# The Asymmetric leader election algorithm: number of survivors near the end of the game

Guy Louchard

March 14, 2014

# Outline

# Introduction

The following classical asymmetric leader election algorithm has obtained quite a bit of attention lately.
Starting with $n$ players, each one throws a coin, and the $k$ of them which have thrown head (with probability $q$) go on, and the leader will be found amongst them, using the same strategy. Should nobody advance, will the party repeat the procedure.
One of the most interesting parameter here is the number $J(n)$ of rounds until a leader has been identified. Without being exhaustive, let us mention the following related papers: Prodinger [19], Fill, Mahmoud, Szpankowski [1], Janson, Szpankowski [5], Knessel [10], Lavault, Louchard [11], Janson, Lavault, Louchard [4], Louchard, Prodinger [15], Kalpathy, Mahmoud, Ward [9], Louchard, Martinez, Prodinger [12], Louchard, Prodinger [16], Louchard, Prodinger, Ward [17], Kalpathy [6], Kalpathy [7].

Our attention was recently attracted by an interesting paper by Kalpathy, Mahmoud and Rosenkrantz [8], where they consider the number of survivors $S_{n,t}$, after $t$ election rounds, in a broad class of fair leader election algorithms starting with $n$ candidates. They give sufficient conditions for $\frac{S_{n,t}}{n}$ to converge to a product of independent identically distributed random variables. Their analysis starts from the beginning of the game. They give two illustrative examples, one with classical leader election and one with uniform splitting protocol.

We found it interesting to investigate, in the classical asymmetric leader election algorithm (with possible resurrections), what happens near the end of the game, namely we fix an integer $\kappa$ and we would like to study the behaviour of the number of survivors $\mathcal{L}$ at level $J(n) - \kappa$.

In our asymptotic analysis (for $n \to \infty$) we are focusing on the *limiting distribution function.* It might look paradoxical at first sight, that the asymptotic formulae involve, on the right side, some quantities $P(i,k), G(i,k,\ell)$ defined by some recursions. However, this happens often, and convergence is quite good, so that with only a few terms (obtained directly from the recursion) a good approximation of the numerical values can be obtained.

Further, we investigate what happens, if the parameter $p = 1 - q$ gets small ($p \to 0$) or large ($p \to 1$). We use three efficient tools: an urn model, a Mellin-Laplace technique for Harmonic sums and some asymptotic distributions related to one of the extreme-value distributions: the Gumbel law.

The paper is organized as follows: Sec. 2 presents our model and notations, Sec. 3 analyses $\mathcal{L} :=$ number of survivors at position $J(n) - \kappa$. Sec. 4 considers the case where the parameter $p \to 0$ and Sec. 5 the case where $p \to 1$. Sec. 6 concludes the paper which fits within the framework of Analytic Combinatorics.

# Model and Notations

Let the random variable $X$ be geometrically distributed, i. e., $\mathbb{P}(X = j) = pq^{j-1}$, with $q = 1 - p$: We interpret $p$ as the killing probability, and $q$ as the survival probability. Let us consider the game as an urn model, with urns labelled $1, 2, \ldots$, where we throw $n$ balls, and the probability of each ball falling into urn $j$ being given by $pq^{j-1}$. The balls at level $j$ represent the candidates who are killed at this level. Let us denote by $J(n)$ the number of rounds, when we start with $n$ players. Note that, when all the players are killed, they are magically resurrected and try again. This increases the parameter $J$, but leaves the party at the same level (same urn). Note that we often speak synonymously about players, balls, candidates, etc., and also about urns, levels, positions, time etc. We drop the $n$-dependency when there is no ambiguity, to ease the notation. We will use the following notations:

$$
\begin{aligned}
\Pi(\lambda, u) &:= e^{-\lambda}\lambda^{u}/u!, \text{ (Poisson distribution)},\\
J(n) &:= \text{ number of rounds until a leader has been identified,}\\
&\quad \text{starting with } n \text{ candidates },\\
n^* &:= n\frac{p}{q},\\
Q &:= 1/q,\\
M &:= \log p,\\
\chi_t &:= \frac{2t\pi\mathbf{i}}{L},\\
\tilde{\alpha} &:= \alpha/L,\\
\log &:= \log_Q,\\
\eta &:= j - \log n^*,
\end{aligned}
$$

$$\begin{aligned}
L &:= \ln Q, \\
\{x\} &:= \text{ fractional part of } x, \\
\mathcal{L} &:= \text{ number of survivors at position } J(n) - \kappa, \\
P(i,k) &:= \mathbb{P}(J(i) = k | \text{ we start with } i \text{ candidates}), \\
G(i,k,\ell) &:= \mathbb{P}(\text{ number of survivors at step } k \text{ is given by } \ell, \\
&\quad \text{starting with } i \text{ candidates})(\ \textit{including possible resurrections}\ ), \\
G^*(i,k,\ell) &:= \mathbb{P}(\text{ number of survivors at step } k \text{ is given by } \ell, \\
&\quad \text{starting with } i \text{ candidates})(\ \textit{with no resurrections}\ ).
\end{aligned}$$

We recall the main properties of such a model.

