

A bijection for rooted maps on general surfaces

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joint work with

Guillaume Chapuy, CNRS & LIAFA, Université Paris Diderot

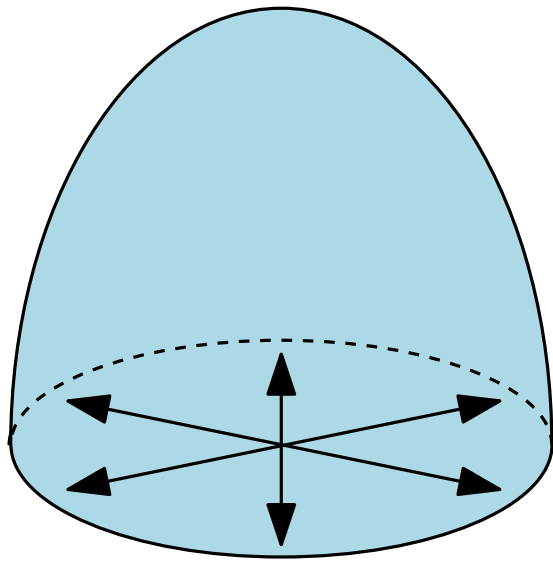
I. Maps

Maps

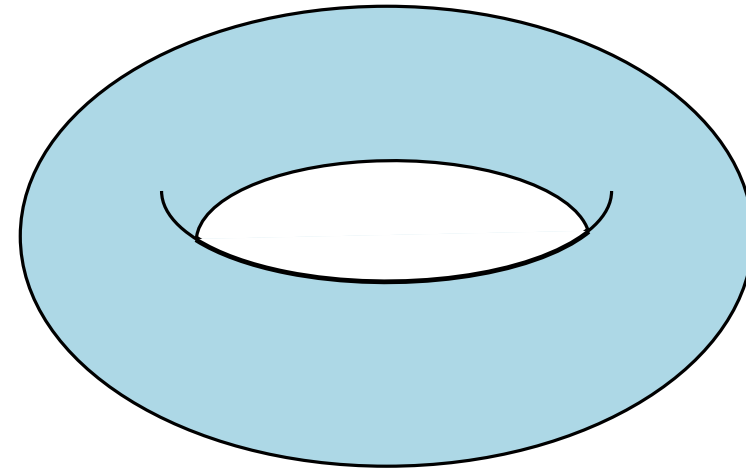
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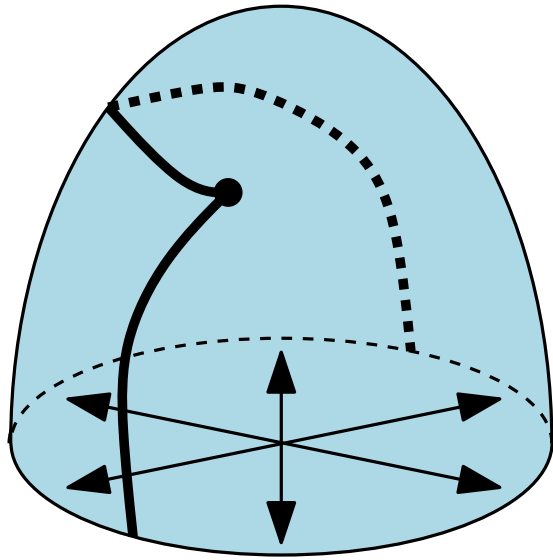
Projective plane



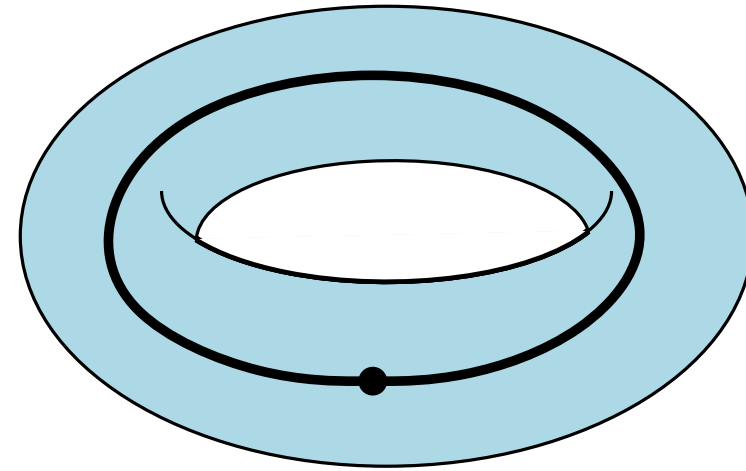
Torus

Maps

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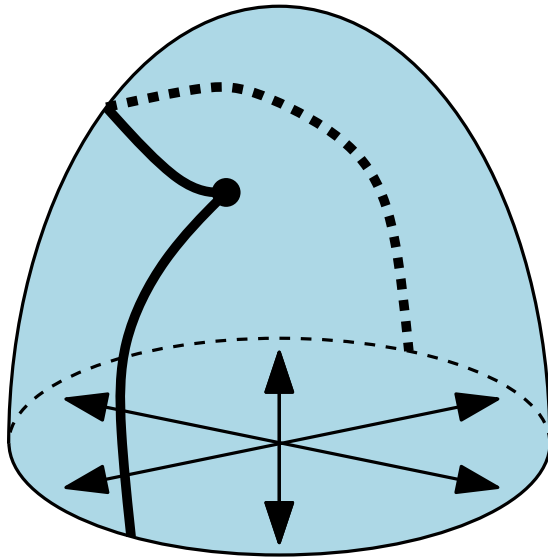
This is a map



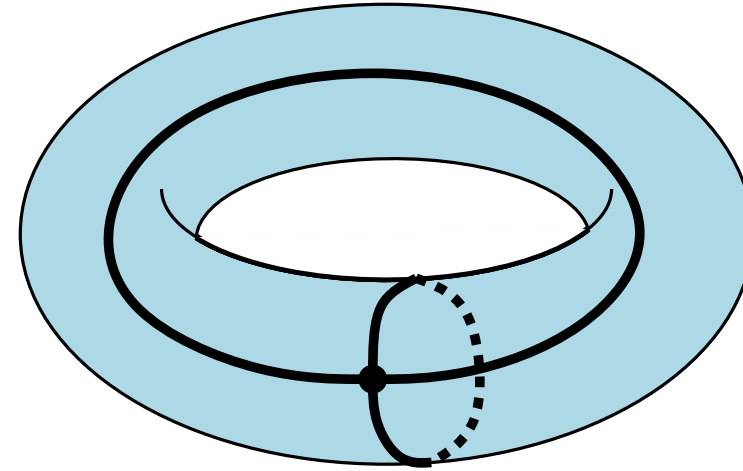
This is not a map!

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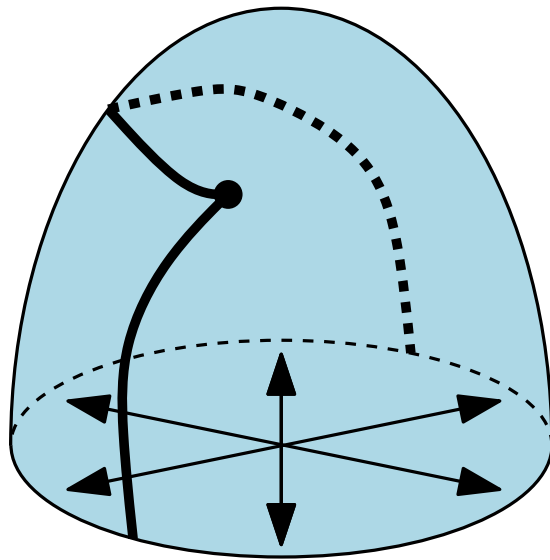
This is a map



This is a map too.

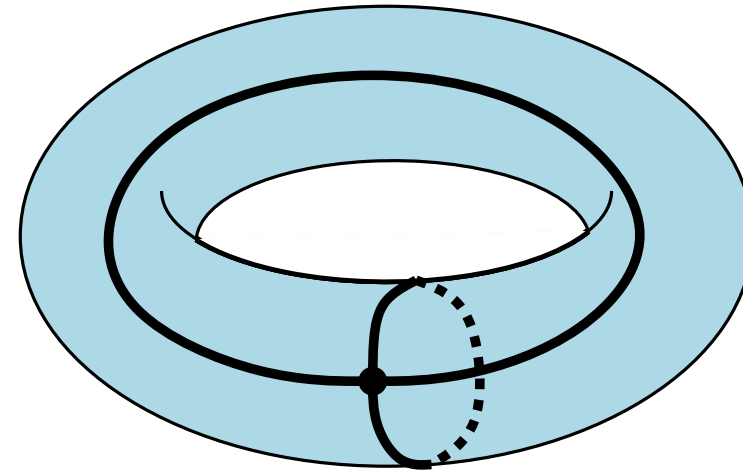
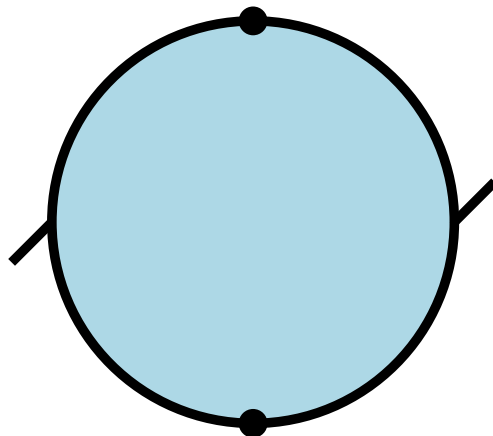
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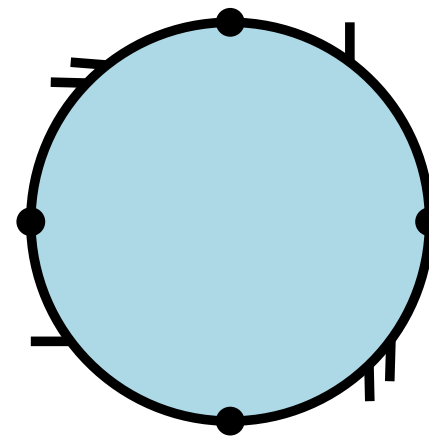
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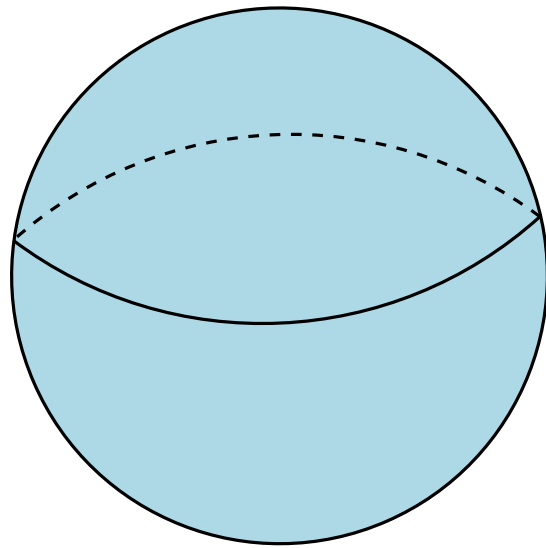
Orientable vs. non-orientable

Surfaces are classified by their **Euler characteristic**: $\chi(\mathbb{S})$. The number g is the **type** of surface \mathbb{S} if $\chi(\mathbb{S}) = 2 - 2g$. Surfaces can be:

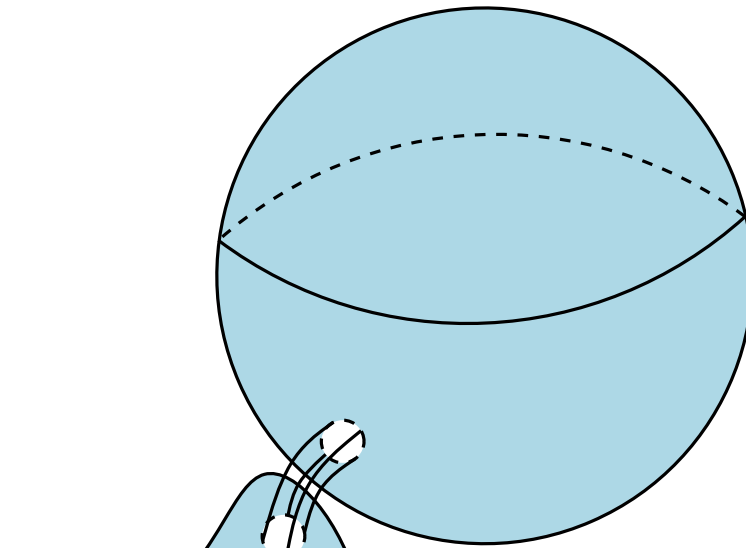
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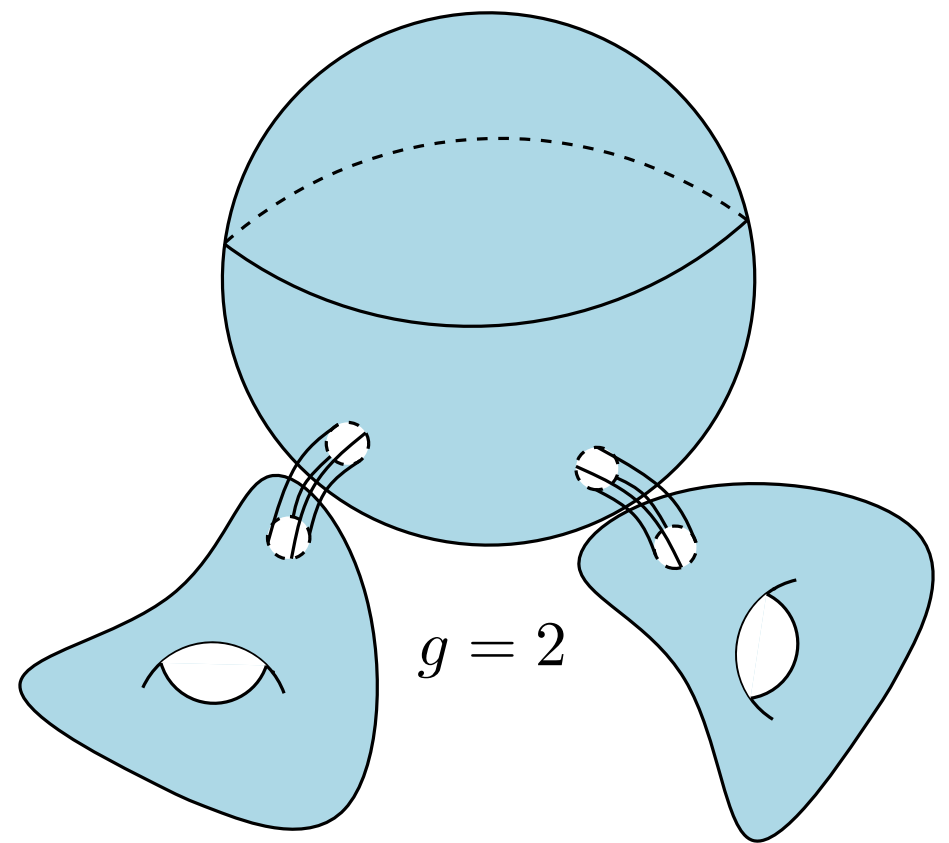
- **orientable**



$g = 0$



$g = 1$

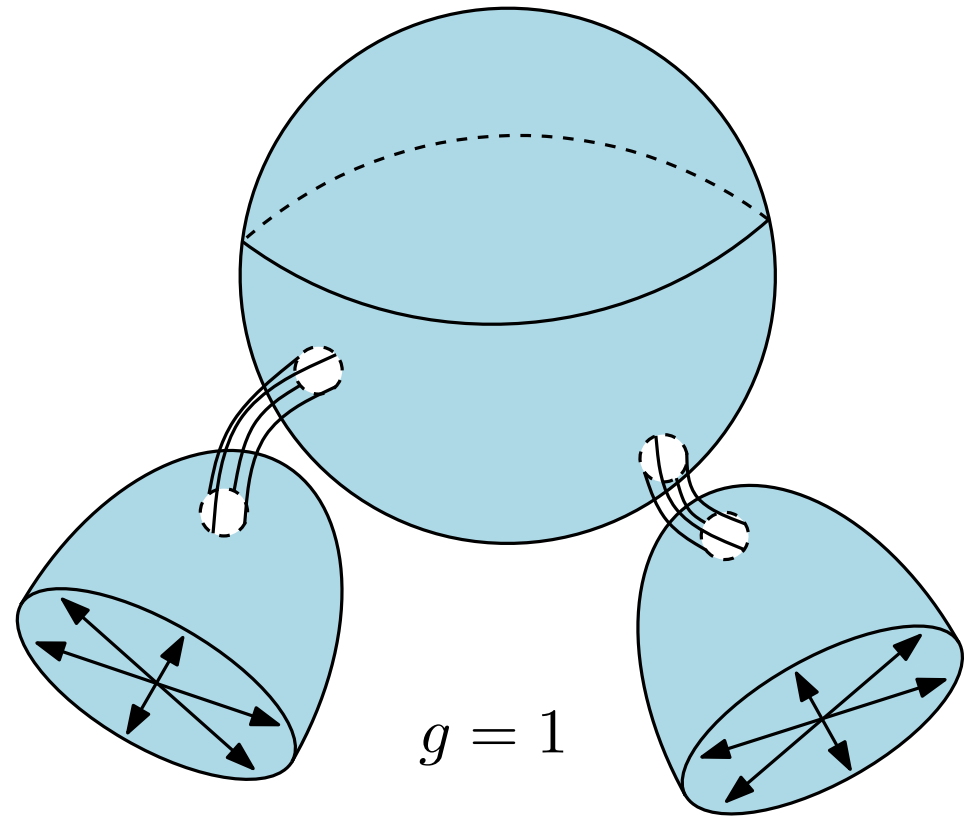
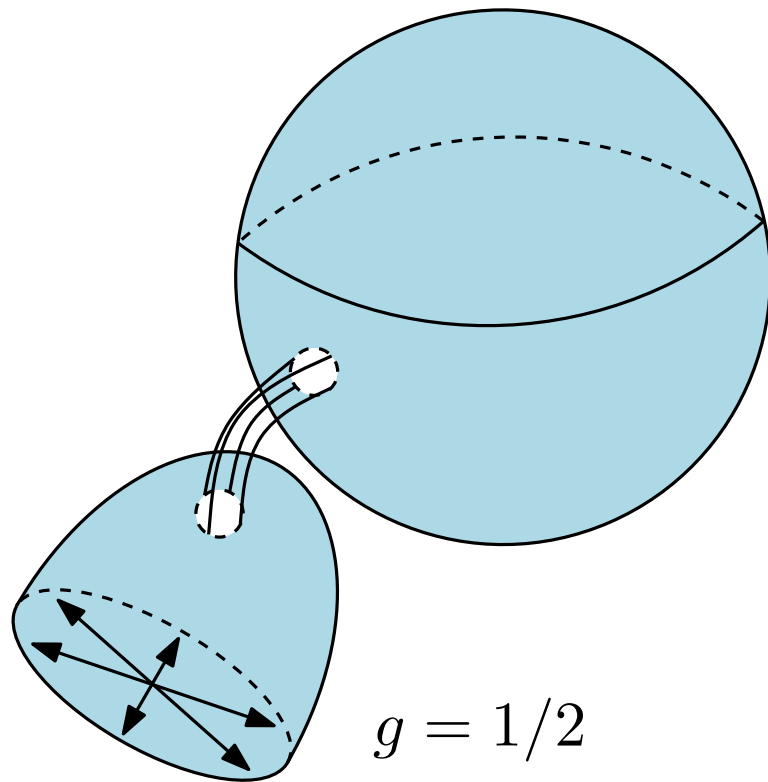


$g = 2$

Orientable vs. non-orientable

Surfaces are classified by their **Euler characteristic**: $\chi(\mathbb{S})$. The number g is the **type** of surface \mathbb{S} if $\chi(\mathbb{S}) = 2 - 2g$. Surfaces can be:

- **non-orientable**



Orientable vs. non-orientable

Surfaces are classified by their **Euler characteristic**: $\chi(\mathbb{S})$. The number g is the **type** of surface \mathbb{S} if $\chi(\mathbb{S}) = 2 - 2g$. Surfaces can be:

- **orientable**,
- **non-orientable**.

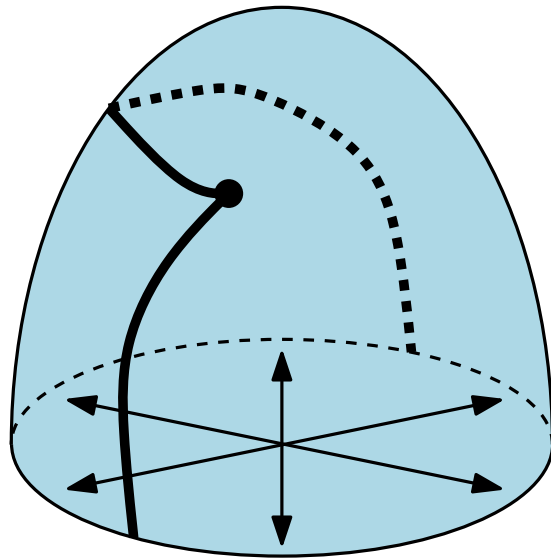
We will say that a map M is **orientable/non-orientable of type g** if the underlying surface is orientable/non-orientable of type g .

