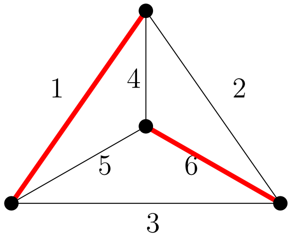

$$(1, 0, 0, 0, 0, 1)$$

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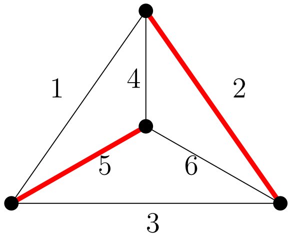

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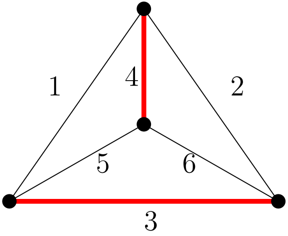

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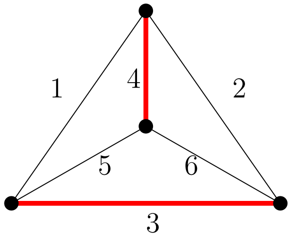

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$\{(a, b, c, c, a, b)$ with $a + b + c = 1$ and $a, b, c \geq 0\}$

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Theorem (Edmonds 1965)

A vector $w \in \mathbb{R}^E$ is in the perfect matching polytope if and only if

1. for each edge e, $w_e \geq 0$,
2. for each vertex v, $\sum_{e \ni v} w_e = 1$, and
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Equivalently, there is a probability distribution on the perfect matchings of G, such that each edge of G has probability $\frac{1}{3}$ to be in a random perfect matching.

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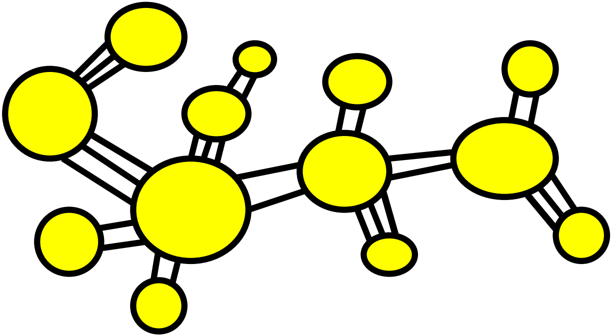

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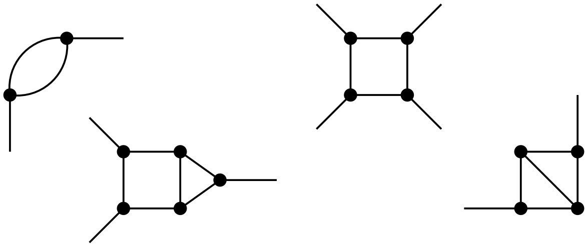

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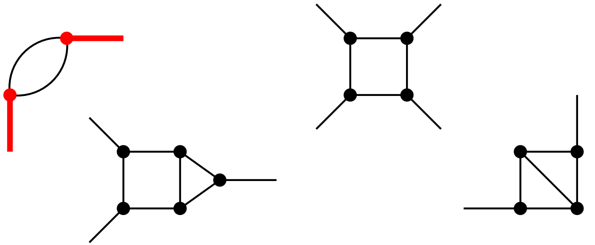

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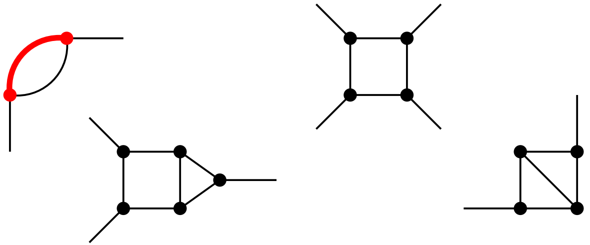

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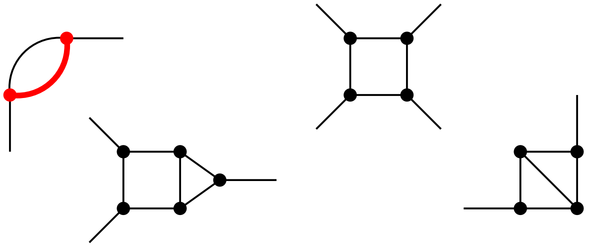

