The densest subgraph of sparse random graphs

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The objective method (Aldous-Steele, 2004)

Context:
given a large interacting system (graph), one is interested in a macroscopic quantity which depends on the microscopic contribution of each particle (vertices).

Key assumption:
no long-range interactions, i.e. the microscopic contribution of each particle is essentially insensitive to remote perturbations of the system.

Expected consequences:
1. efficient approximability by local distributed algorithms;
2. existence of an infinite-volume limit.

Idea:
formalize that via local weak convergence, and use this framework to replace the asymptotic study of large graphs by the direct analysis of their infinite-volume limits.
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Local convergence around a fixed root

Let $G$, $o$, and $R$ be a countable, locally finite, connected rooted graph, a root, and a ball of radius $R$ around $o$ in $G$, respectively. We denote by $G_n$, $o_n$, and $R_n$ their respective finite subgraphs around $o$, and by $\equiv$ the equality of rooted graphs. We say that $\{G_n, o_n : n \in \mathbb{N}\} \to G, o$ locally converges if for each fixed $R$, there is $n_R \in \mathbb{N}$ such that $n \geq n_R \Rightarrow [G_n, o_n] \equiv [G, o]_{R_R}$. This means that the finite subgraphs of $G_n$ around $o_n$ converge to the graph $G$ with root $o$ in the neighborhood of radius $R$. The condition ensures that the local structure around the root is preserved under this convergence.
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\[ n \geq n_R \implies [G_n, o_n]_R \equiv [G, o]_R \]
Local weak convergence (Benjamini-Schramm, 2001)

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Consider the law on \( G_\star \) obtained by choosing a root \( o \in V \) uniformly at random, and restricting to its connected component:
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\mathcal{L}_G := \frac{1}{|V|} \sum_{o \in V} \delta_{[G,o]}.
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\( \{ G_n \}_{n \geq 1} : \text{sequence of finite graphs.} \)
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\( \triangleright \mathcal{L} \) describes the local geometry of \( G_n \) around a random node
Examples of local weak limits

Note: graphs must be sparse, i.e. $|E| \sim |V|$

$G_n$ = box of size $n \times \ldots \times n$ in $\mathbb{Z}^d$

$L$ = dirac at $(\mathbb{Z}^d, 0)$

$G_n$ = random $d$-regular graph on $n$ nodes

$L$ = dirac at the $d$-regular infinite rooted tree

$G_n$ = Erdős-Rényi graph with $p_n = c$ on $n$ nodes

$L$ = law of a Galton-Watson tree with degree Poisson$(c)$

$G_n$ = random graph with degree distribution $\pi$ on $n$ nodes

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$G_n$ = uniform random tree on $n$ nodes

$L$ = Infinite Skeleton Tree (Grimmett, 1980)

$G_n$ = preferential attachment graph on $n$ nodes

$L$ = Polya-point graph (Berger-Borgs-Chayes-Sabery, 2009)
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$$\mu_G(\{0\}) = \frac{\dim \ker(A_G)}{|V|}.$$ Asymptotics when $G$ is large?

Conjecture (Bauer-Golinelli 2001). For $G_n$: Erdős-Rényi $(n, cn)$,

$$\mu_{G_n}(\{0\}) \xrightarrow{n \to \infty} \lambda^* + e^{-c\lambda^*} + c\lambda^* e^{-c\lambda^*} - 1,$$

where $\lambda^* \in [0, 1]$ is the smallest root of $\lambda = e^{-c\lambda} - c\lambda$.

Theorem (Bordenave-Lelarge-S., 2011)

1. $G_n \to L = \Rightarrow \mu_{G_n}(\{0\}) \to \mu_L(\{0\})$.
2. When $L = \text{Galton-Watson}(\pi)$, 

$$\mu_L(\{0\}) = \min_{\lambda} \left\{ \lambda^* + f'(1)\lambda^* + f(1 - \lambda^*) + f(1 - \lambda) - 1 \right\},$$

where $f(z) = \sum n \pi_n z^n$ and $\lambda^* = f'(1)/(f'(1) - f(1))$. 


An illustration: the nullity of large graphs

\[ \mu_G(\{0\}) = \frac{\dim \ker(A_G)}{|V|}. \]

Asymptotics when \( G \) is large?

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where \( f(z) = \sum_n \pi_n z^n \) and \( \lambda^* = \frac{f'(1 - \lambda)}{f'(1)} \).
Continuity with respect to local weak convergence

In the sparse regime, many important graph parameters $\Phi$ are essentially determined by the local geometry only.