Orientable vs. non-orientable

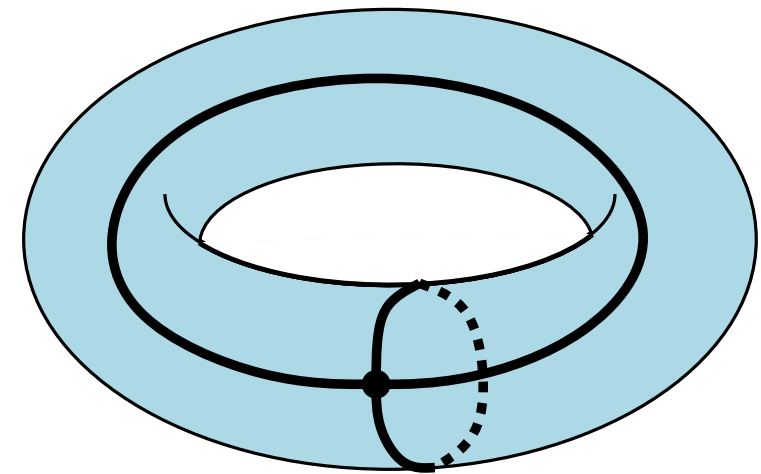
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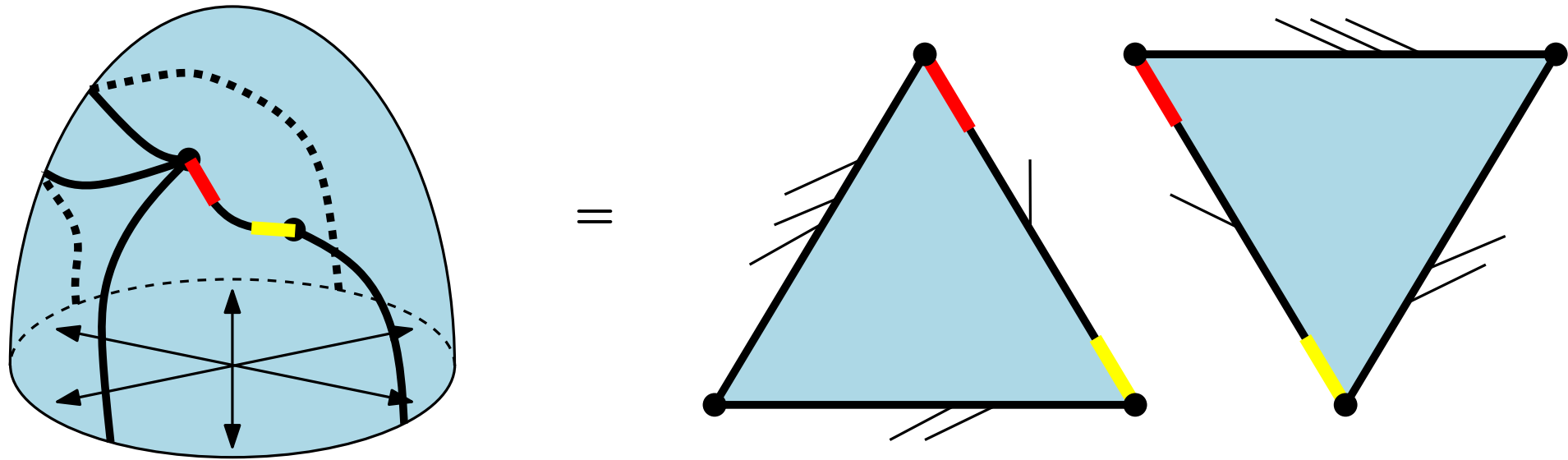
Non-orientable map of type 1/2



Orientable map of type 1

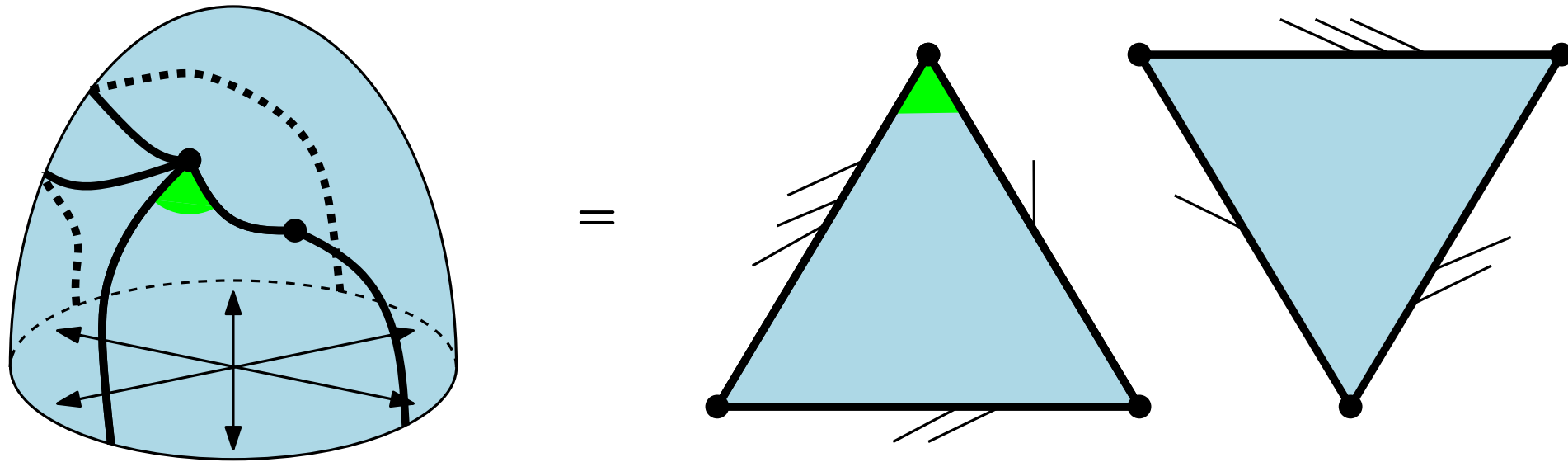
Rooted maps

Each edge consists of two **half-edges**.



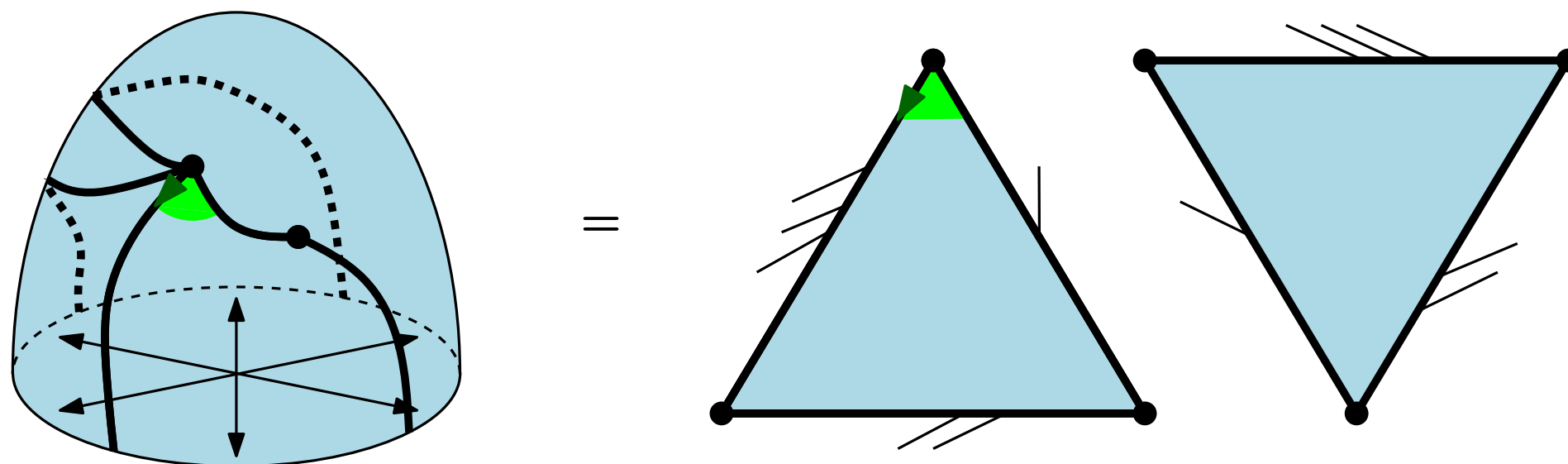
Rooted maps

Each edge consists of two half-edges. A region between two consecutive half-edges attached to a vertex is called **corner**.



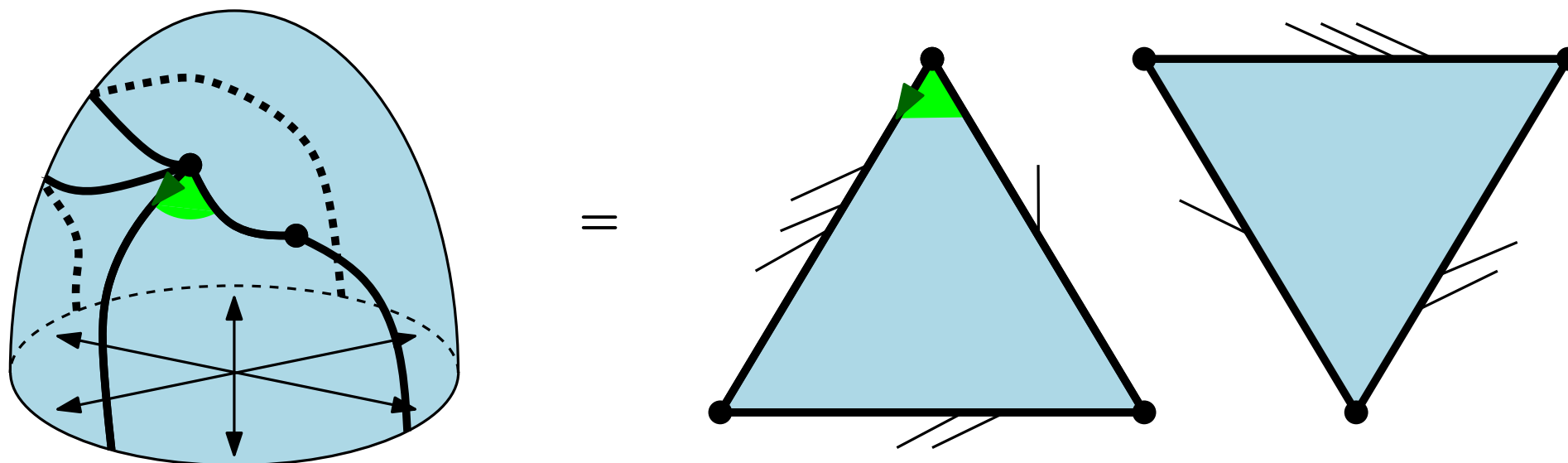
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Remark:

Tutte noticed that maps are much simpler to enumerate, when **rooted**, because of the lack of symmetry. From now on, all maps will be **rooted**!

Maps with n edges vs. bipartite quadrangulations with n faces

Map M is **bipartite** if vertices can be colored by two different colors ($V_{\bullet}(M)$ - set of black vertices, $V_{\circ}(M)$ - set of white vertices, root vertex is black by convention) such that each edge connects two vertices of different colors.

Quadrangulation is a map with all faces of degree 4.

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Theorem [Tutte 1960]

There is a bijection between

- the set of rooted maps on \mathbb{S} with n edges, l vertices and k faces of degree $\lambda_1, \dots, \lambda_k$,
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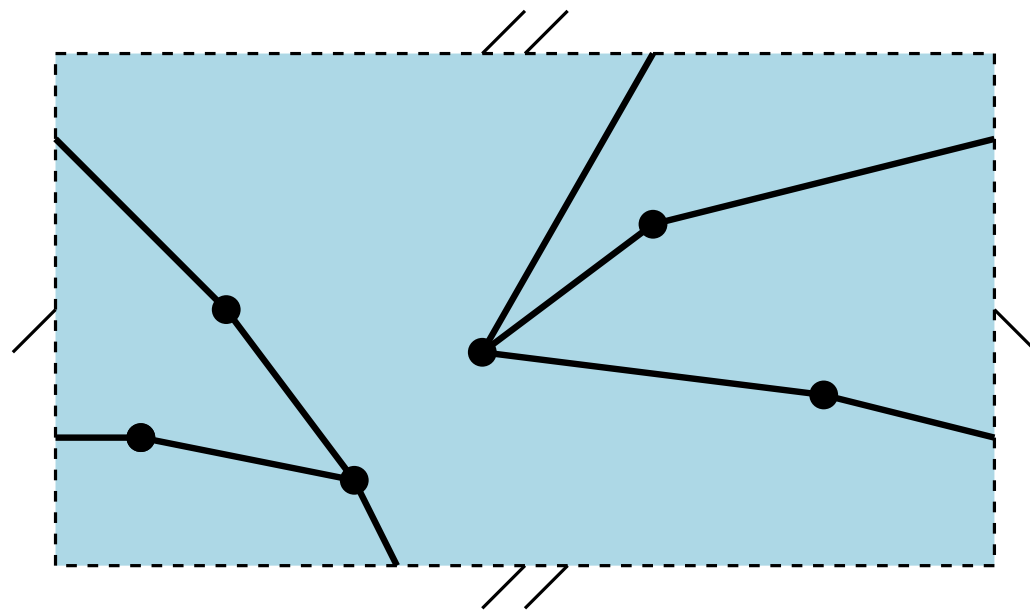
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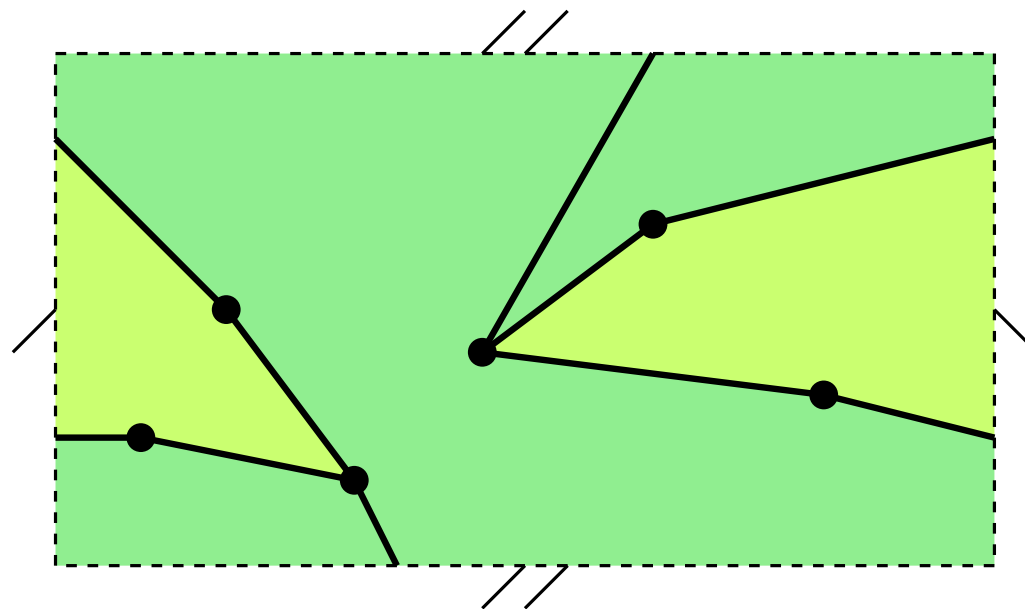
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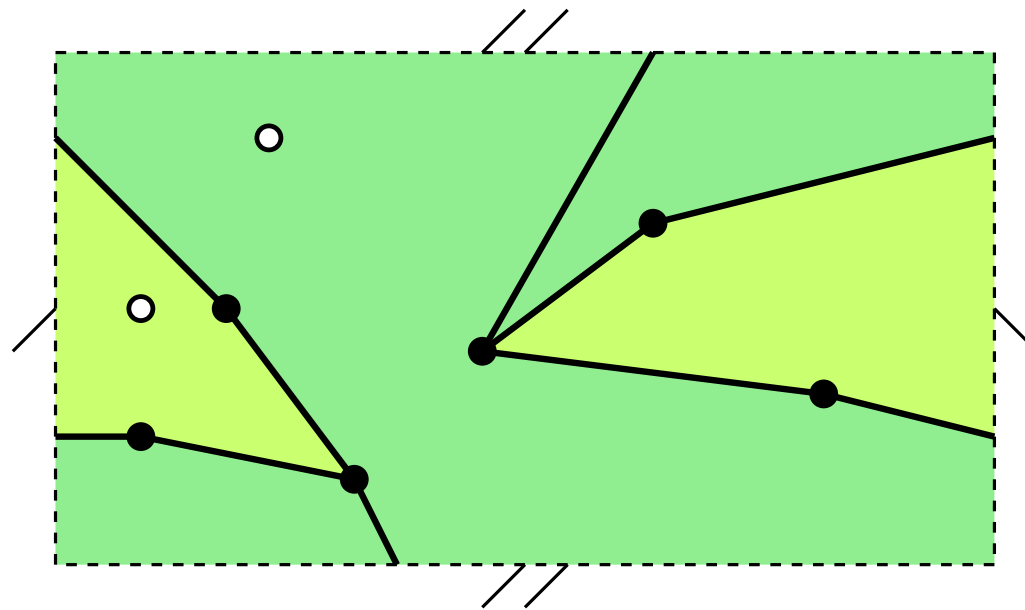
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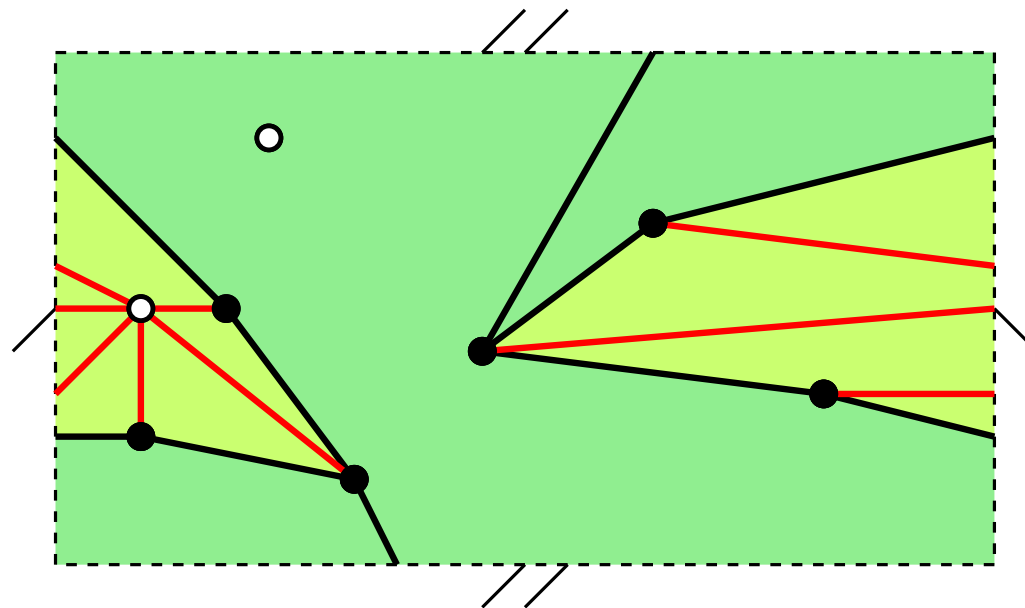
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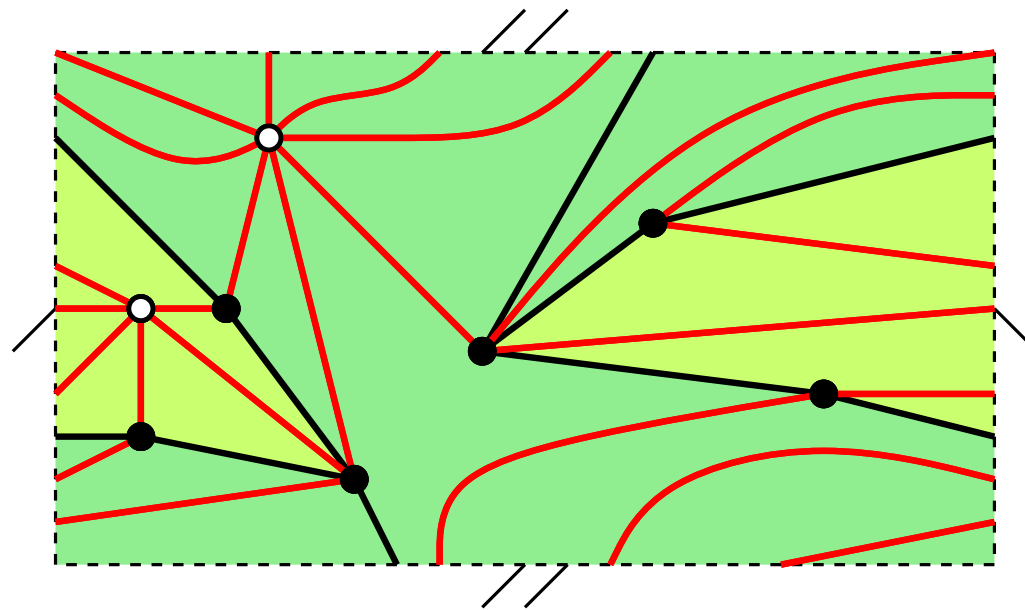
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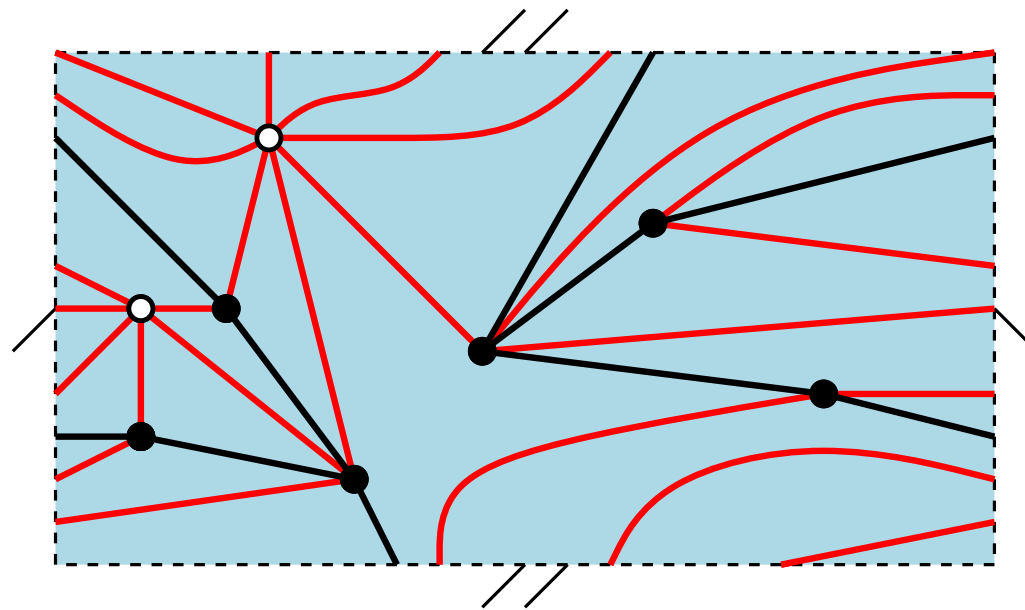
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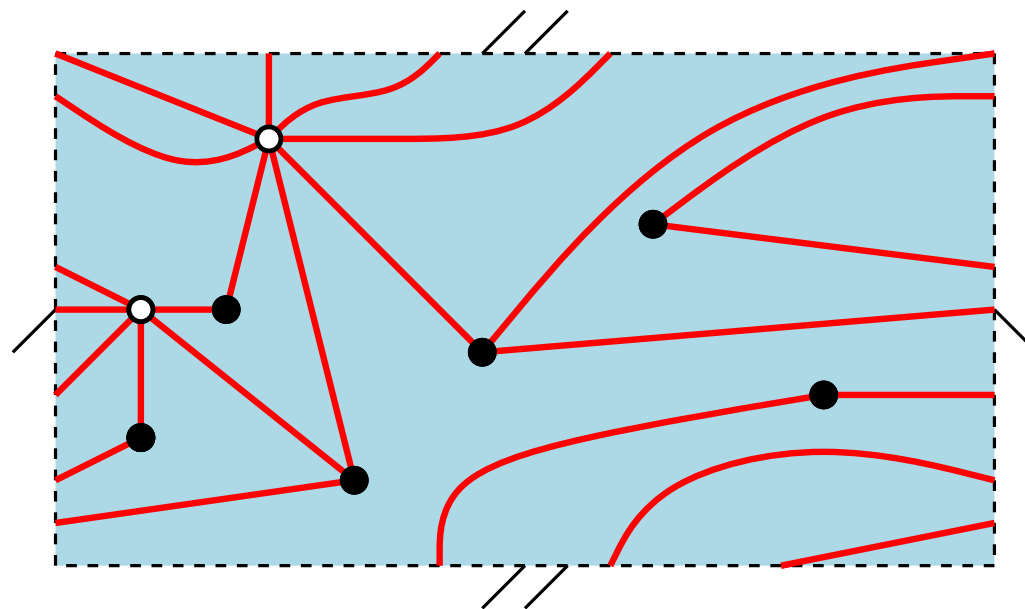
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II. Bijections for bipartite quadrangulations