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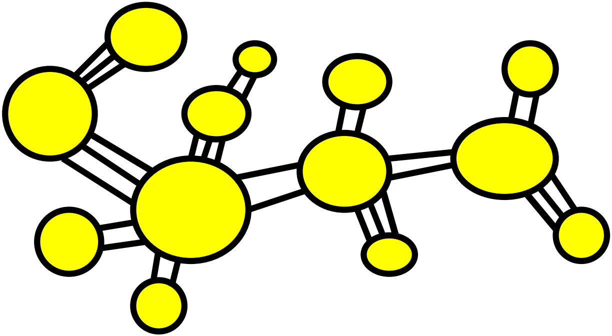

Perfect matchings in k-regular graphs

Theorem

For every $k \geq 3$, every $(k-1)$-edge-connected k-regular graph on n vertices (n even) has at least $2^{(1-\frac{1}{k})(1-\frac{2}{k})n/3656} \geq 2^{n/9750}$ perfect matchings.

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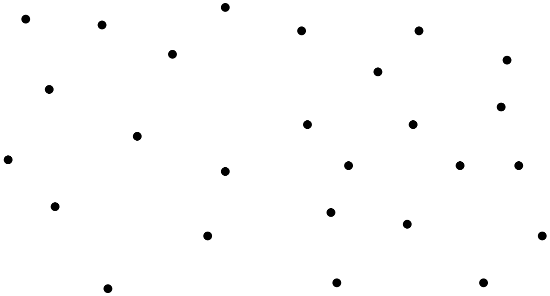

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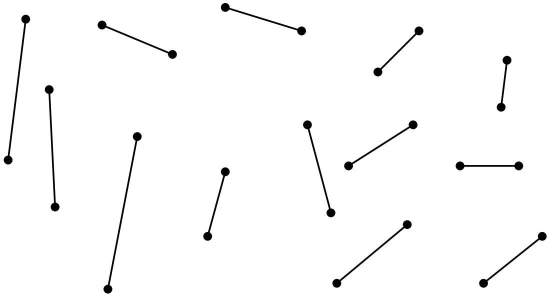

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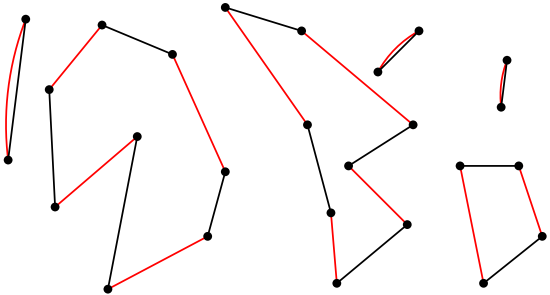

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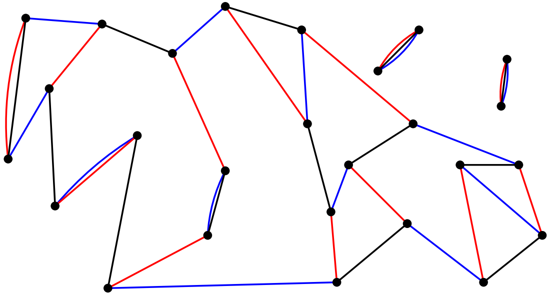

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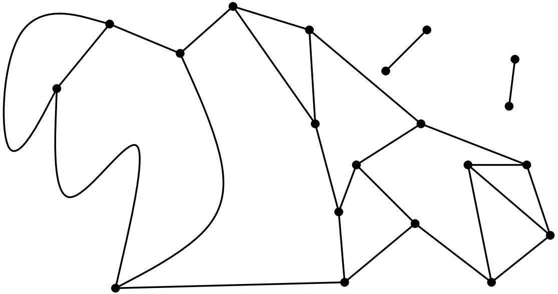

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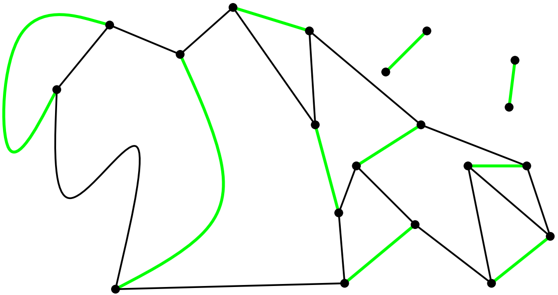

