Feuille d'exercices

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Exercise 1. *What is the local weak limit of a complete binary tree T_n of height n ?*

Exercise 2. *Let g be a finite graph. We define the $\mathcal{G} \to \mathbb{R}$ function*

$$f(G) = \frac{1}{|V|} \sum_S \mathbf{1}(G \cap S \simeq g),$$

where the sum is over all subsets $S \subset V$ of cardinal p. Find a function $\psi : \mathcal{G}_* \to \mathbb{R}$ such that $f(G) = \mathbb{E}_{U(G)} \psi(G, o)$.

Exercise 3. *Using the involution invariance lemma, show directly from their definitions that the unimodular Galton-Watson tree with degree distribution P and the skeleton tree are unimodular.*

Exercise 4. *Is the infinite 2-ary tree a unimodular graph ?*

Exercise 5. *Let $c > 1$, does there exists a sequence of trees T_n such that $U(T_n)$ converges to* UGW(Poi(c)) *?*

Exercise 6. *We define an* end *of a rooted infinite tree (T, o) as a semi-infinite self-avoiding path on T starting from o. Let $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ be supported on infinite rooted graph. Show that if $\mathbb{E}_\rho \deg_G(o) = 2$ then ρ-a.s. (G, o) is a tree and it has one or two ends.*

Exercise 7. *We consider the skeleton tree and we put iid uniform $[0,1]$ weights on its edges. What is its minimal spanning forest ?*

Exercise 8. *For $G = (V, E)$ a finite graph, $\mathcal{I}(G)$ is the set of independent sets of G (largest non-adjacent subset $S \subset V$). We are interested by the function*

$$I(G) = \max_{S \in \mathcal{I}(G)} H(S)$$

with $H(S) = \sum_{v \in S} w_v$ and $w_v > 0$. We define the payoff at $o \in V$ as

$$X(G, o) = \max_{S \in \mathcal{I}(G)} H(S) - \max_{S \in \mathcal{I}(G-o)} H(S)$$

We assume that $\sum_{v \in A} w_v \neq \sum_{v \in B} w_v$ for any $A \neq B \subset V$.

1. If $H(S^*) = I(G)$ show that $o \in S^*$ if and only if $X(G, o) > 0$.

2. If (T, o) is a rooted tree, show that

$$X(T, o) = \left(w_o - \sum_{v \sim o} X(T_v, v) \right)_+$$

where T_v is ...

3. Let $P \in \mathcal{P}(\mathbb{N})$. We define the operator $\Theta : \mathcal{P}(\mathbb{R}_+) \mapsto \mathcal{P}(\mathbb{R}_+)$ which maps μ to the law of

$$\left(w_o - \sum_{i=1}^{N} X_i \right)_+ ,$$

where w_o is an $\exp(1)$ variable, N has law P, X_i has law μ, all variables being independent. For $P = \delta_r$ and $P = \mathrm{Poi}(c)$, for which values of r and c, Θ and Θ^2 have a unique fixed point ?

Exercise 9. *1. Find a greedy algorithm which computes $\nu(T)$, the size of a maximal matching, for any tree T.*

2. (Difficult !) If T_n is a uniformly sampled random labeled tree with n vertices, prove that $\mathbb{E}\nu(T_n)/n = 1 - x^* + o(1) \simeq 0.43$ where x^* is the real root of $x = e^{-x}$.