Labeled and well-labeled maps

A map is called **labeled** if its vertices are labeled by integers such that:

- the root vertex has label 1;
- if two vertices are linked by an edge, their labels differ by at most 1.

If in addition we have:

- all the vertex labels are positive,

then the unicellular map is called **well-labeled**.

Labeled and well-labeled maps

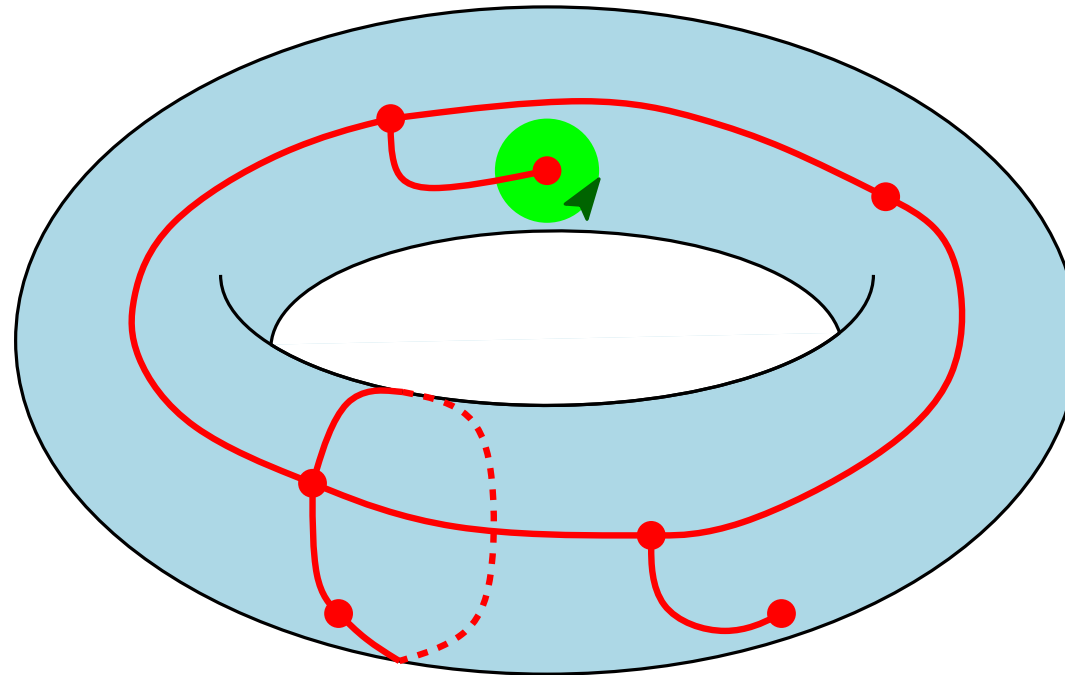
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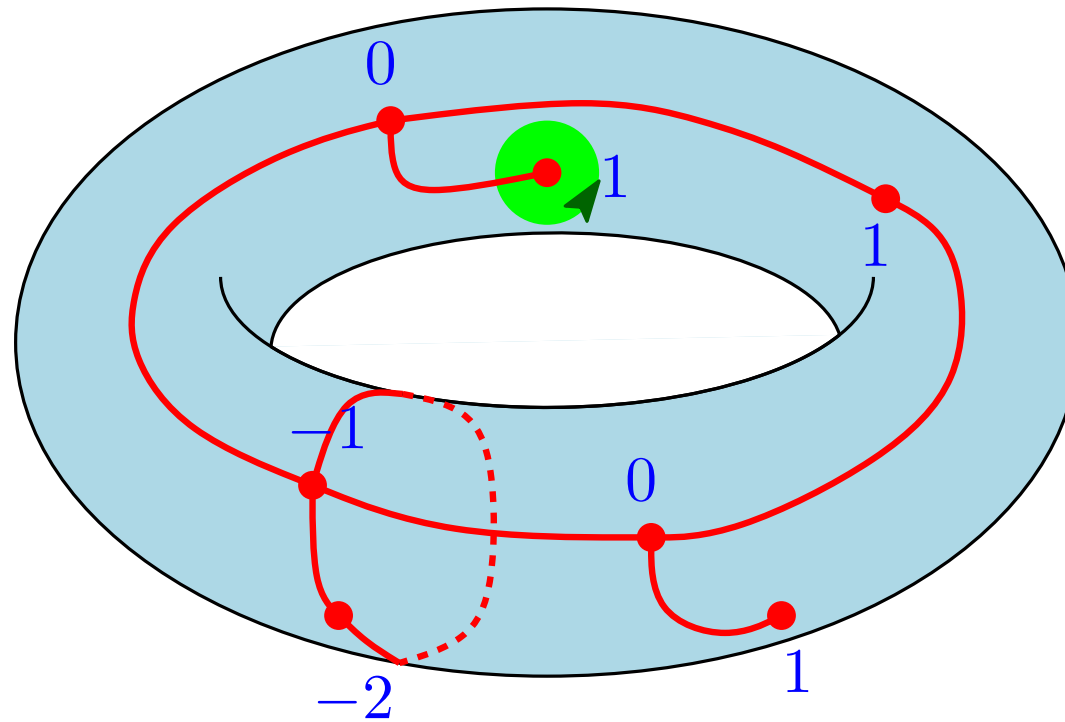
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labeled map on a torus

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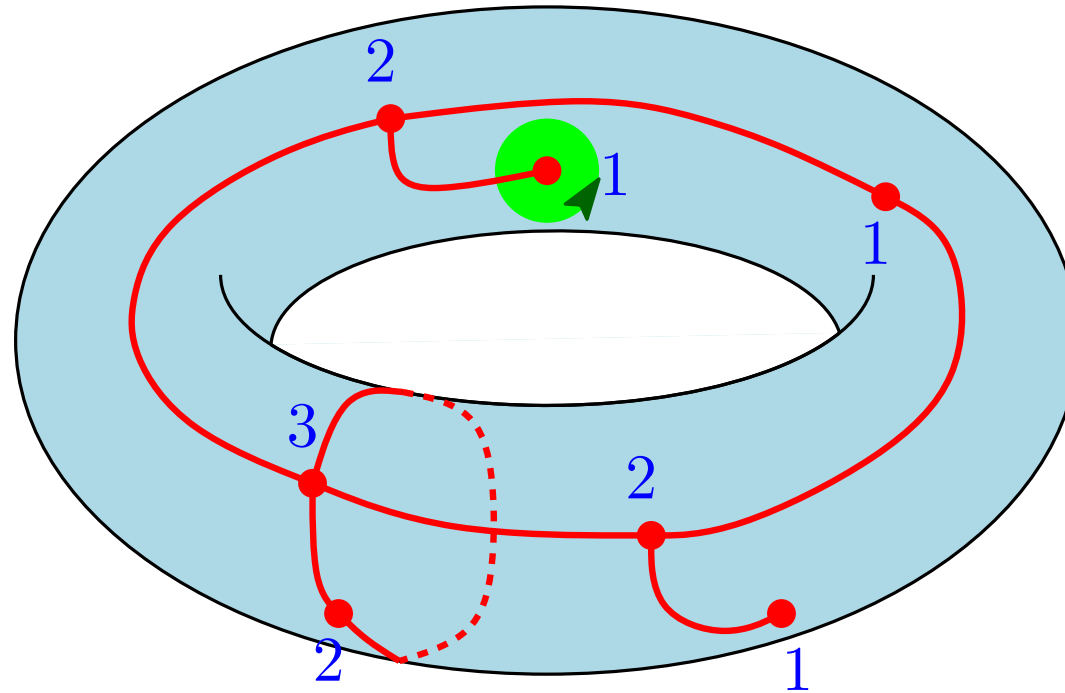
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Orientable case

Theorem [Marcus, Schaeffer 1996]

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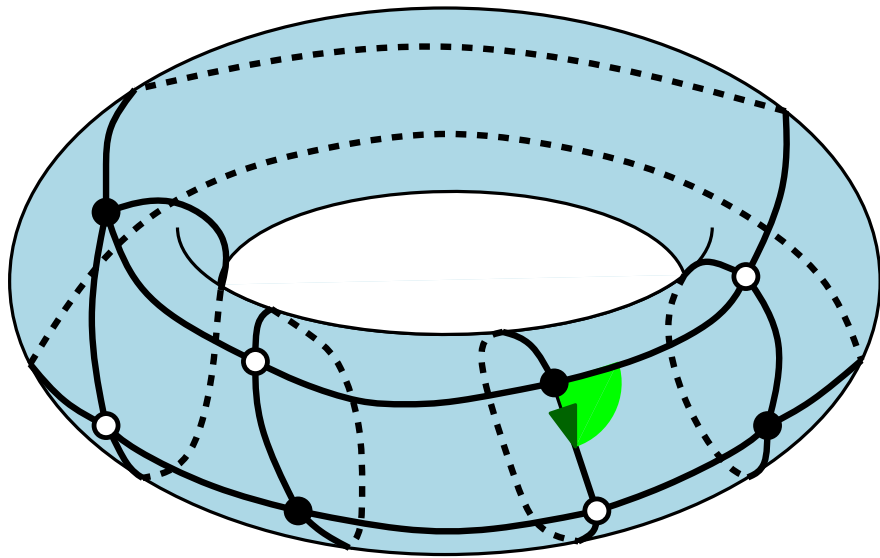
- rooted, **bipartite quadrangulations** on **ORIENTABLE** surface \mathbb{S} with n faces and N_i vertices at distance i from the root vertex ($i \geq 1$);
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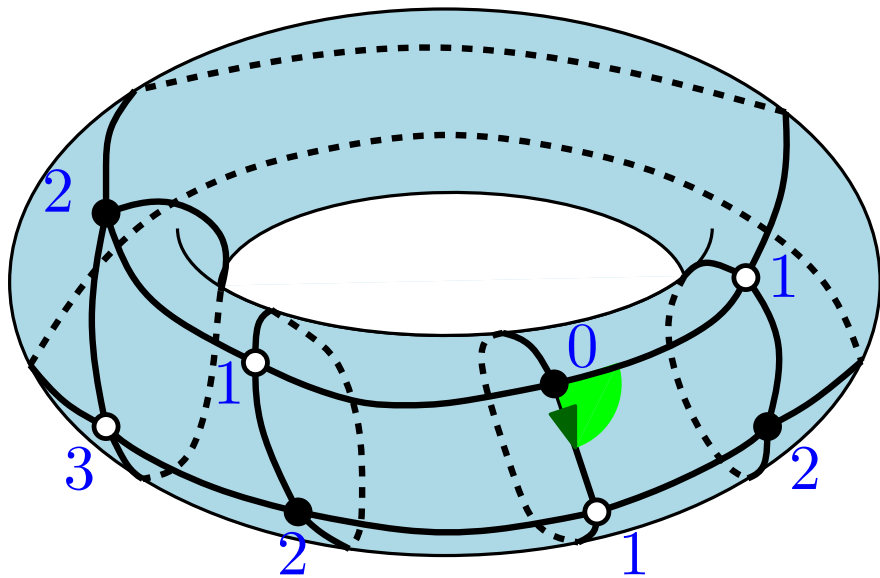


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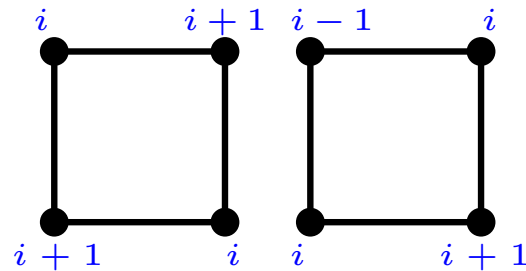
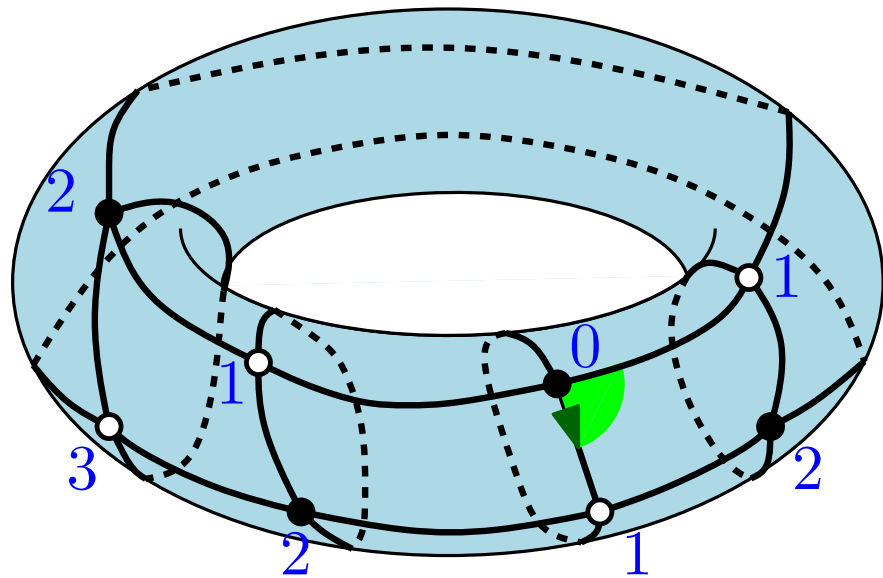


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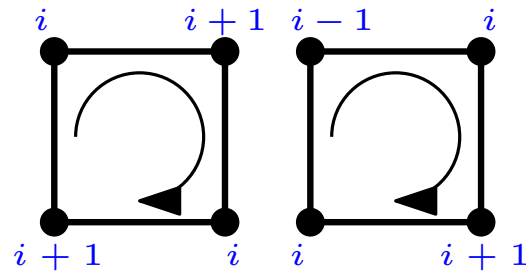
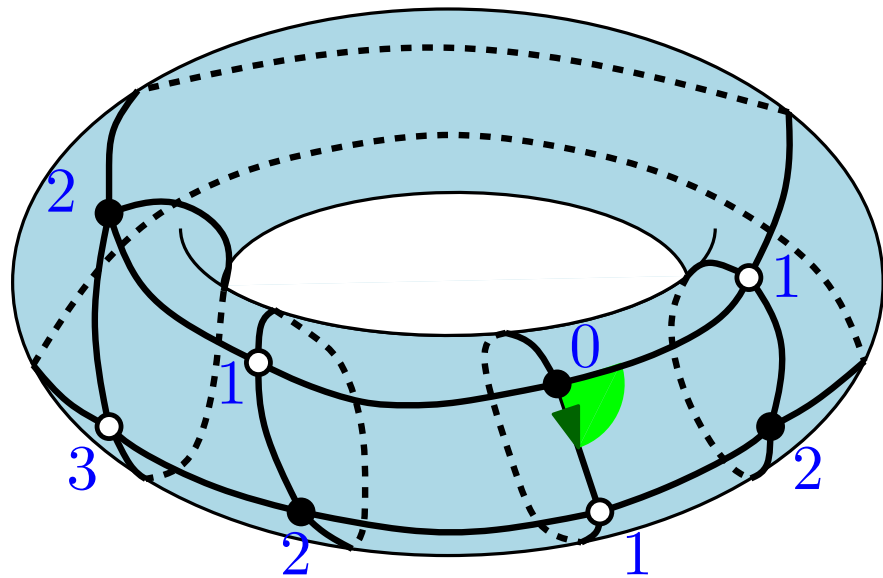


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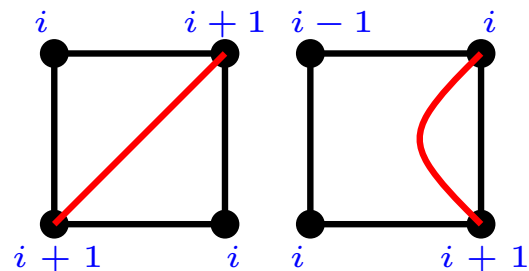
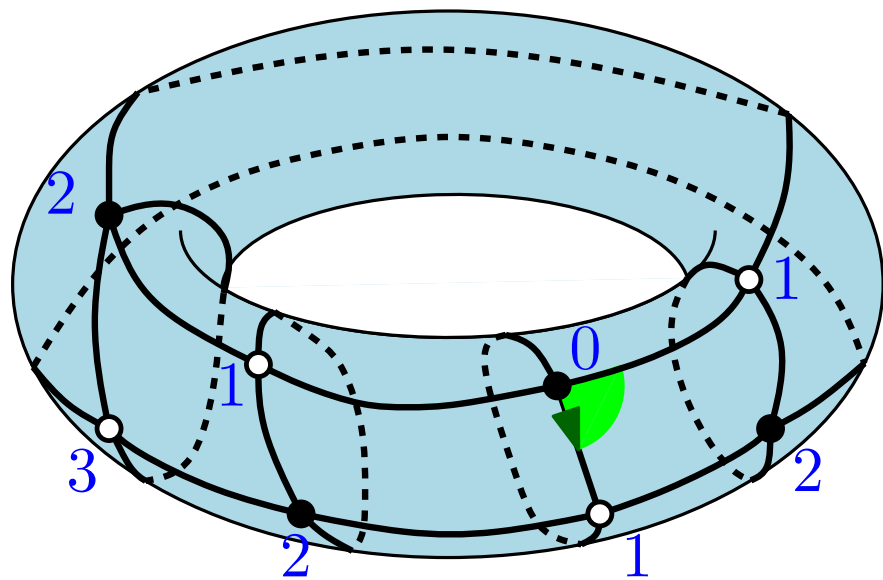