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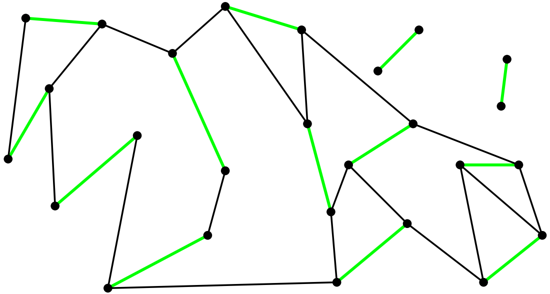

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Equivalently:

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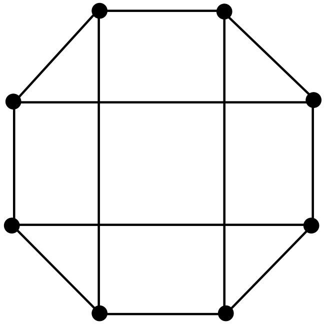

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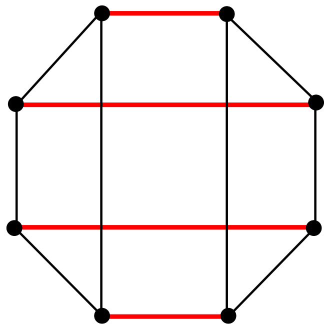

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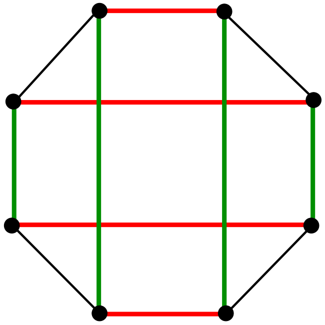

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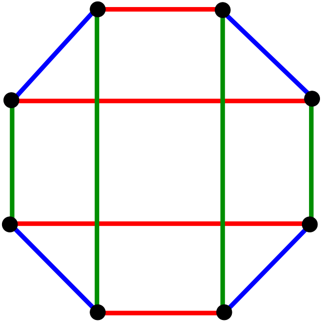

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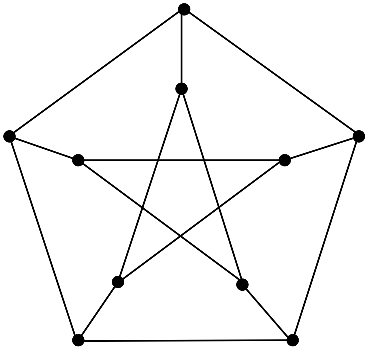

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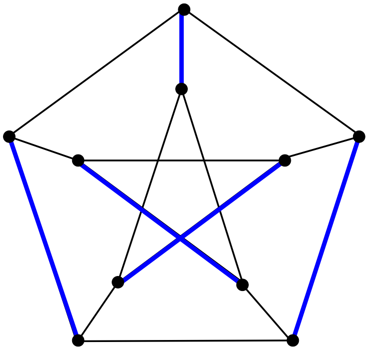

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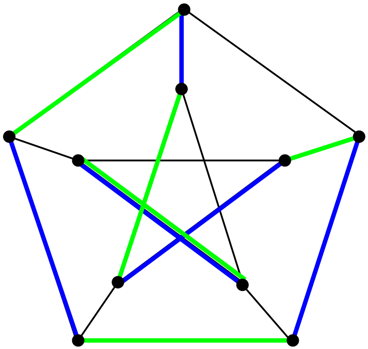

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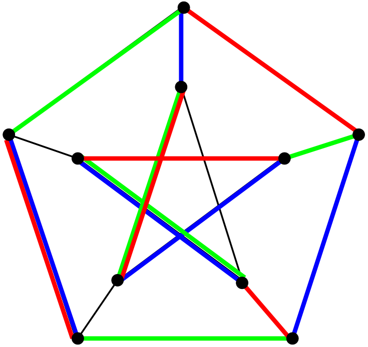

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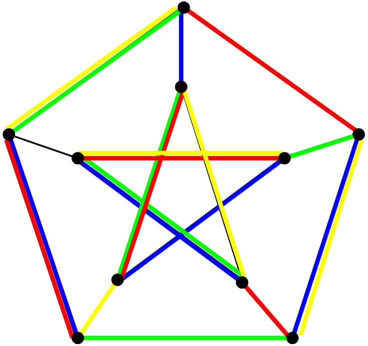