This can be rigorously formalized by a continuity theorem:

$$G_n^{\text{loc}} \xrightarrow{n \to \infty} L \Rightarrow \Phi(G_n) \xrightarrow{n \to \infty} \Phi(L)$$

Algorithmic implication: $\Phi$ is efficiently approximable via local distributed algorithms, independently of network size.

Analytic implication: $\Phi$ admits a limit along most sparse graph sequences. The distributional self-similarity of $L$ may even allow for an explicit determination of $\Phi(L)$.

Examples: number of spanning trees (Lyons, 2005), spectrum and rank (Bordenave-Lelarge-S, 2011), matching polynomial (idem, 2013), Ising models (Dembo-Montanari-Sun, 2013)...

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Continuity with respect to local weak convergence

- In the sparse regime, many important graph parameters Φ are essentially determined by the **local geometry** only.
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\[ G_n \xrightarrow{\text{loc.}} \mathcal{L} \quad \Rightarrow \quad \Phi(G_n) \xrightarrow{n \to \infty} \Phi(\mathcal{L}) \]

- **Algorithmic implication:** Φ is efficiently approximable via local distributed algorithms, independently of network size.
- **Analytic implication:** Φ admits a limit along most sparse graph sequences. The distributional self-similarity of \( \mathcal{L} \) may even allow for an explicit determination of \( \Phi(\mathcal{L}) \).
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- This can be rigorously formalized by a continuity theorem:

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G_n \xrightarrow{\text{loc.}} L \quad \implies \quad \Phi(G_n) \xrightarrow{n \to \infty} \Phi(L)
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- **Algorithmic implication**: $\Phi$ is efficiently approximable via local distributed algorithms, independently of network size.
- **Analytic implication**: $\Phi$ admits a limit along most sparse graph sequences. The distributional self-similarity of $L$ may even allow for an explicit determination of $\Phi(L)$.
- **Examples**: number of spanning trees (Lyons, 2005), spectrum and rank (Bordenave-Lelarge-S, 2011), matching polynomial (idem, 2013), Ising models (Dembo-Montanari-Sun, 2013)
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Load balancing
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An allocation on $G$ is a function $\theta: \vec{E} \rightarrow [0, 1]$ satisfying

$$\theta(i, j) + \theta(j, i) = 1$$
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The allocation is balanced if for each $(i, j) \in \vec{E}$

$$\partial \theta(i) < \partial \theta(j) \implies \theta(i, j) = 0$$
From local to global optimality

Claim.
For an allocation \( \theta \), the following are equivalent:
1. \( \theta \) is balanced
2. \( \theta \) minimizes \( \sum_i (\partial \theta(i)) \)
3. \( \theta \) minimizes \( \sum_i f(\partial \theta(i)) \) for any convex function \( f: \mathbb{R} \to \mathbb{R} \).

Corollary 1.
Balanced allocations always exist.

Corollary 2.
They all induce the same loads \( \partial \theta: V \to [0, \infty) \).

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Balanced loads solve the densest subgraph problem: \( \max_{i \in V} \partial \theta(i) = \varrho^\star \) and \( \arg\max_{i \in V} \partial \theta(i) = H^\star \).
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Example
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How do those ‘densities’ look on a large sparse graph?
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Density profile of a random graph with average degree 3
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Density profile of a random graph with average degree 4
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Define the **density profile** of \( G = (V, E) \) as

\[ \Lambda_G = \frac{1}{|V|} \sum_{o \in V} \delta_{\partial \Theta(G, o)} \in \mathcal{P}(\mathbb{R}). \]
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2. \( \varrho^*(G_n) \xrightarrow{\mathbb{P}}_{n \to \infty} \sup\{ t \in \mathbb{R} : \Lambda(t, +\infty) > 0 \} \)
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Specifically, the excess function $\Phi: t \mapsto \int_{\mathbb{R}} (x - t)^+ \Lambda_{\mathcal{L}}(dx)$
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**Theorem.** Assume that $L[\deg(G, o)] < \infty$. Then,

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Specifically, the excess function $\Phi: t \mapsto \int_{\mathbb{R}} (x - t)^+ \Lambda_L(dx)$ solves

$$\Phi(t) = \max_{f: G^* \to [0, 1]} \left\{ \frac{1}{2} L \left[ \sum_{i \sim o} f(G, i) \wedge f(G, o) \right] - t L[f(G, o)] \right\}$$
Result 2: maximum subgraph density of sparse graphs

Extend the definition of $\varrho^\star$ to local weak limits by

$$\varrho^\star(L) := \sup \text{ess} \Lambda L = \sup \left\{ t : \Phi(t) > 0 \right\}$$

In light of previous result, one expects a continuity principle:

$$G_n \text{loc} \rightarrow \frac{}{\rightarrow} L \Rightarrow \varrho^\star(G_n) \rightarrow \varrho^\star(L)$$

Counter-example: adding a large but fixed clique to $G_n$ will arbitrarily boost $\varrho^\star(G_n)$ without affecting convergence $G_n \rightarrow L$.