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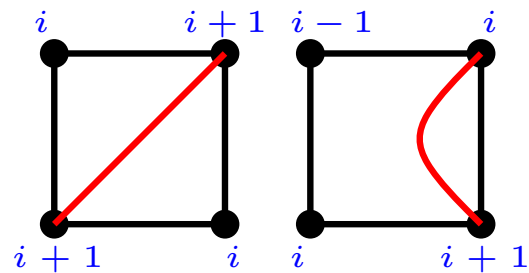
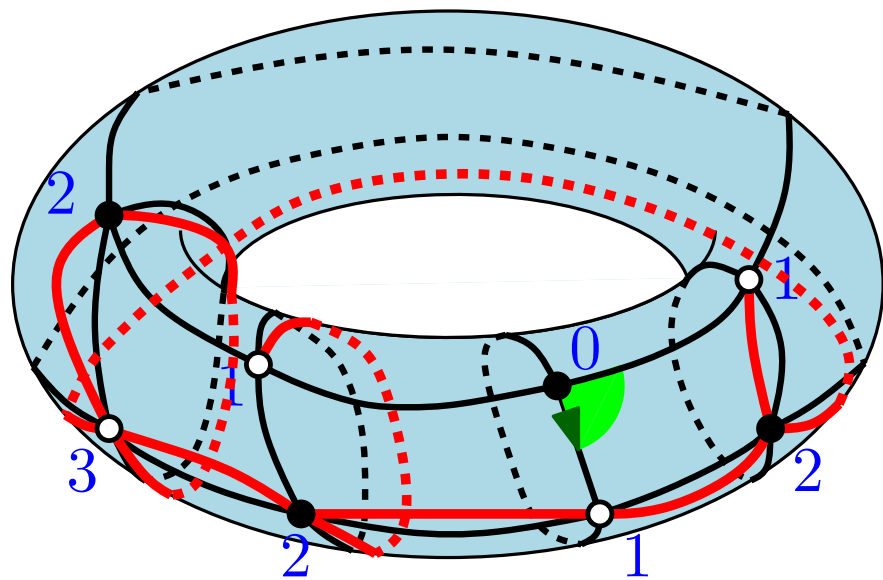


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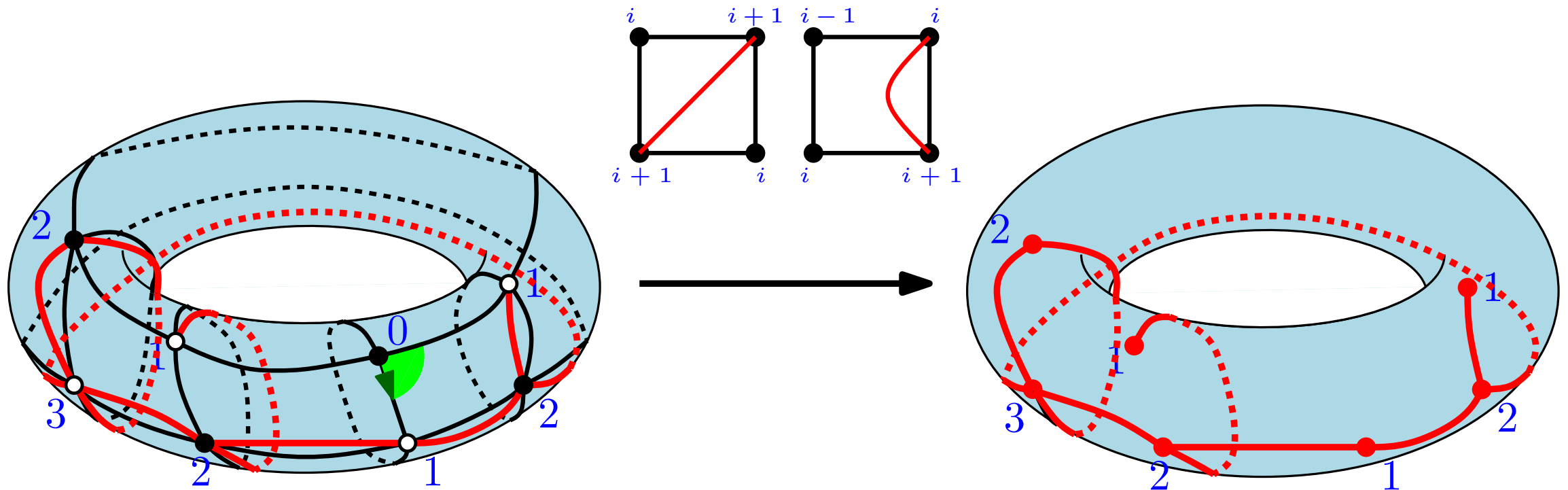


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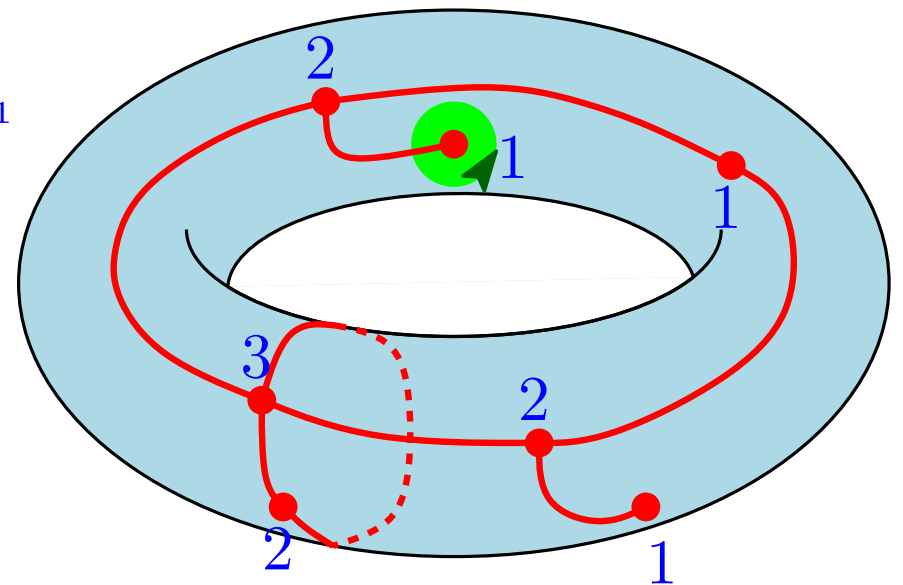
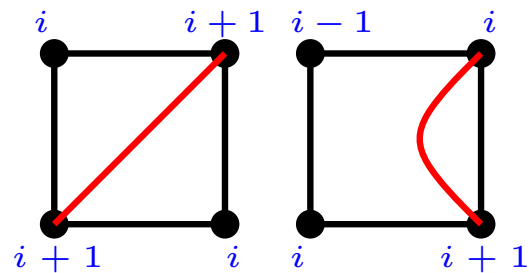
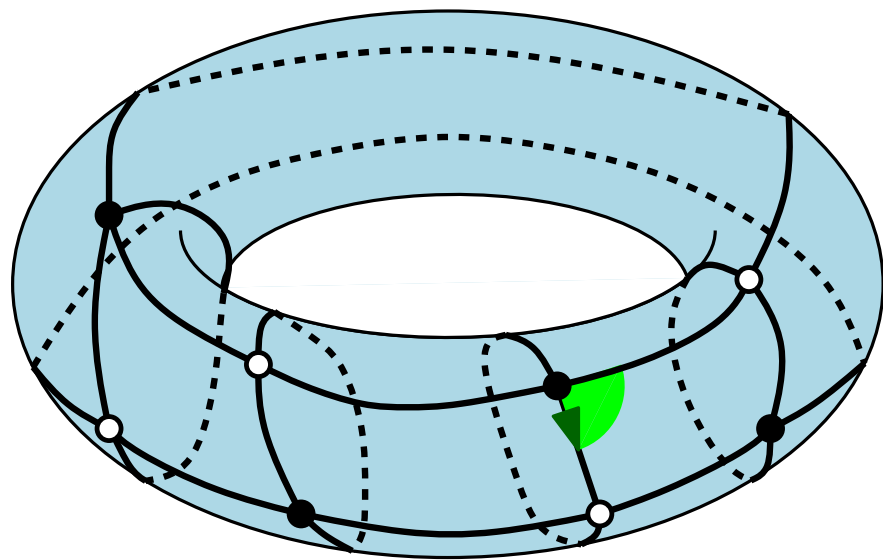


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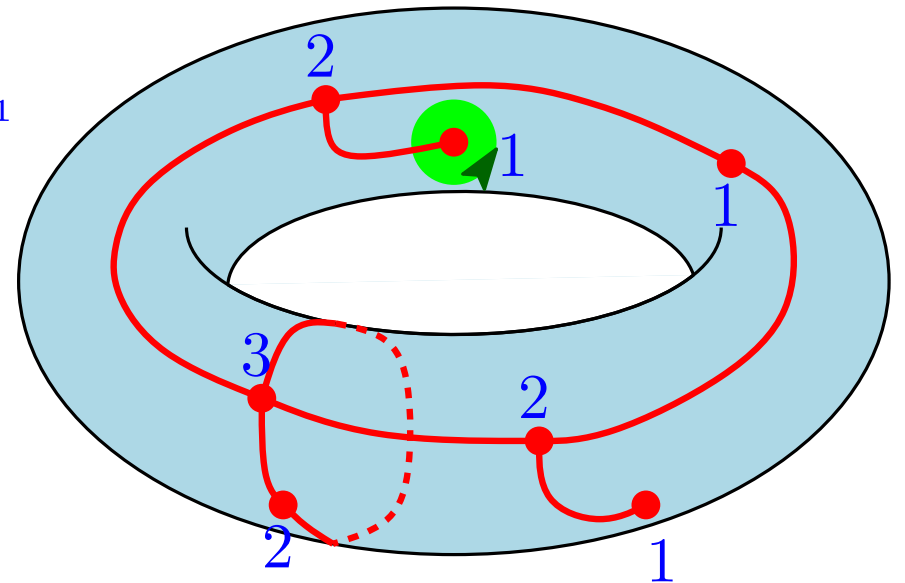
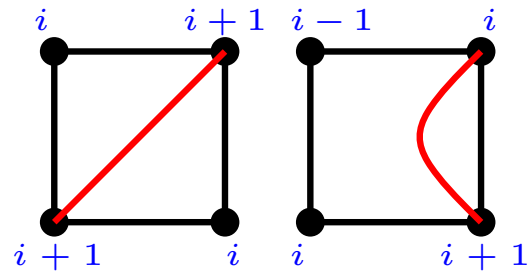
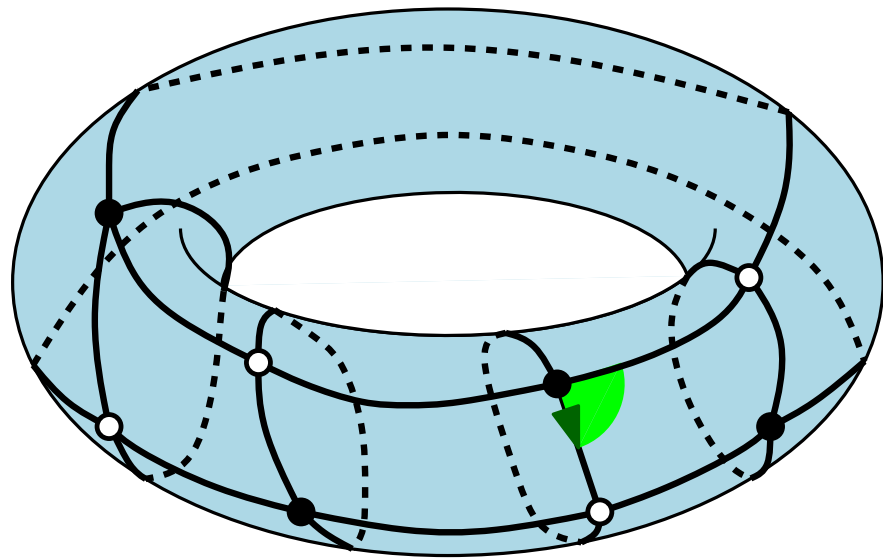


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- rooted, **bipartite quadrangulations** on **ORIENTABLE** surface \mathbb{S} with n faces and N_i vertices at distance i from the root vertex ($i \geq 1$);
- rooted, **one-face, well-labeled** maps on **ORIENTABLE** surface \mathbb{S} with n edges and N_i vertices of label i ($i \geq 1$);



What about
non-orientable maps?

General case

Theorem [Chapuy, D. 2015]

There is a bijection between:

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Idea:

To extend Marcus-Schaeffer bijection we have to use the same rules locally, but in non-orientable case we have to orient each face in some **CANONICAL** way.

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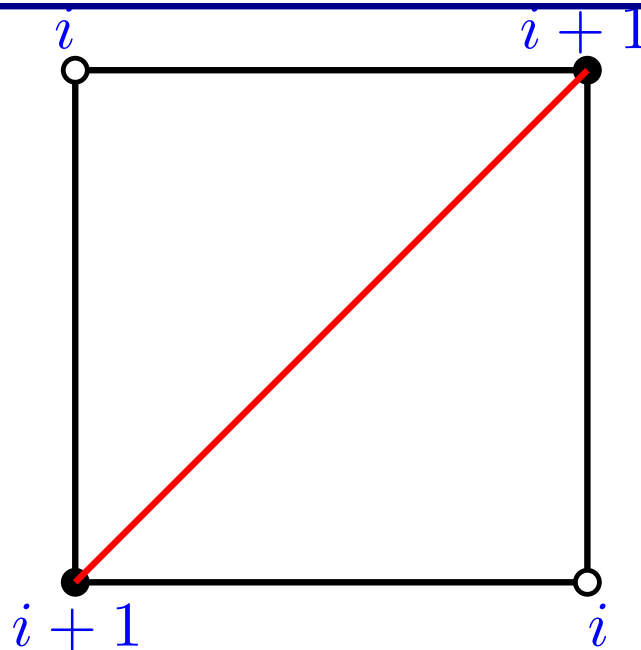
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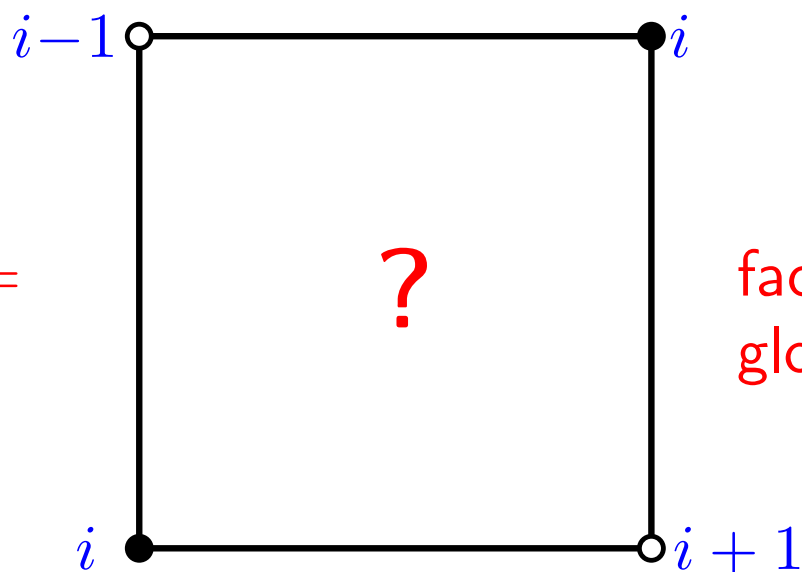
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Map is non-orientable =



faces **CANNOT** be globally oriented!