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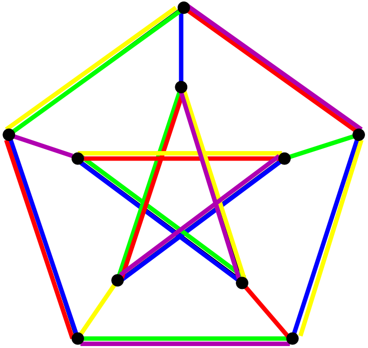

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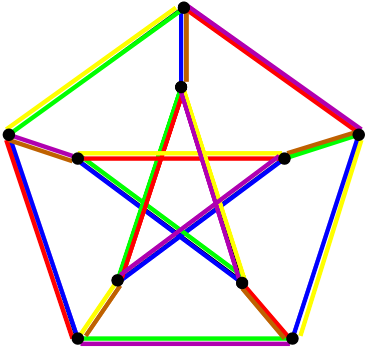

Covering the edge-set with perfect matchings

Berge-Fulkerson conjecture would imply

Conjecture (Berge)

The edge-set of every cubic bridgeless graph can be covered by at most 5 perfect matchings.

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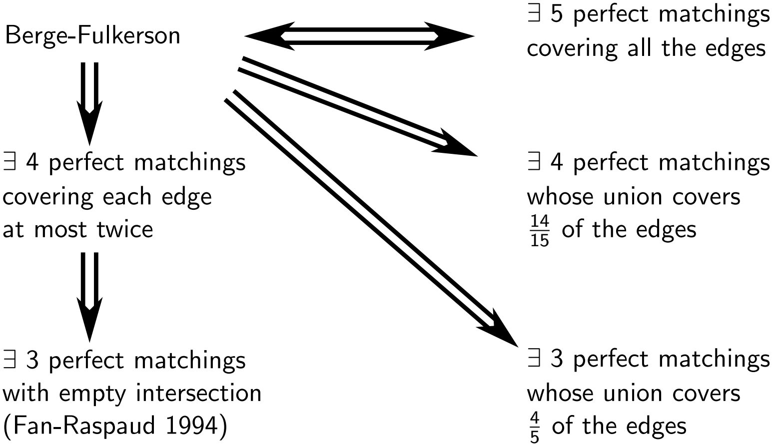
To achieve $\log n$: Draw random perfect matchings from the $\frac{1}{3}$-distribution until all edges are covered.

Consequences of Berge-Fulkerson



Consequences of Berge-Fulkerson

Berge-Fulkerson $\Longleftrightarrow$ $\exists$ 5 perfect matchings covering all the edges

Berge-Fulkerson $\Downarrow$ $\exists$ 4 perfect matchings covering each edge at most twice

$\exists$ 5 perfect matchings covering all the edges $\Downarrow$ $\exists$ 4 perfect matchings whose union covers $\frac{14}{15}$ of the edges

$\exists$ 4 perfect matchings covering each edge at most twice $\Longleftarrow$ $\exists$ 4 perfect matchings whose union covers $\frac{14}{15}$ of the edges

$\exists$ 4 perfect matchings covering each edge at most twice $\Downarrow$ $\exists$ 3 perfect matchings with empty intersection (Fan-Raspaud 1994)

$\exists$ 4 perfect matchings whose union covers $\frac{14}{15}$ of the edges $\Downarrow$ $\exists$ 3 perfect matchings whose union covers $\frac{4}{5}$ of the edges

$\exists$ 3 perfect matchings with empty intersection (Fan-Raspaud 1994) $\Longleftarrow$ $\exists$ 3 perfect matchings whose union covers $\frac{4}{5}$ of the edges

Extended formulations

Theorem (Edmonds 1965)

A vector $w \in \mathbb{R}^E$ is in the perfect matching polytope if and only if

1. for each edge e, $w_e \geq 0$,
2. for each vertex v, $\sum_{e \ni v} w_e = 1$, and
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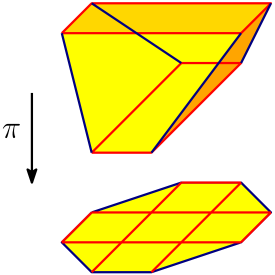

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Theorem (Rothvoß 2014)

Any extended formulation of the perfect matching polytope of a complete graph needs an exponential number of inequalities.