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Assume degrees have light tail, i.e., $\limsup_{k \rightarrow \infty} \pi_1/k < 1$.

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\[
\varrho^* (G_n) \xrightarrow{n \to \infty} \varrho^* (\mathcal{L}), \text{ where } \mathcal{L} = \text{GALTON-WATSON}(\pi).
\]
Result 3: the case of Galton-Watson trees

Theorem. In the case where $L_{\text{Galton-Watson}}(\pi)$, $\Phi(t) = \max_{Q \in \mathcal{P}([0,1])} \{E[D]_{2P(\xi_1 + \xi_2 > 1) - tP(\xi_1 + \cdots + \xi_D > t)}\}$ where $D \sim \pi$ and $\{\xi_k\}_{k \geq 1}$ are iid with law $Q$, independent of $D$. The maximum is over all choices of $Q \in \mathcal{P}([0,1])$ such that $\xi_d = \lceil 1 - t + \xi_1 + \cdots + \xi_D \rceil_0$ where $\lfloor \cdot \rfloor_0$ denotes projection onto $[0,1]: \lfloor x \rfloor_0 = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \in [0,1] \\ 1 & \text{if } x > 1 \end{cases}$.
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where $D \sim \pi$ and $\{\xi_k\}_{k \geq 1}$ are IID with law $Q$, independent of $D$. 

**Distributional fixed-point equation:** can be solved numerically.
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An explicit formula

$G_n$: Erdős-Rényi $(n, c)$

$k \in \mathbb{N}^*$ fixed

Question: does $G_n$ contain a $k$-dense subgraph?

Define $f_k(x) = e^x - (1 + x + \cdots + x^k) + \frac{x^k}{k!}$

Set $c^* = xe^{f_k^{-1}(x)}$, where $x$ is the unique solution to $xf_k^{-1}(x)f_k(x) = 2^k$.

Theorem:
With probability tending to one as $n \to \infty$,

- If $c < c^*$ then $G_n$ does not contain a $k$-dense subgraph
- If $c > c^*$ then $G_n$ contains a $k$-dense subgraph

$k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, k_{11}, k_{12}, k_{13}, k_{14}, k_{15}, k_{16}, k_{17}, k_{18}, k_{19}, k_{20}$
An explicit formula

\[ G_n : \text{Erdős-Rényi} \left( n, \frac{c}{n} \right) \]
An explicit formula

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Define \( f_k(x) = e^x - \left( 1 + x + \cdots + \frac{x^k}{k!} \right) \)
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**Theorem.** With probability tending to one as \( n \to \infty \),
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**Theorem.** With probability tending to one as \( n \to \infty \),

\[ \begin{align*}
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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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</thead>
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<td>5.76</td>
<td>7.84</td>
<td>9.90</td>
<td>11.93</td>
<td>13.95</td>
<td>15.97</td>
<td>17.98</td>
<td>19.98</td>
</tr>
</tbody>
</table>
A few words on the proof

Microscopic contribution:

\[ \partial \Theta(G, o) \]

Hope:

\[ \partial \Theta(G, o) \]

is insensitive to what lies far away from \( o \):

\[ \[ G, o \] R = \[ G', o' \] R \Rightarrow |\partial \Theta(G, o) - \partial \Theta(G', o')| \leq f(R), \]

where \( f(R) \rightarrow 0 \) as \( R \rightarrow \infty \).

Counter-example:

let \( G \) be a \( d \)-regular graph with girth \( > R \) • \( \partial \Theta(G, o) = d^2 \)

\( R \) is a tree • \( \partial \Theta \leq \varrho \ast < 1 \) on any tree

▶ Balanced loads exhibit long-range dependences!
A few words on the proof

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A few words on the proof

Microscopic contribution: $\partial \Theta(G, o)$
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$\left[ G, o \right] R \equiv \left[ G', o' \right] R \Rightarrow |\partial \Theta(G, o) - \partial \Theta(G', o')| \leq f(R), \quad \text{where} \quad f(R) \to 0 \quad \text{as} \quad R \to \infty.$

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Balanced loads exhibit long-range dependences!
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▶ Balanced loads exhibit long-range dependences!
Solution: relaxed load balancing
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$\varepsilon > 0$: perturbative parameter
Solution: relaxed load balancing

\[ \varepsilon > 0 : \text{perturbative parameter} \]