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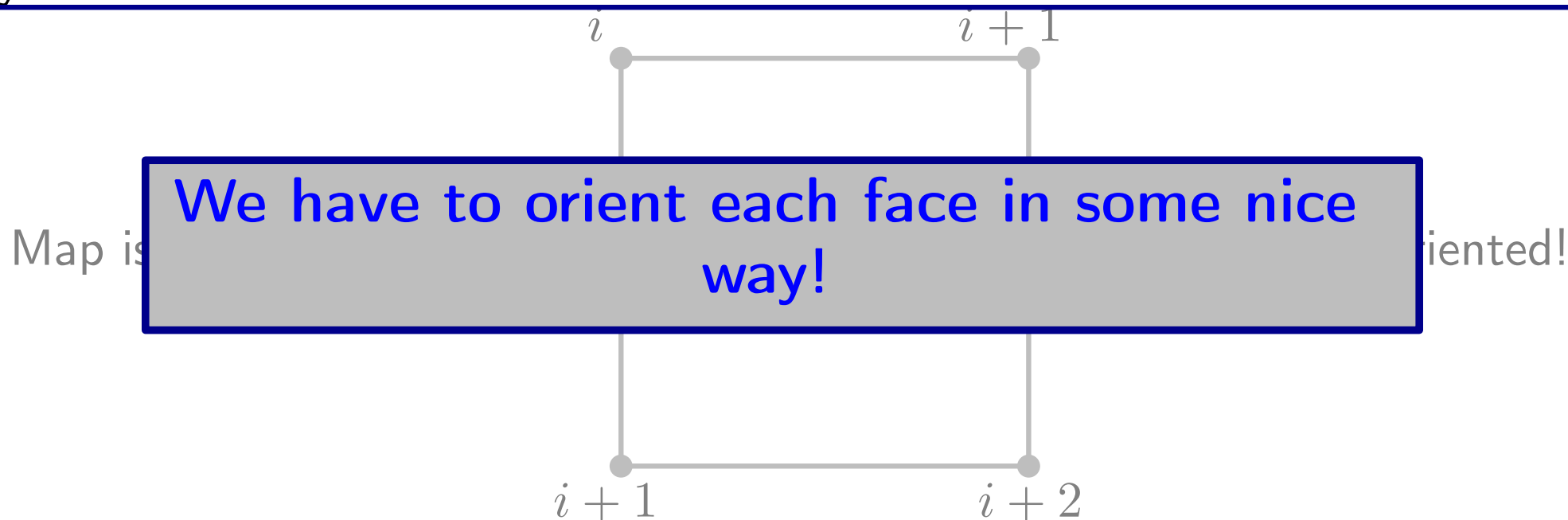
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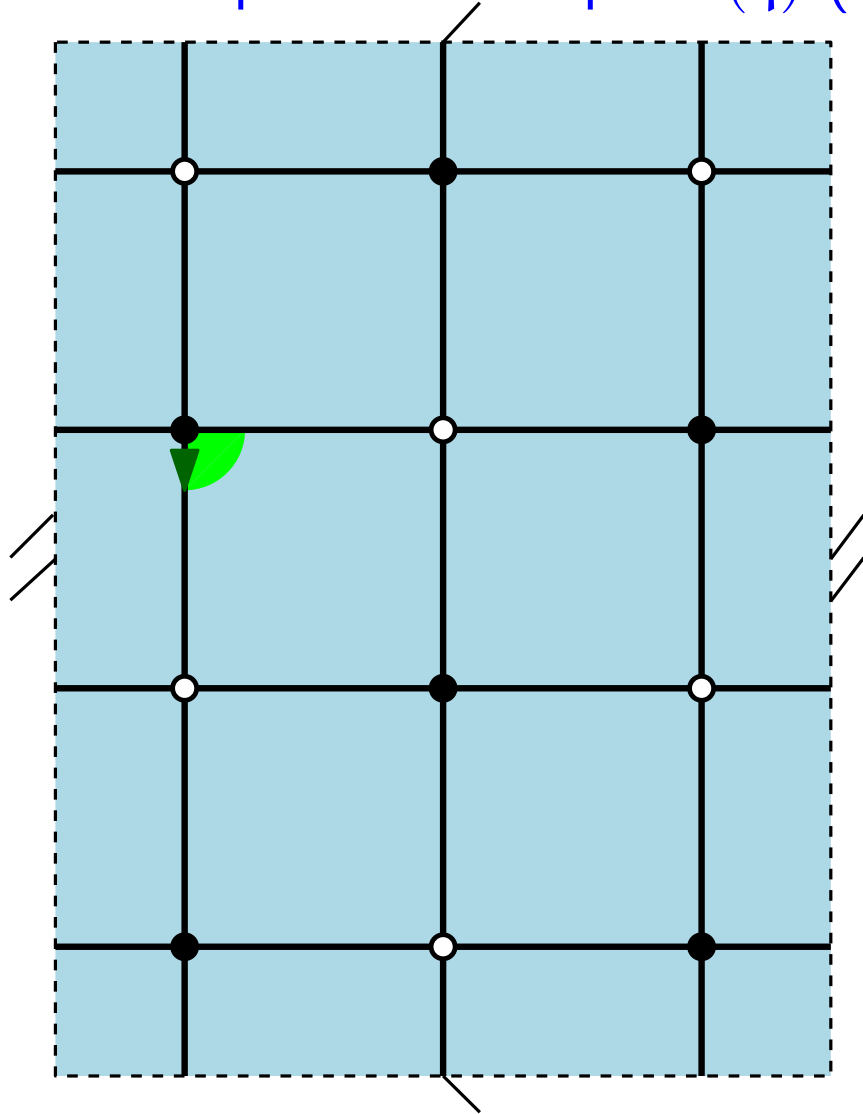
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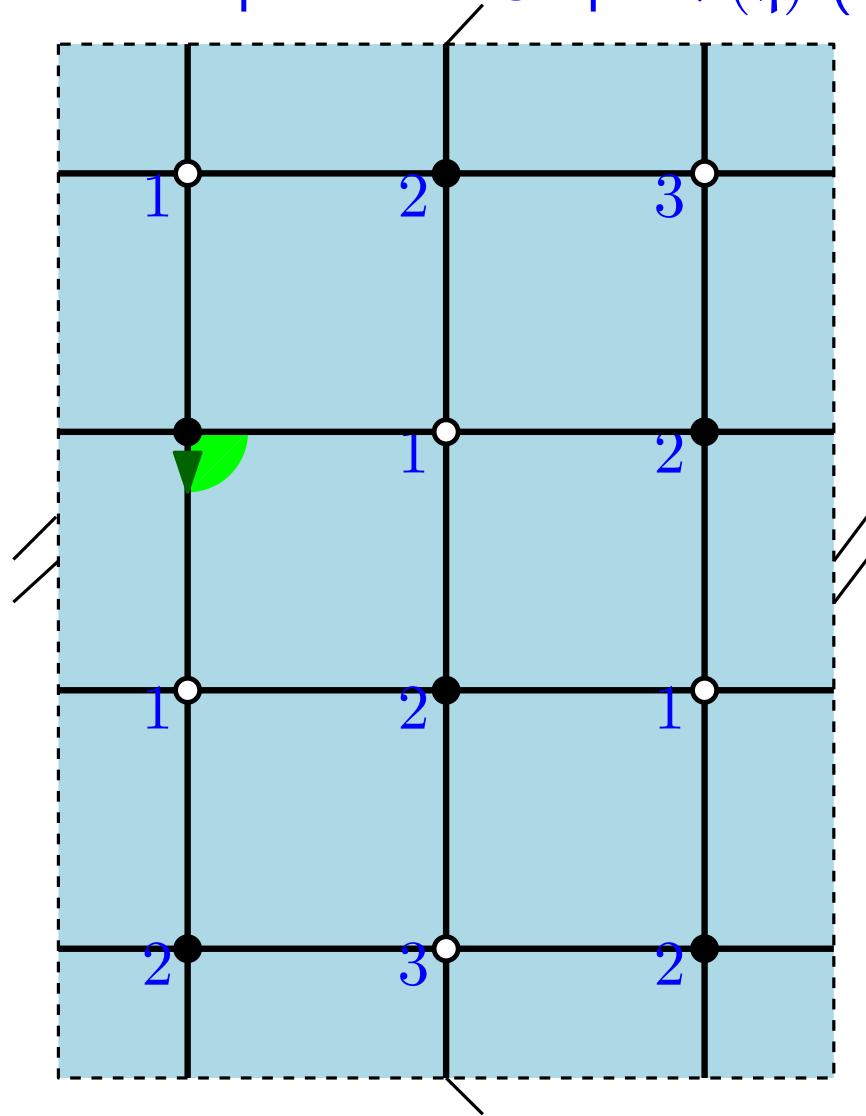
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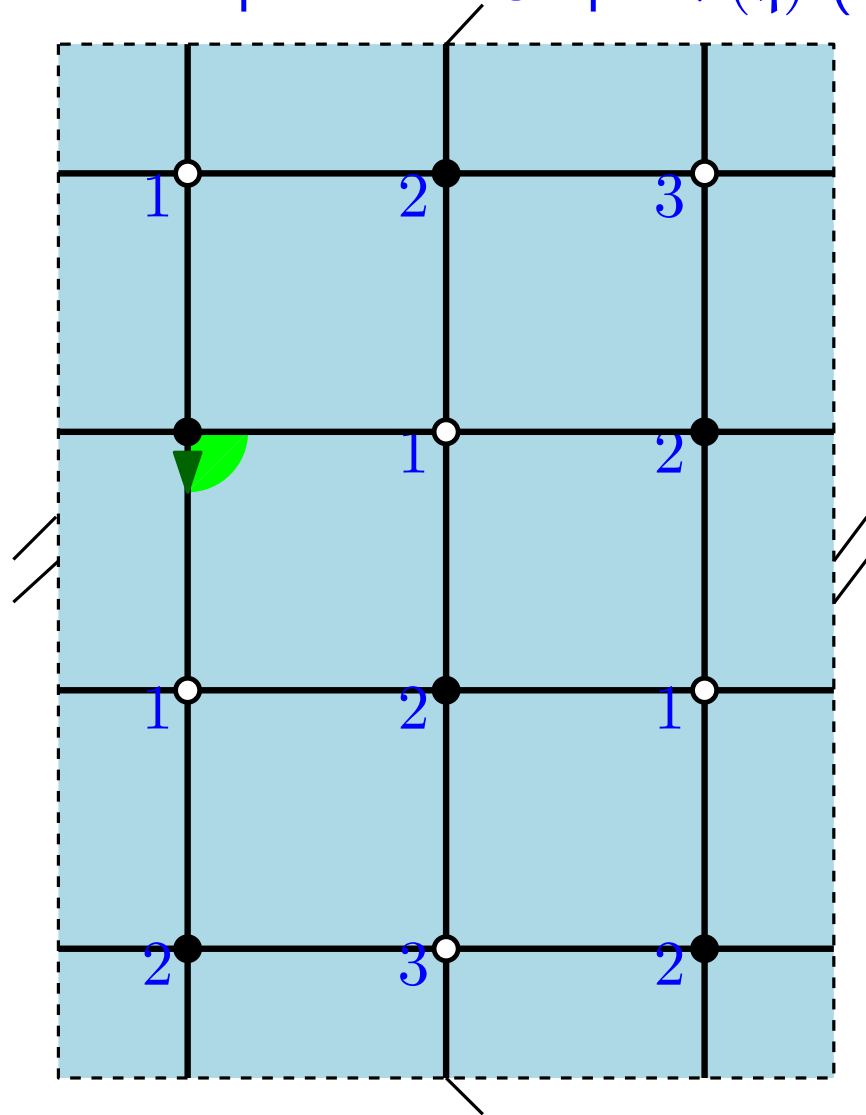
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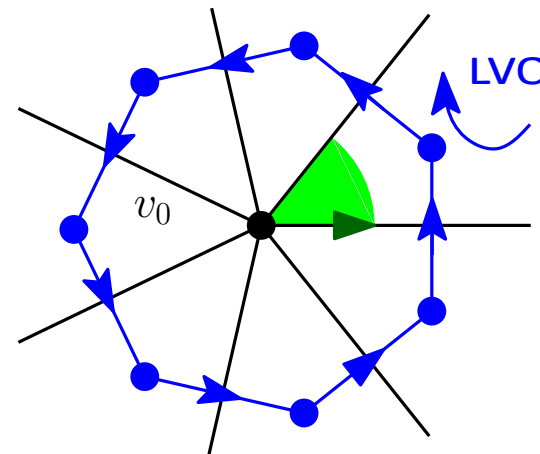


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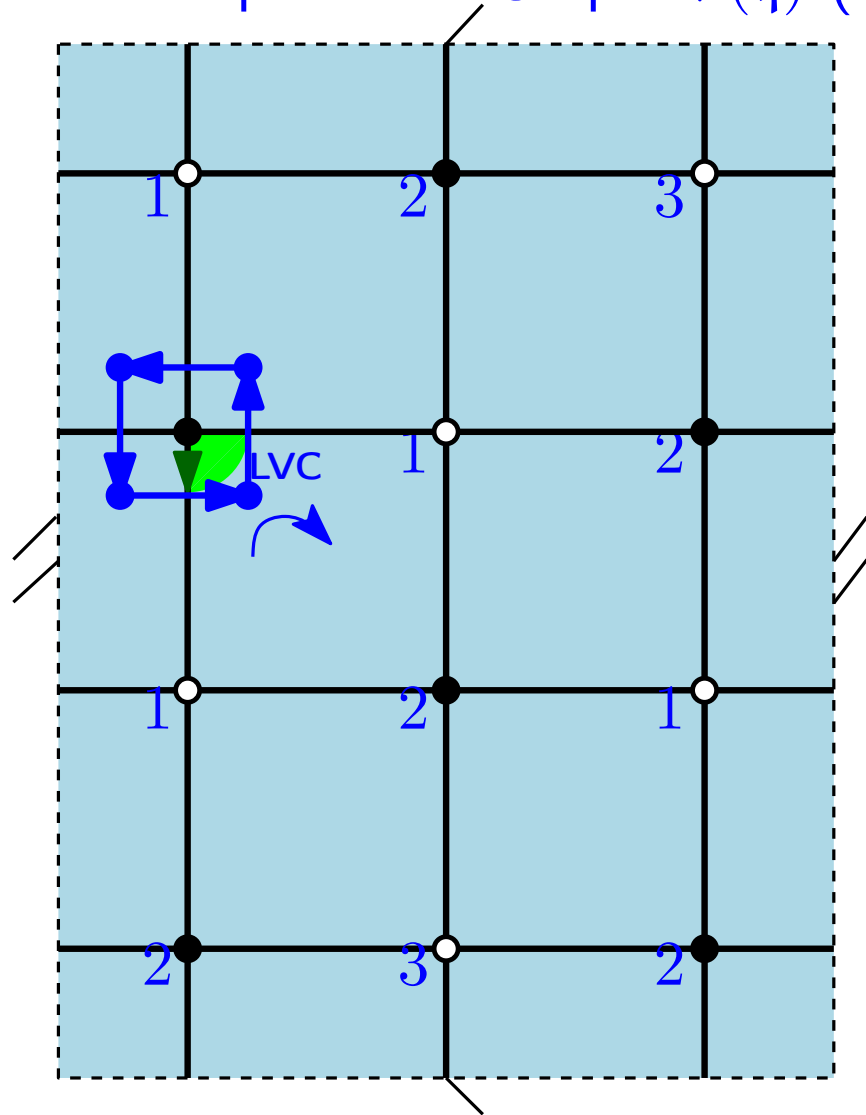


Step 0: Initialization

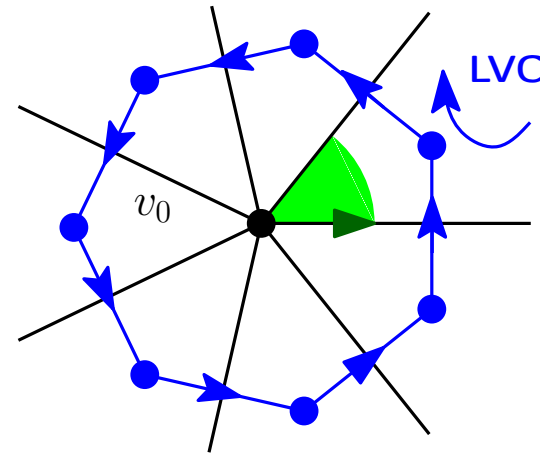


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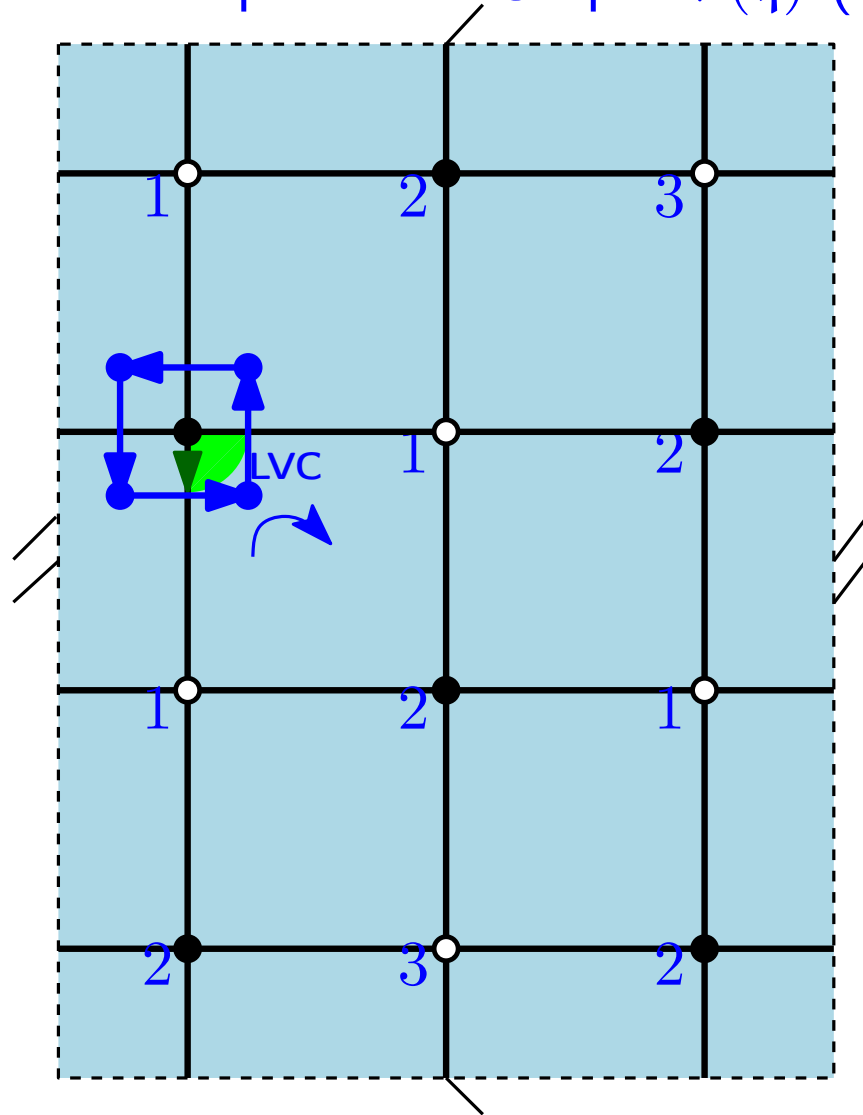


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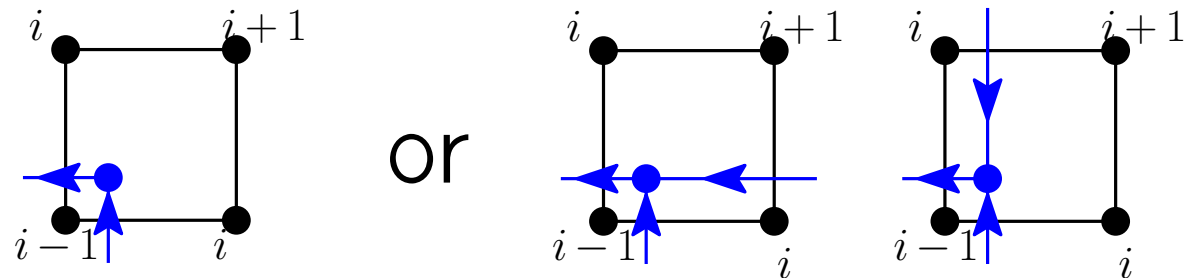
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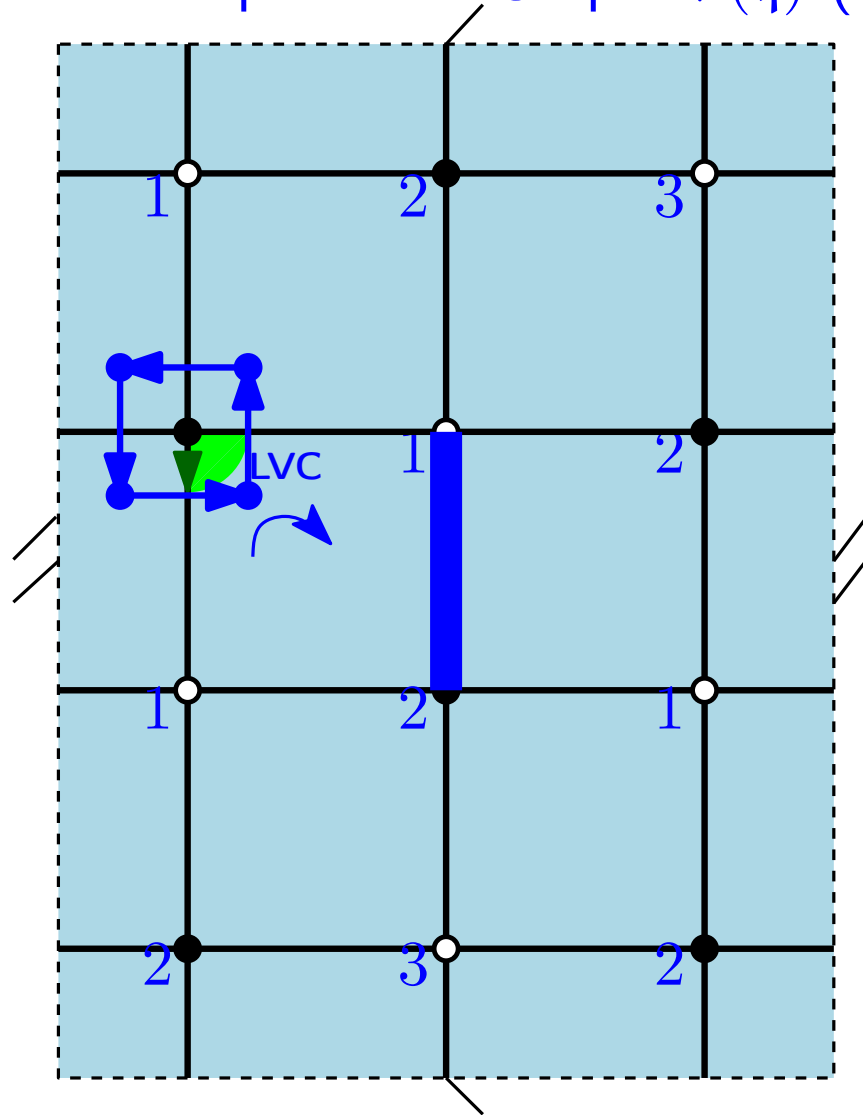
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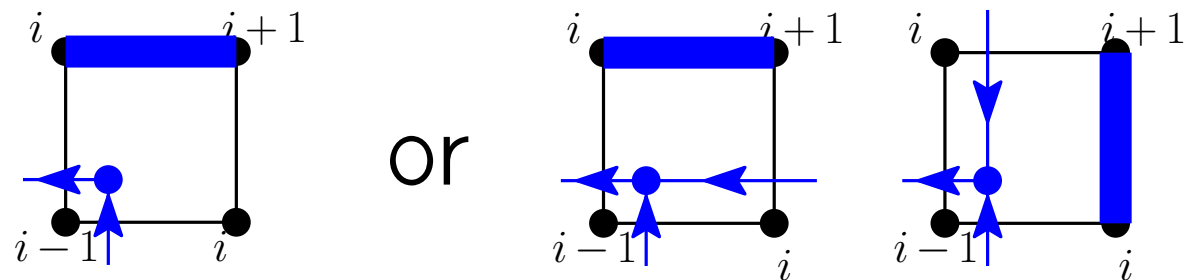
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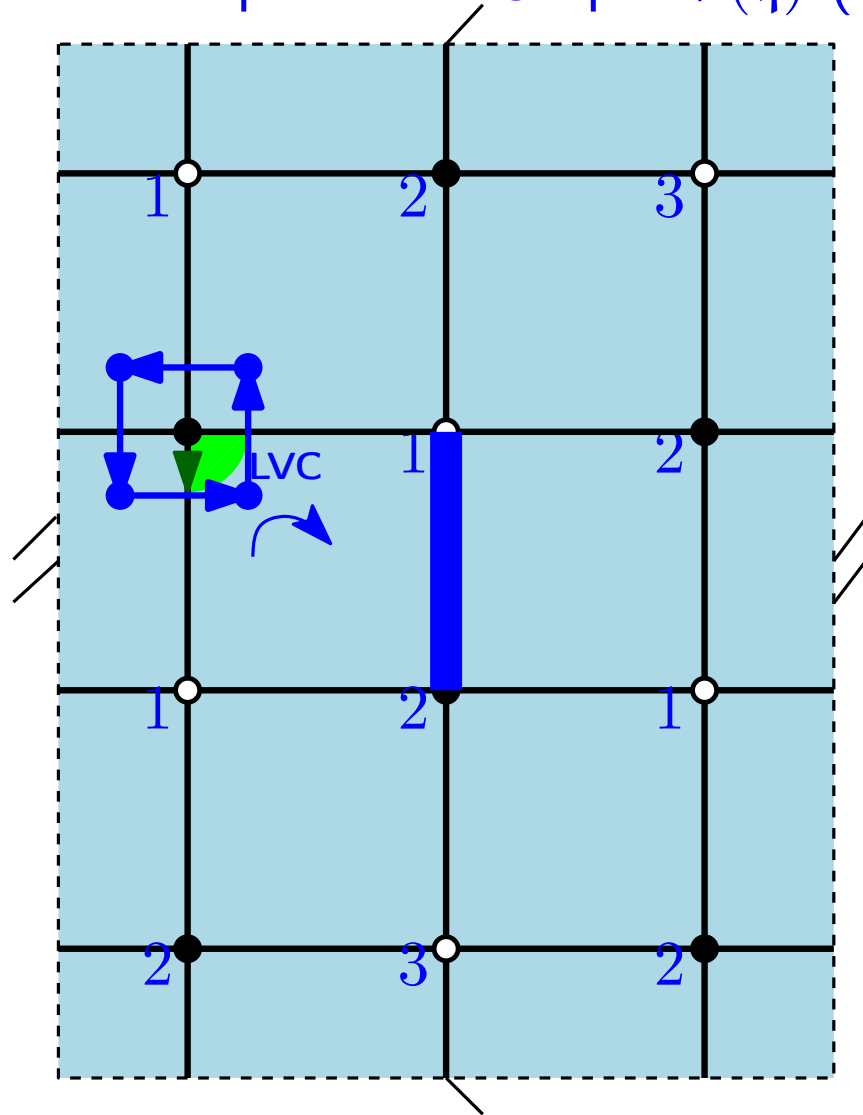
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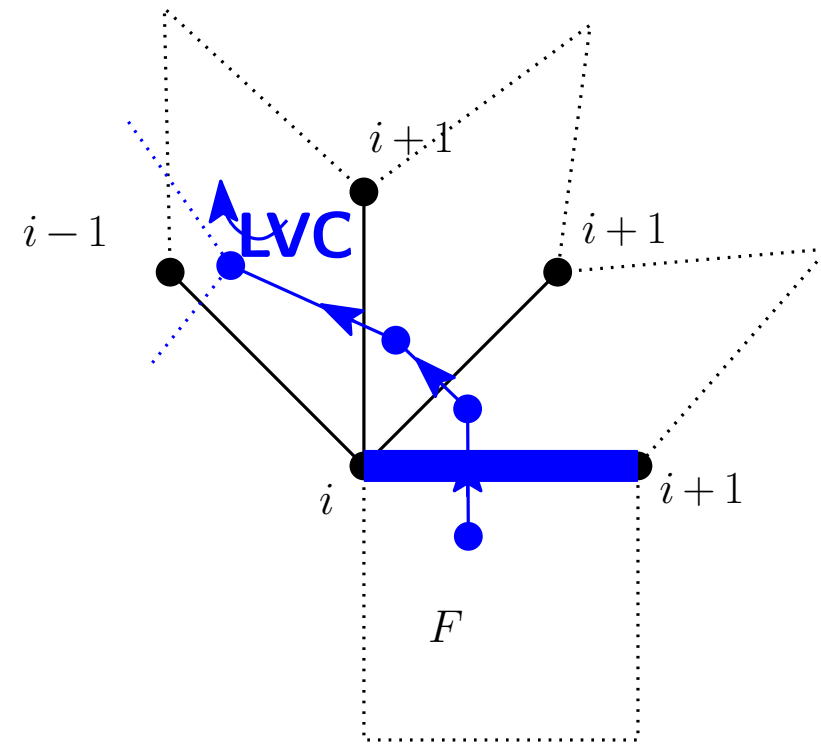


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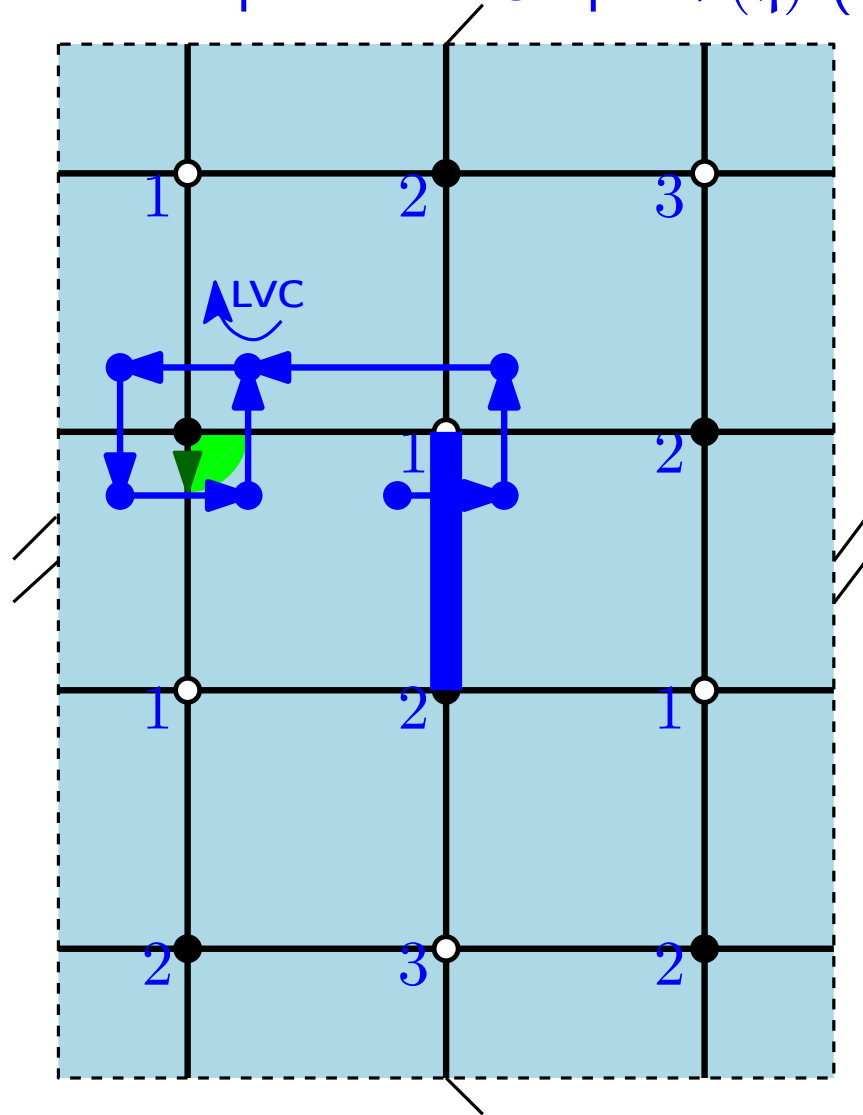


Step 2: Attaching a new branch of blue edges labeled by i starting across e

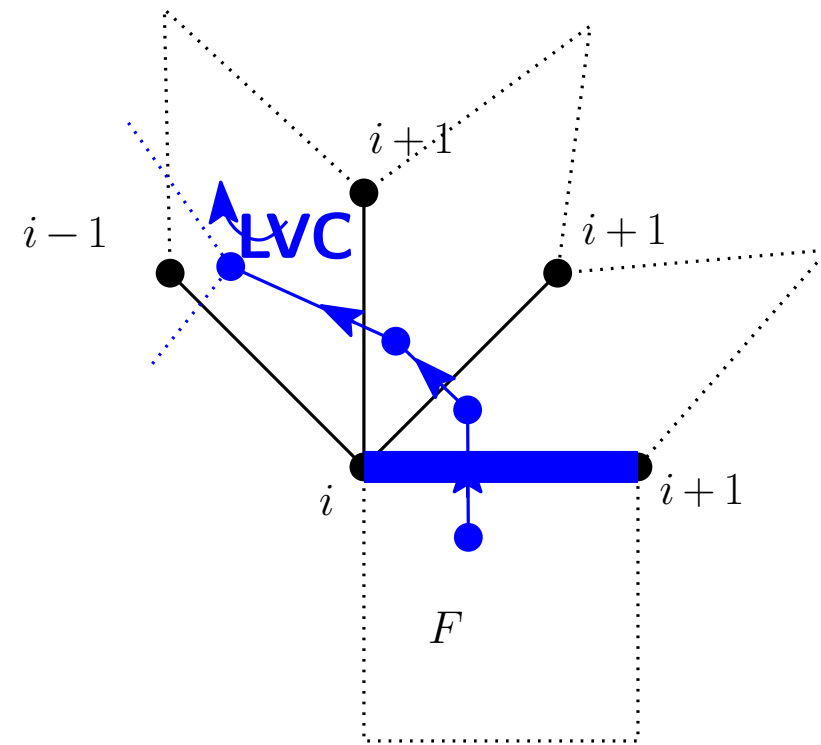


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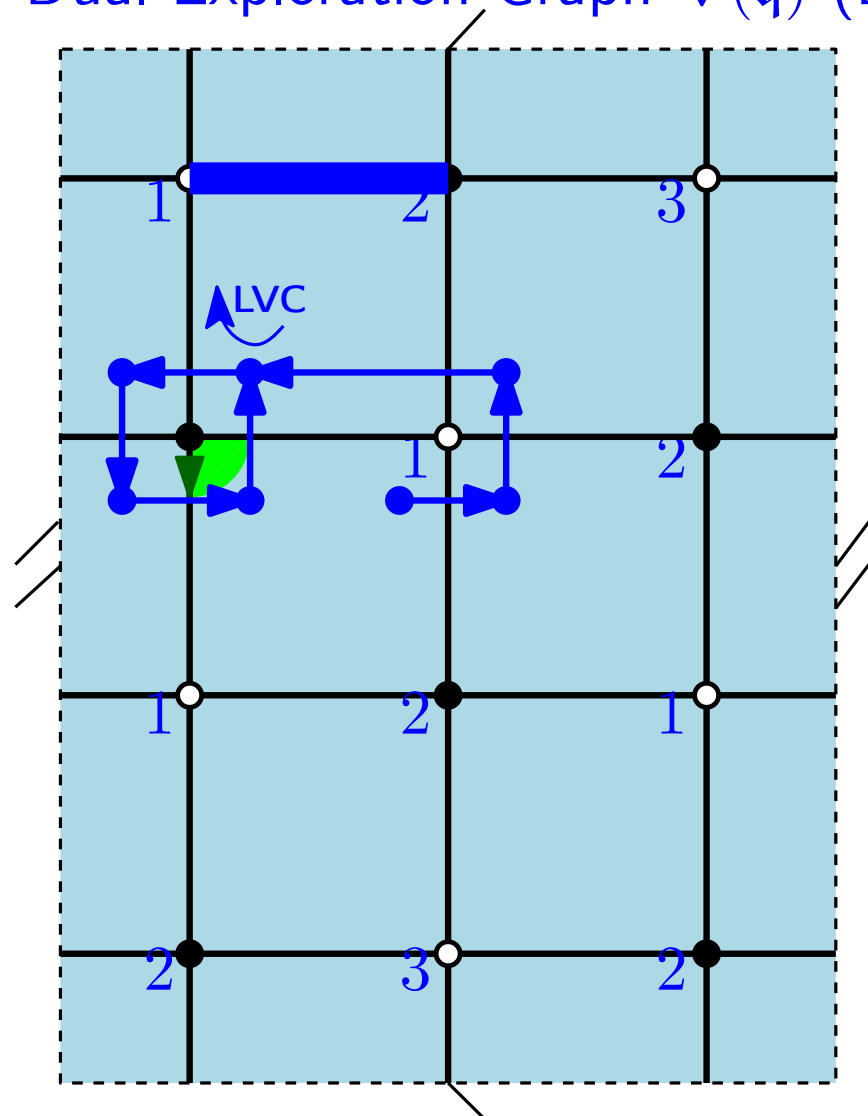


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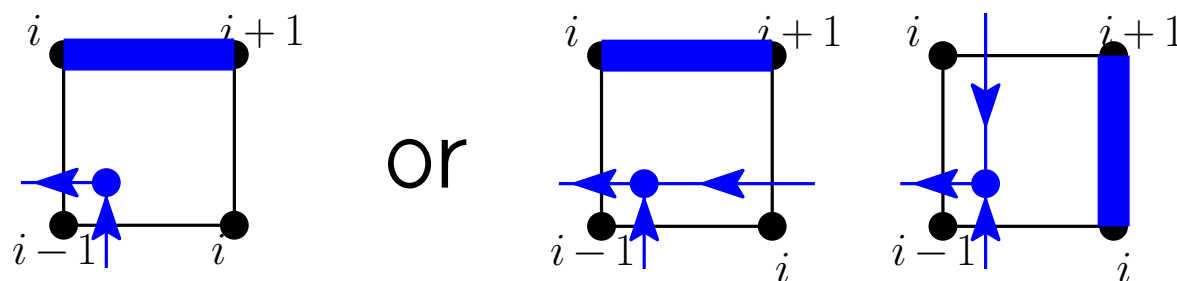
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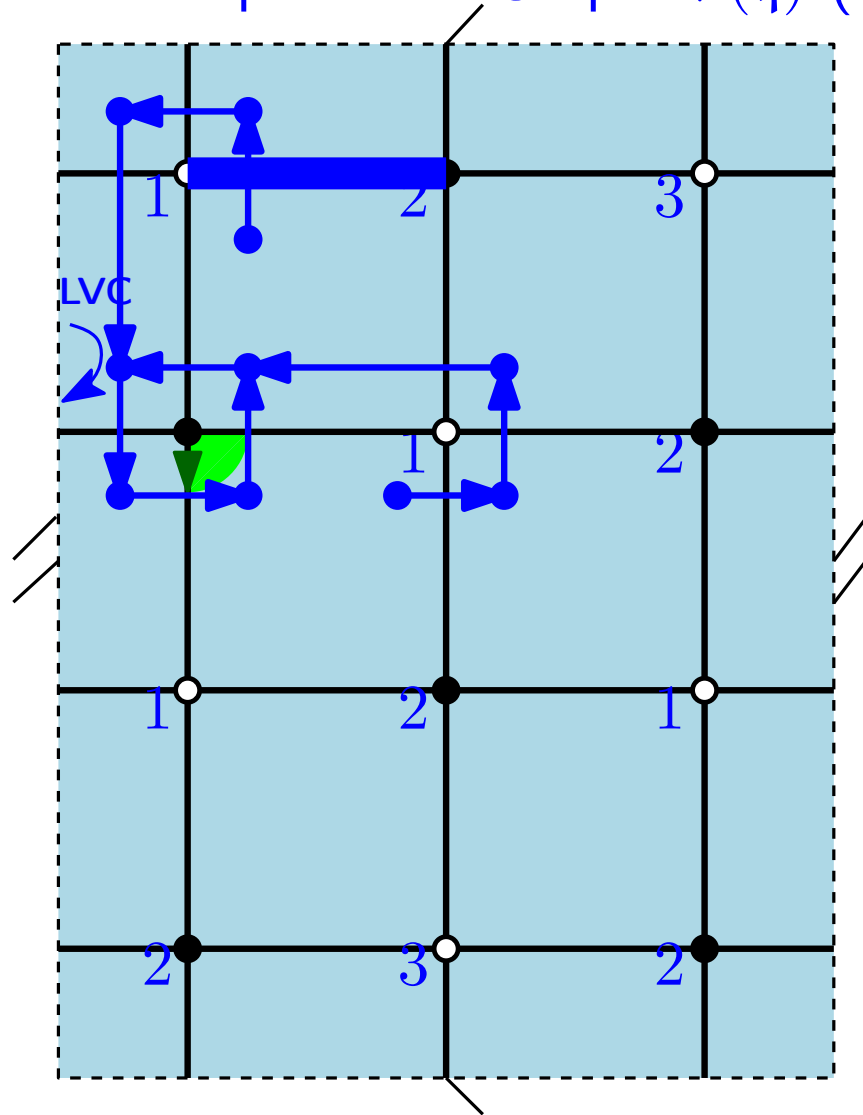
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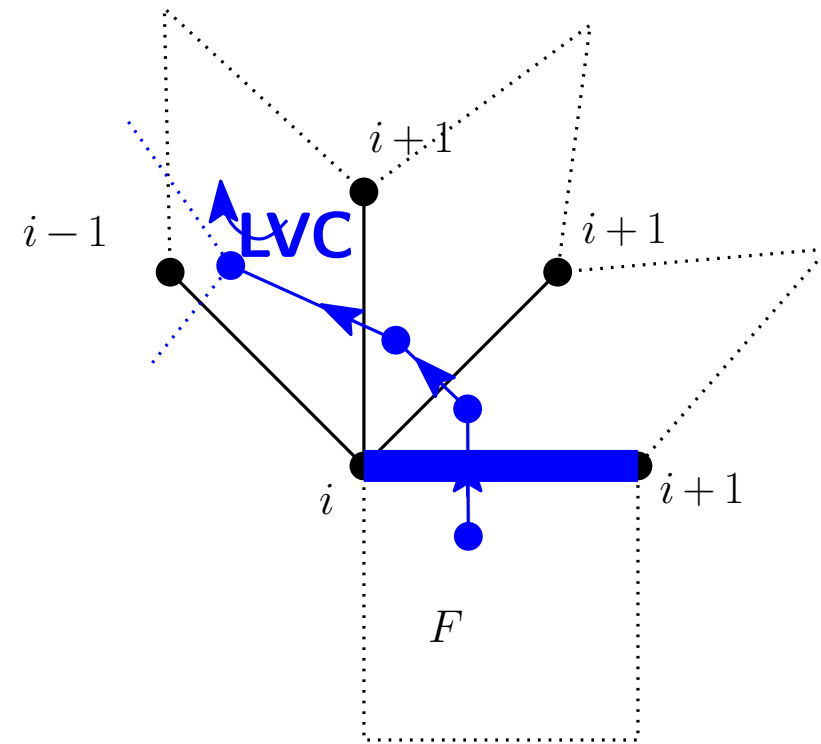


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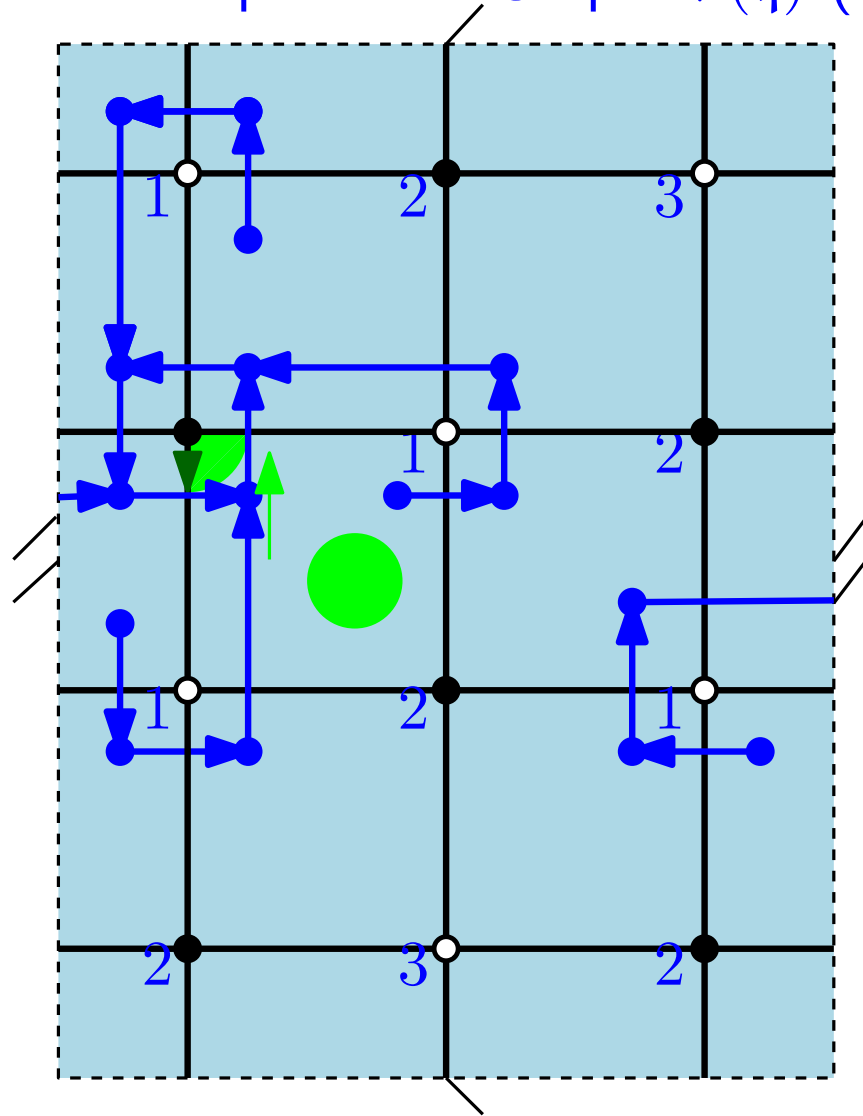