**Definition.** An allocation \( \theta \) on \( G = (V, E) \) is \( \varepsilon \)-balanced if

\[
\theta(i,j) = \left[ \frac{1}{2} + \frac{\partial \theta(i) - \partial \theta(j)}{2\varepsilon} \right]_0^1
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Solution: relaxed load balancing

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$$|\partial \Theta_\varepsilon(G, o) - \partial \Theta_\varepsilon(G', o')| \leq \Delta \left(1 + \frac{2\varepsilon}{\Delta} \right)^{-R}.$$

**Corollary.** $\Theta_\varepsilon$ extends continuously to infinite graphs!
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**Corollary.** \( \Theta_\varepsilon \) extends continuously to infinite graphs!
Proof outline

Assume $G_n \xrightarrow{n \to \infty} L$ with $\int G \star \deg dL < \infty$. Consider a test function $\psi : \mathbb{R} \to \mathbb{R}$ (bounded, Lipschitz). 

$|V_n| \sum_{o \in V_n} \psi(\partial \Theta(G_n, o)) \xrightarrow{n \to \infty} \int G \star (\psi \circ \partial \Theta)$.
Proof outline

Assume \( G_n \xrightarrow{\text{loc.}} G \xrightarrow{n \to \infty} L \) with \( \int_{G^*} \deg dL < \infty \).
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Assume $G_n \xrightarrow{\text{loc.}} L$ with $\int_{G_n} \deg dL < \infty$.

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Assume $G_n \xrightarrow{\text{loc.}} L$ with $\int_{G_n} \deg d\mathcal{L} < \infty$.

Consider a test function $\psi: \mathbb{R} \rightarrow \mathbb{R}$ (bounded, Lipschitz)

$$\frac{1}{|V_n|} \sum_{o \in V_n} \psi(\partial \Theta(G_n, o)) \xrightarrow{??} \int_{G_n} (\psi \circ \partial \Theta)d\mathcal{L}$$
Proof outline

Assume $G_n \xrightarrow{\text{loc.}} \mathcal{L}$ with $\int_{G_\ast} \deg d\mathcal{L} < \infty$.

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$$\frac{1}{|V_n|} \sum_{o \in V_n} \psi (\partial \Theta_{\varepsilon}(G_n, o)) \xrightarrow{n \to \infty} \int_{G_\ast} (\psi \circ \partial \Theta_{\varepsilon}) d\mathcal{L}$$
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\[\varepsilon \to 0\]

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\[\varepsilon \to 0\]
Conclusion

In the sparse regime, many important graph parameters $\Phi$ are essentially determined by the local geometry of the graph. This can be rigorously formalized by a continuity theorem:

$$G_n \xrightarrow{n \to \infty} L = \Rightarrow \Phi(G_n) \xrightarrow{n \to \infty} \Phi(L)$$

Algorithmic implication: $\Phi$ is efficiently approximable via local distributed algorithms, independently of network size.

Analytic implication: $\Phi$ admits a limit along most sparse graph sequences. The distributional self-similarity of $L$ may sometimes even allow for an explicit determination of $\Phi(L)$.

Many examples: spanning trees, spectrum and rank, matching polynomial, Ising models, dense subgraphs...
Conclusion

▶ In the sparse regime, many important graph parameters $\Phi$ are essentially determined by the *local geometry* of the graph.
Conclusion

- In the sparse regime, many important graph parameters \( \Phi \) are essentially determined by the **local geometry** of the graph.
- This can be rigorously formalized by a continuity theorem:

\[
\begin{align*}
G_n & \xrightarrow{\text{loc.}} \mathcal{L} \\
\xrightarrow{n \to \infty} & \Phi(G_n) \xrightarrow{n \to \infty} \Phi(\mathcal{L})
\end{align*}
\]

- Algorithmic implication: \( \Phi \) is efficiently approximable via local distributed algorithms, independently of network size.
- Analytic implication: \( \Phi \) admits a limit along most sparse graph sequences. The distributional self-similarity of \( \mathcal{L} \) may sometimes even allow for an explicit determination of \( \Phi(\mathcal{L}) \).
- Many examples: spanning trees, spectrum and rank, matching polynomial, Ising models, dense subgraphs...
Conclusion

- In the sparse regime, many important graph parameters $\Phi$ are essentially determined by the local geometry of the graph.
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- In the sparse regime, many important graph parameters $\Phi$ are essentially determined by the **local geometry** of the graph.
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$$G_n \xrightarrow{loc. \ n \to \infty} \mathcal{L} \implies \Phi(G_n) \xrightarrow{n \to \infty} \Phi(\mathcal{L})$$

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Conclusion

- In the sparse regime, many important graph parameters $\Phi$ are essentially determined by the local geometry of the graph.
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Thank you!