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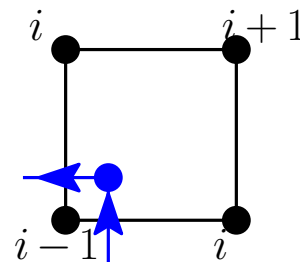
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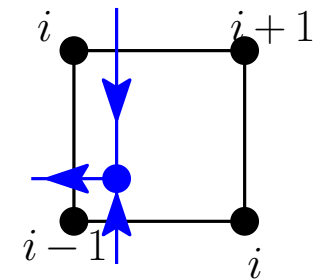
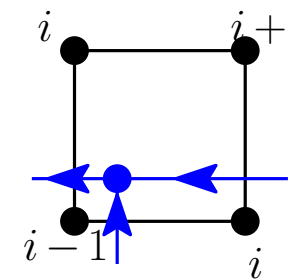


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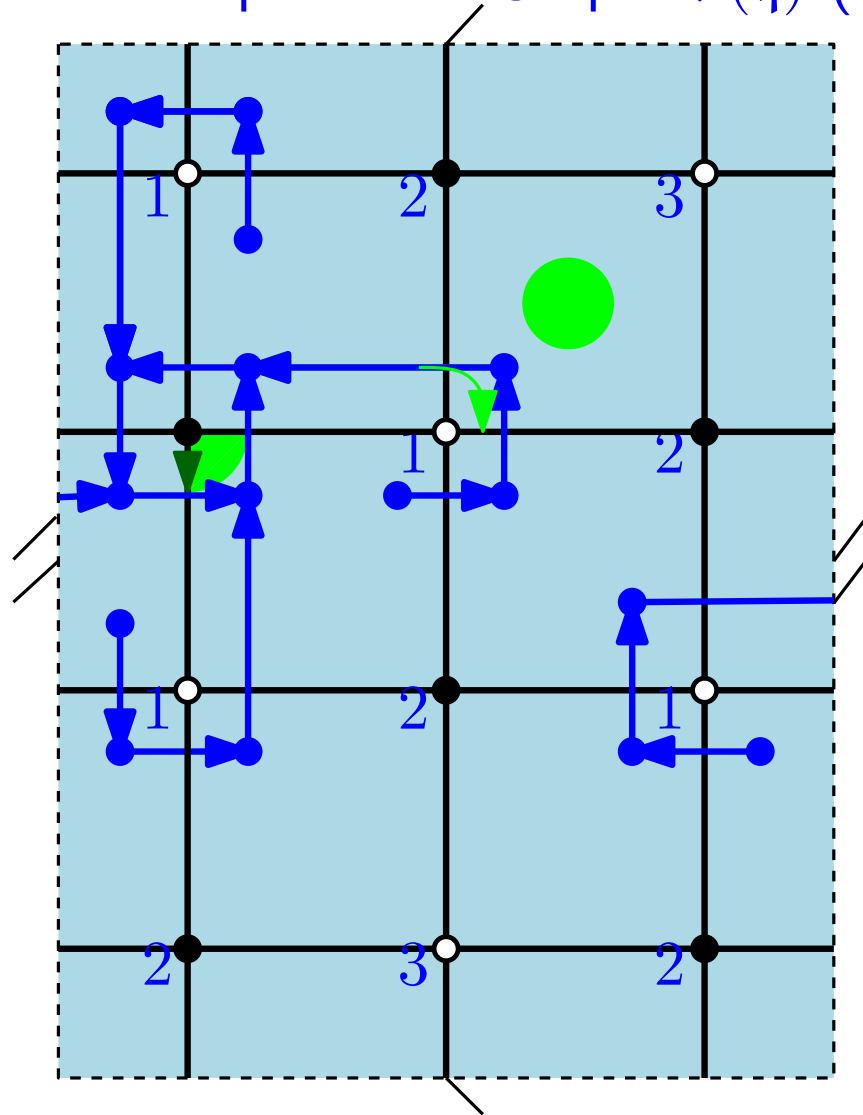


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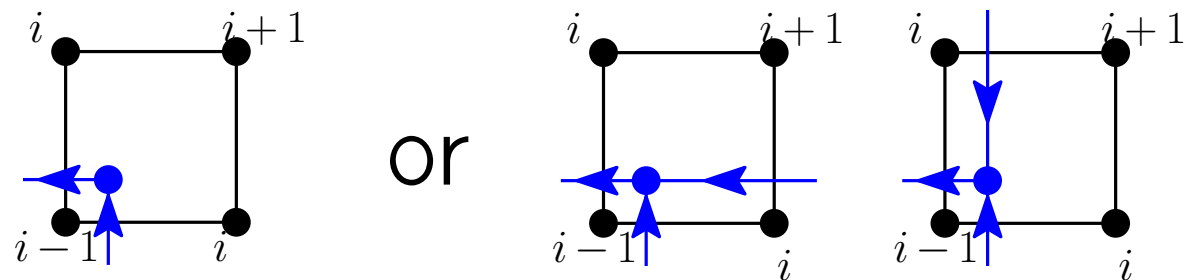
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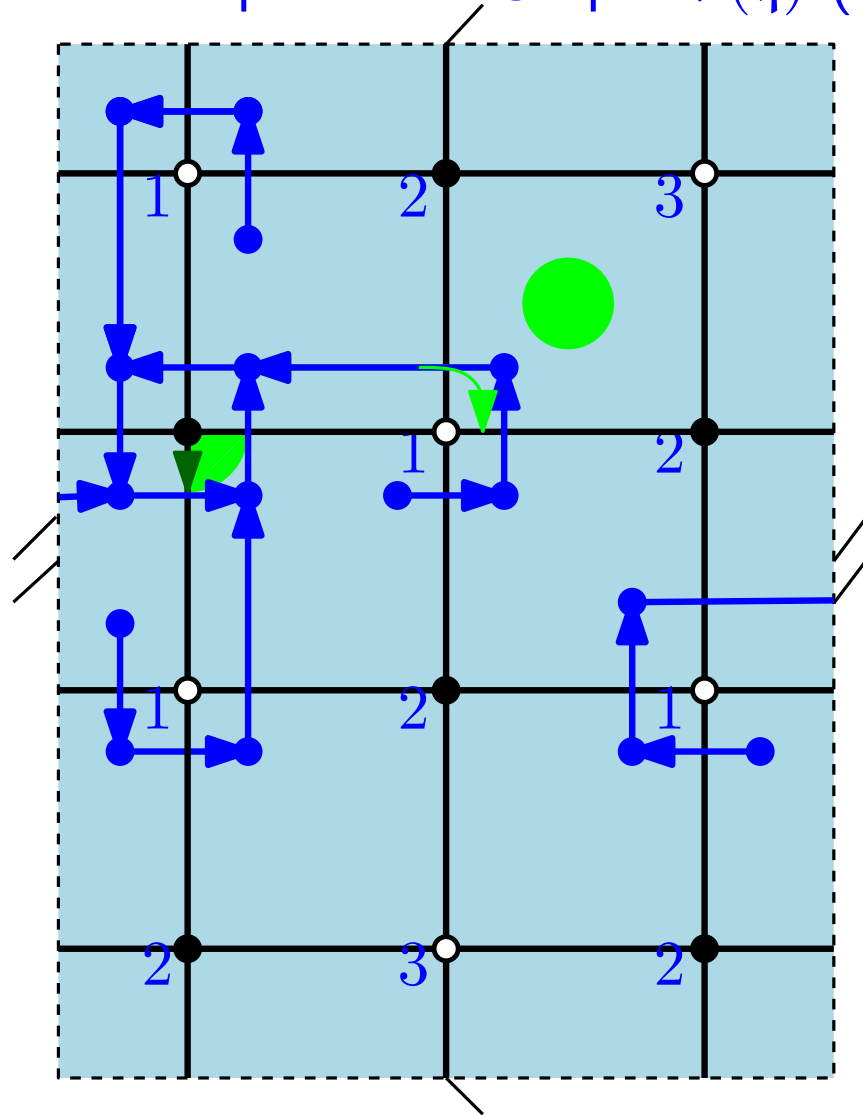
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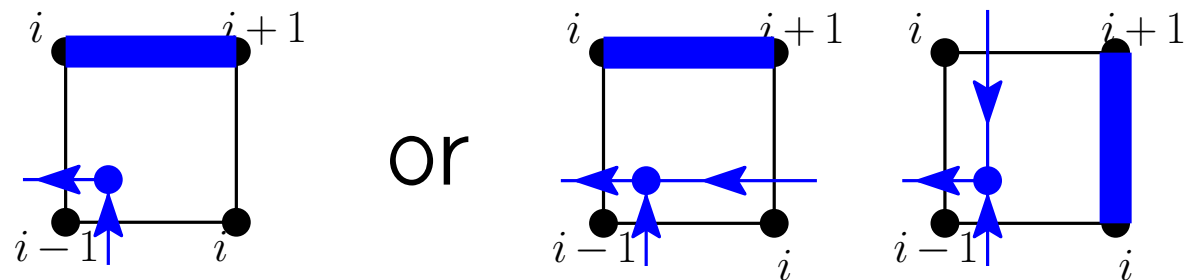
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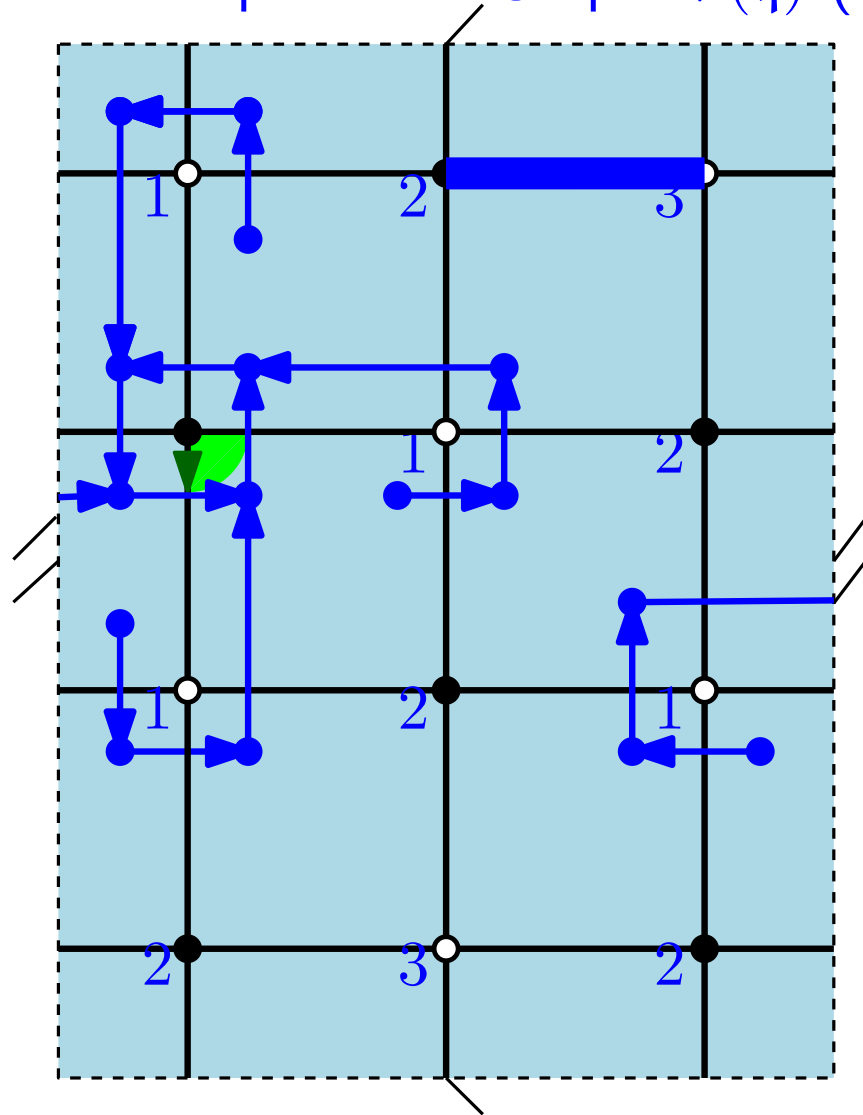
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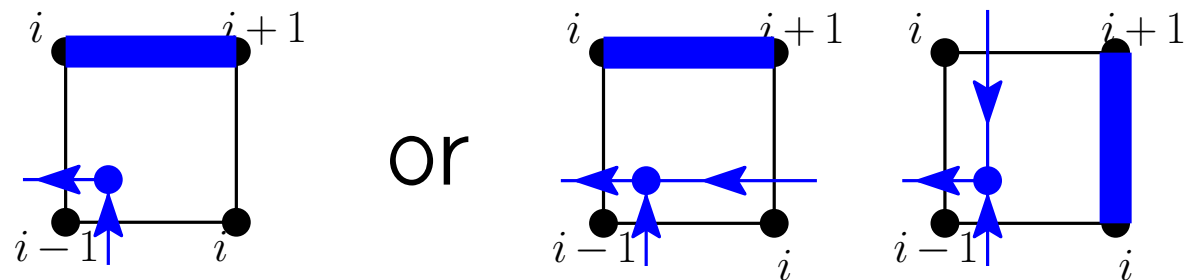
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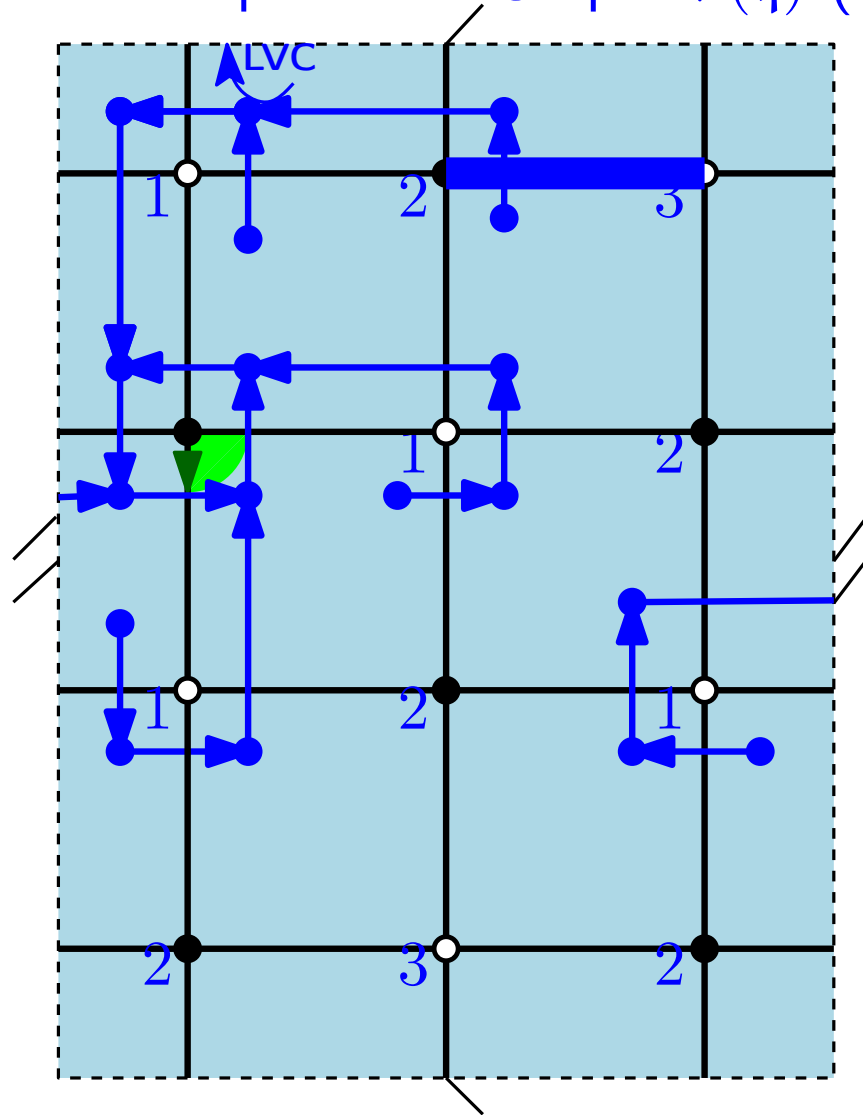
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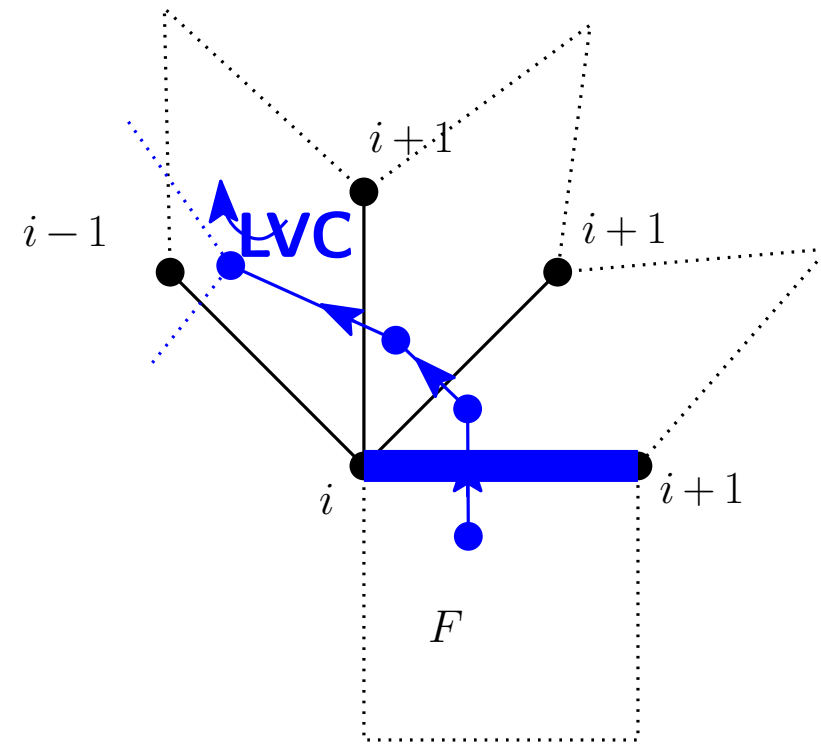


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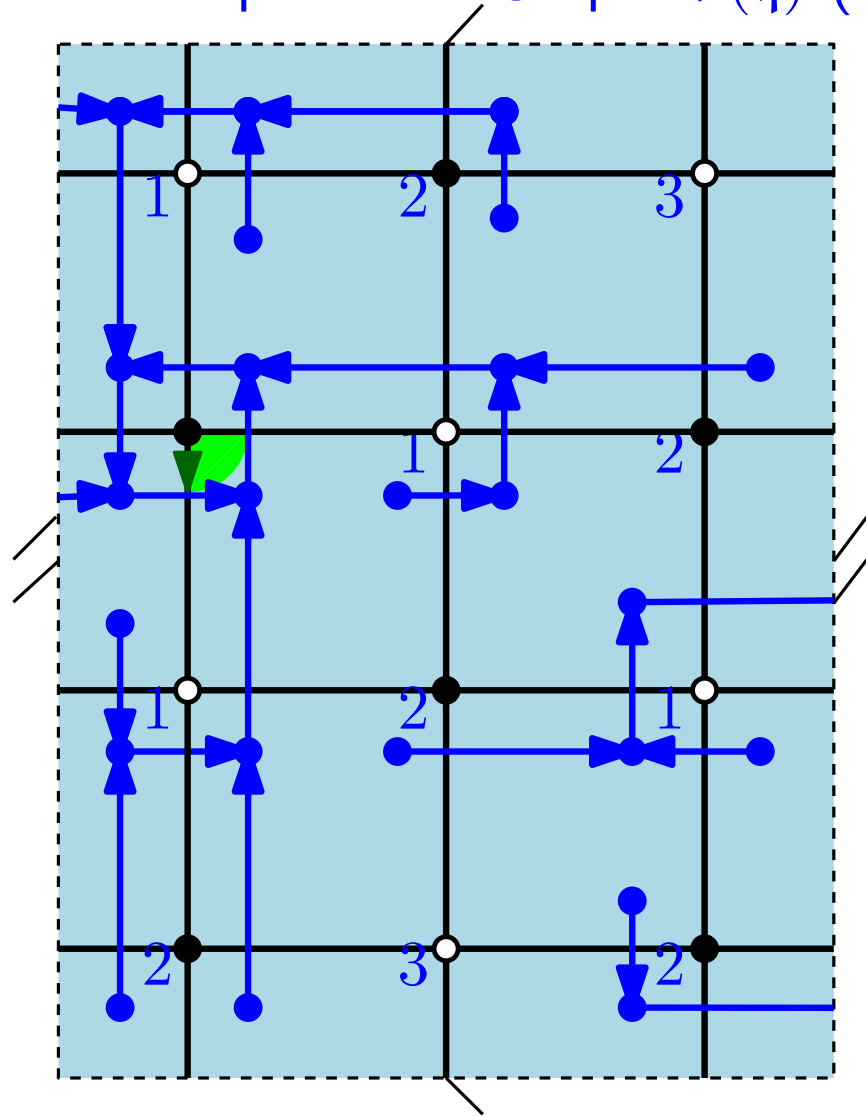


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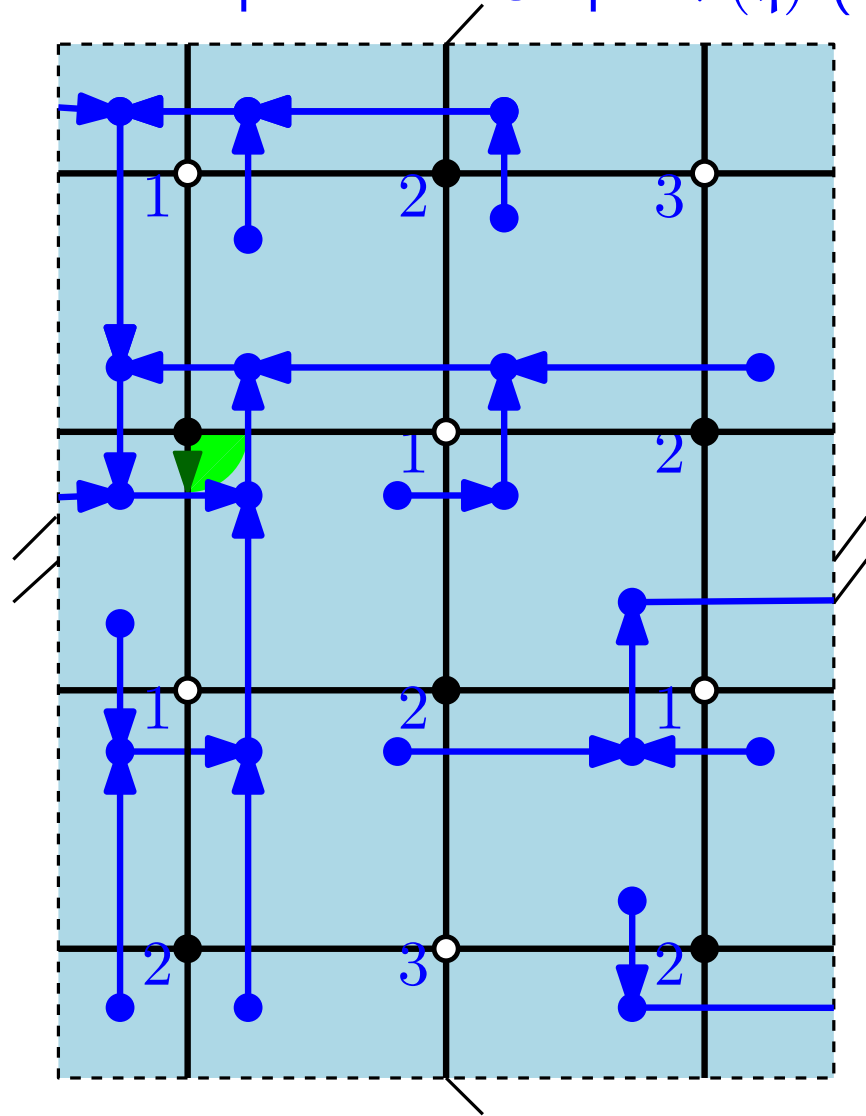
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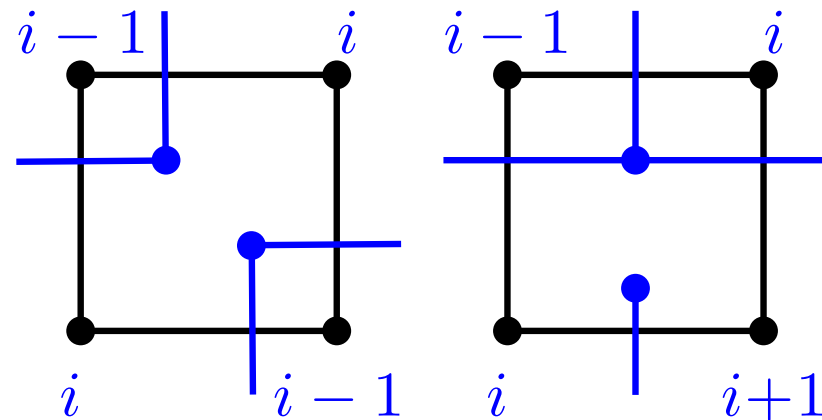
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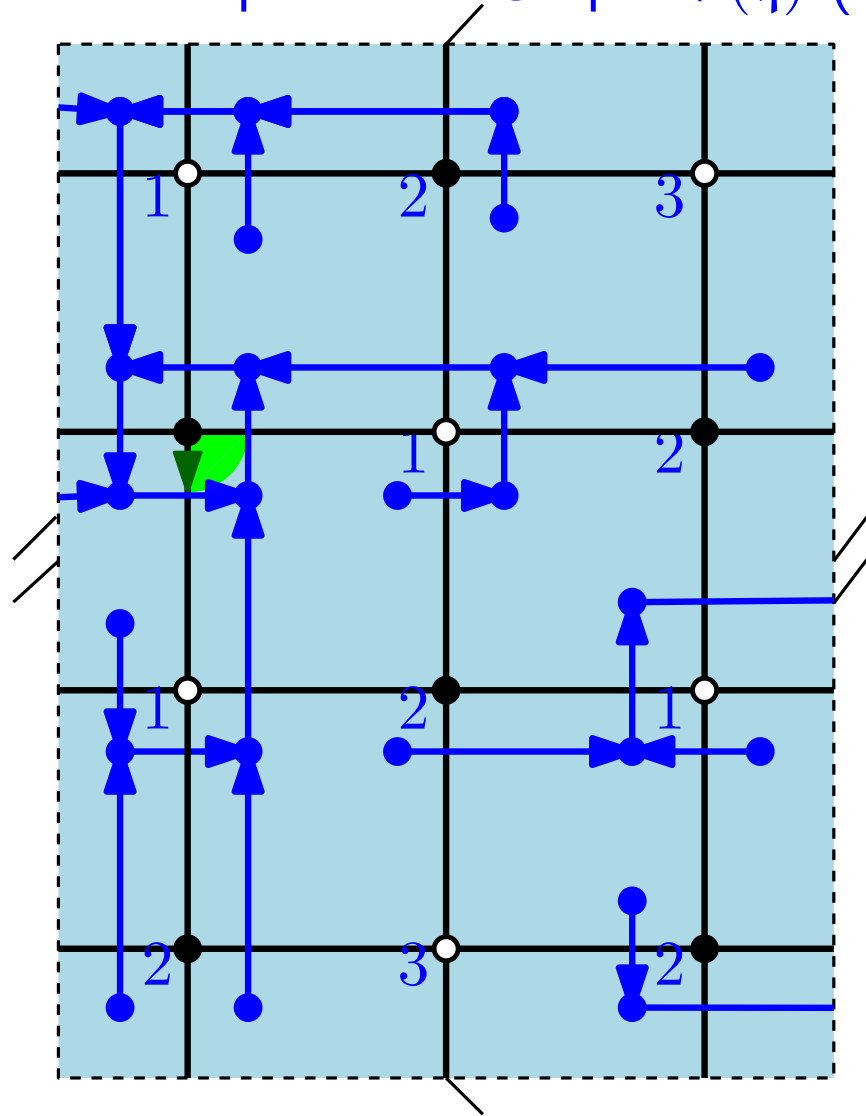
Proposition:

DEG $\nabla(q)$ is formed by a unique oriented cycle encircling root vertex v_0 , to which oriented trees are attached. After the construction of $\nabla(q)$ is complete, each face of q is of one of the two types:



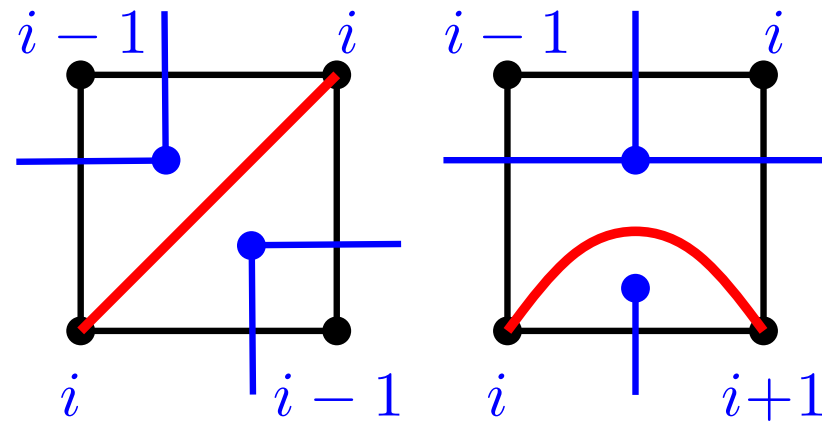
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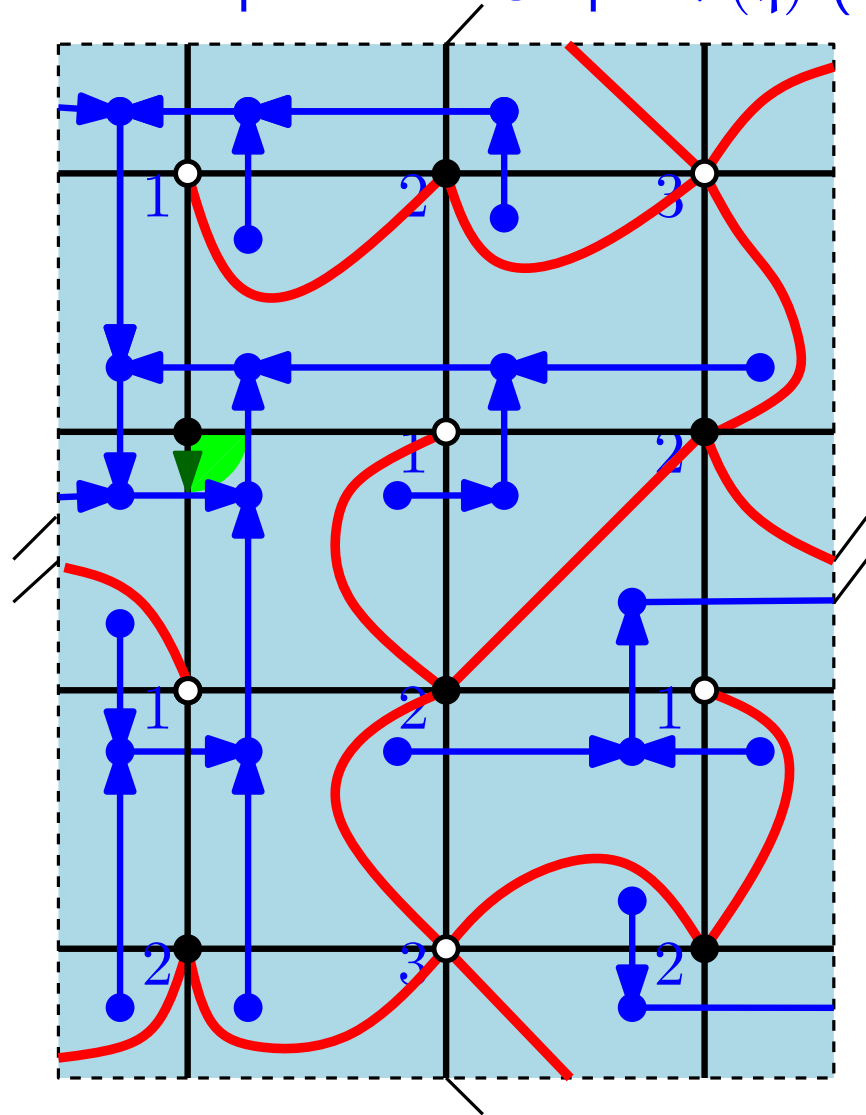
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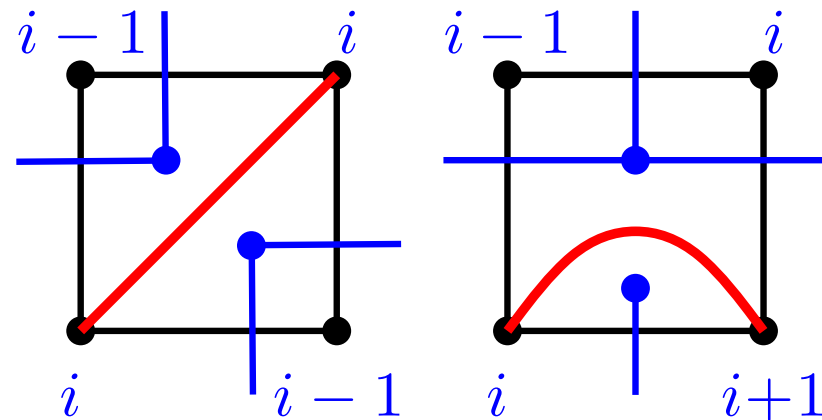
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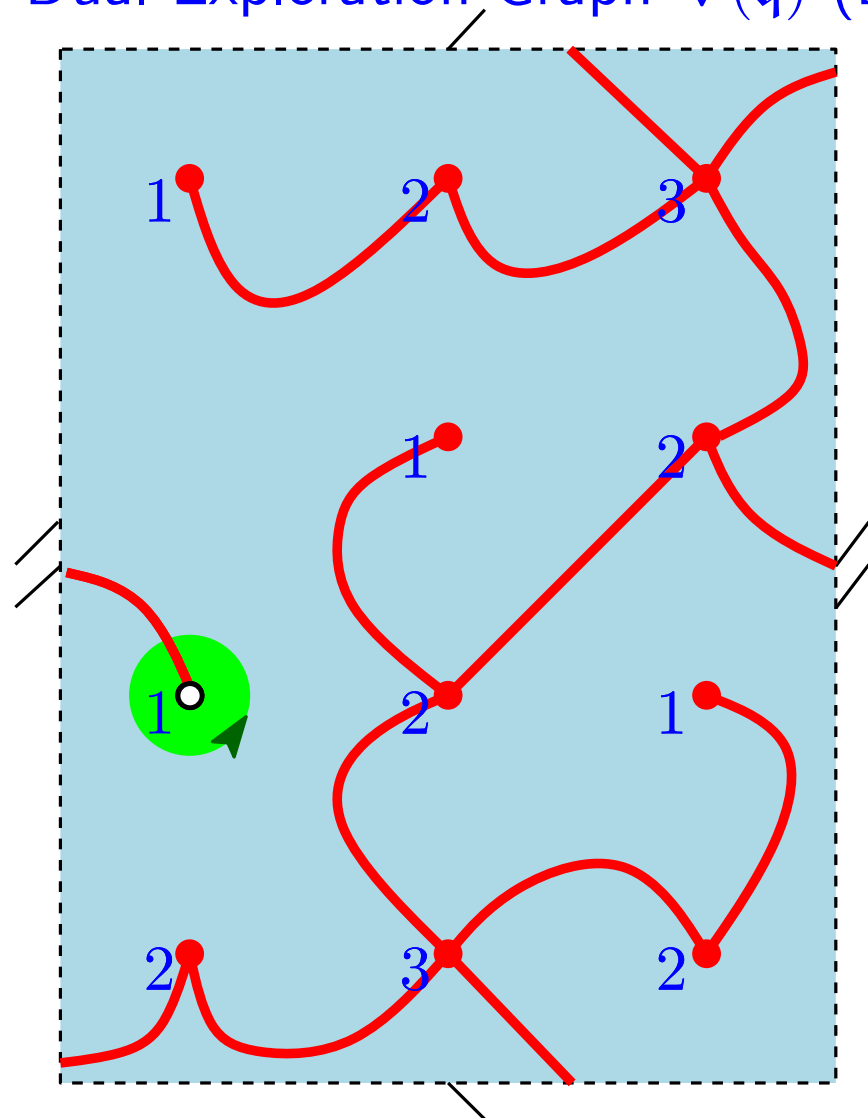
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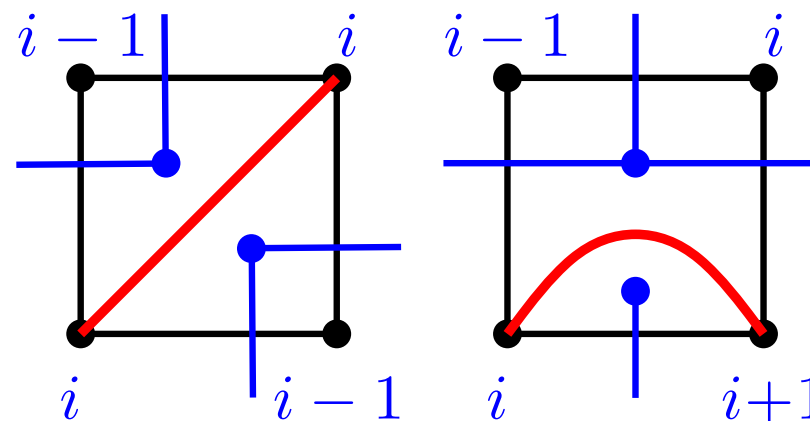
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Corollary:

Red map $\phi(q)$ is a one-face well-labeled rooted map with n edges, where n is the number of faces of q .

General case (III)

{rooted, **bipartite quadrangulations** on \mathbb{S} with n faces and N_i vertices
at distance i from the root vertex ($i \geq 1$)}

\Leftrightarrow

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Double rooting trick and Hall's marriage theorem.

III. Applications

Enumeration

Theorem [Bender, Canfield 1986]

Let

$$Q_S(t) := \sum_{n \geq 0} \vec{q}_{S, \bullet} t^n = \sum_{n \geq 0} (n + 2 - 2g) \vec{q}_S(n) t^n$$

be the generating function of rooted maps of type g pointed at a vertex or a face, by the number of edges. Moreover let $U \equiv U(t)$ and $T \equiv T(t)$ be the two formal power series defined by: $T = 1 + 3tT^2$, $U = tT^2(1 + U + U^2)$. Then $Q_S(t)$ is a rational function in U .

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Corollary [Bender, Canfield 1986]

For each $g \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$, there exists a constant p_g such that the number of rooted maps with n edges on the non-orientable surface of type g satisfies:

$$m_g(n) \sim p_g 12^n n^{\frac{5(g-1)}{2}}.$$

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Remark

Our main theorem allows us to recover Bender and Canfield results. In particular we can give some explicit (but very complicated) formula for the constant p_g .

Random maps

Let (\mathcal{M}, v) be a map with distinguished vertex v . We define:

- **radius** of a map \mathcal{M} centered at v by the quantity

$$R(\mathcal{M}, v) = \max_{u \in V(\mathcal{M})} d_{\mathcal{M}}(v, u);$$

- **profile of distances** from the distinguished point v (for any $r > 0$) by:

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Theorem [Chapuy, D. 2014]

Let q_n be uniformly distributed over the set of rooted, bipartite quadrangulations with n faces on \mathbb{S} . Let v_0 be a root vertex of q_n and let v_* be a vertex of q_n chosen uniformly. Then, there exists a continuous, stochastic process $L^{\mathbb{S}} = (L_t^{\mathbb{S}}, 0 \leq t \leq 1)$ such that:

- $\frac{9}{8n}^{1/4} R(q_n, v_*) \xrightarrow[n \rightarrow \infty]{(d)} \sup L^{\mathbb{S}} - \inf L^{\mathbb{S}};$
- $\frac{9}{8n}^{1/4} d_{q_n}(v_0, v_*) \xrightarrow[n \rightarrow \infty]{(d)} \sup L^{\mathbb{S}};$
- $\frac{I_{(q_n, v_*)}((8n/9)^{1/4})}{n+2-2g} \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{I}^{\mathbb{S}},$

where $\mathcal{I}^{\mathbb{S}}$ is defined as follows: for every non-negative, measurable

$$g : \mathbb{R}_+ \rightarrow \mathbb{R}_+,$$

$$\langle \mathcal{I}^{\mathbb{S}}, g \rangle = \int_0^1 dt g(L_t^{\mathbb{S}} - \inf L^{\mathbb{S}}).$$

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- after normalization by $\frac{9}{8n}^{1/4}$, label process of uniformly chosen pointed, rooted, planar quadrangulation with n faces converges to the so-called **head of the Brownian snake** $L^{\mathbb{S}} = (L_t^{\mathbb{S}}, 0 \leq t \leq 1)$ which is, conditionally on $c^{\mathbb{S}}$, continuous Gaussian process with covariance:

$$\text{Cov}(L_s^{\mathbb{S}}, L_t^{\mathbb{S}}) = \inf\{c_u^{\mathbb{S}} : \min(s, t) \leq u \leq \max(s, t)\}.$$

IV. Further directions

- Generalization of the [Bouttier-Di Francesco-Guitter](#) bijection for non-orientable maps (bijection between bipartite [\$2p\$ -angulations](#), or, more generally bipartite maps with n faces of prescribed degrees and some kind of [non-orientable mobiles?](#))

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- Studying random maps on [ANY](#) surface in Gromov-Hausdorff topology (using our bijection and existing methods we (Bettinelli, Chapuy, D.) can prove a convergence of bipartite quadrangulations up to extraction of [SUBSEQUENCE](#) - what about full convergence?).

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