- ASYMPTOTIC INDEPENDENCE. We have asymptotic independence of urns, for all events related to urn $j$ containing $\mathcal{O}(1)$ balls. This is proved, by Poissonization-De-Poissonization, in [3],[14] and [18]. The error term is $\mathcal{O}(n^{-C})$ where $C$ is a positive constant.

- ASYMPTOTIC DISTRIBUTIONS. We obtain asymptotic distributions of the interesting random variables as follows. The number of balls in each urn is asymptotically Poisson-distributed with parameter $npq^{j-1}$ in urn $j$ containing $\mathcal{O}(1)$ balls (this is the classical asymptotic for the Binomial distribution). This means that the asymptotic number $\ell$ of balls in urn $j$ is given by

$$\exp\left(-npq^{j-1}\right)\frac{\left(npq^{j-1}\right)^{\ell}}{\ell!},$$

and with $\eta = j - \log n^*$, this is equivalent to $\Pi\left(e^{-L\eta}, \ell\right)$. The asymptotic distributions are related to Gumbel distribution functions (given by $\exp\left(-e^{-x}\right)$) or convergent series of such. The error term is $\mathcal{O}(n^{-1})$.

- EXTENDED SUMMATIONS. Some summations now go to $\infty$. This is justified, for example, in [14].
- MELLIN TRANSFORM. Asymptotic expressions for the moments are obtained by Mellin transforms applied to Harmonic sums (see for instance, Flajolet, Gourdon, Dumas [2] for a nice exposition). The error term is $\mathcal{O}(n^{-C})$. We proceed as follows (see [13] for detailed proofs): from the asymptotic properties of the urns, we obtain the asymptotic distributions of our random variables of interest. Next we compute the Laplace transform $\phi(\alpha)$ of these distributions, from which we can derive the dominant part of probabilities as well as the (tiny) periodic part in the form of a Fourier series. This connection will be detailed in the next sections.

- RAPID DECREASE PROPERTY. $\Gamma(s)$ decreases exponentially in the direction $i\infty$:

$$|\Gamma(\sigma + \mathbf{i}t)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}.$$

  Also, we this property is true for all other functions we encounter. So inverting the Mellin transforms is easily justified. (see [13], Sec.4 for example).
- EARLY APPROXIMATIONS. If we compare the approach in this paper with other ones that appeared previously, then we can notice the following. Traditionally, one would stay with exact enumerations as long as possible, and only at a late stage move to asymptotics. Doing this, one would, in terms of asymptotics, carry many unimportant contributions around, which makes the computations quite heavy. Here, however, approximations are carried out as early as possible, and this allows for streamlined (and often automatic) computations.

- CONVERGENCE. Asymptotically, the distribution will be a periodic function of $\{\log n\}$. The distributions do not converge in the weak sense, they do however converge along subsequences $n_m$ for which $\{\log n\}$ is constant. This type of convergence is not uncommon in the Analysis of Algorithms. Many examples are given in [13].

Thus, the present paper represents another application of the powerful technique that was first presented in [13].

# Asymptotic analysis of $\mathcal{L} :=$ number of survivors at position $J(n) - \kappa$

Let us first write down the recursions satisfied by $P(i,k), G(i,k,\ell)$.

$$P(i,k) = (p^i + q^i)P(i,k-1) + \sum_{u=1}^{i-1} \binom{i}{u} q^u p^{i-u} P(u,k-1). \quad (1)$$

Indeed, with probability $q^i$, all advance, we repeat the procedure, with probability $p^i$, all players are killed, they are resurrected and they try again, with probability $\binom{i}{u} q^u p^{i-u}, 1 \le u \le i-1$, $u$ players survive.

$$P(1,0) = 1, \quad P(1,k) = 0, k > 0, \quad P(i,0) = 0, i > 1.$$

$$G(i,k,\ell) = (p^i + q^i)G(i,k-1,\ell) + \sum_{u=1}^{i-1} \binom{i}{u} q^u p^{i-u} G(u,k-1,\ell),$$
$$i \ge 2, \quad 1 \le \ell \le i.$$

The justification is identical.

The following properties are easily checked:

$$G(u,0,u)=1, \quad G(u,0,v)=0, v\neq u.$$

$$G(i,1,v)=\binom{i}{v}q^v p^{i-v}, \quad v\neq i, \quad G(i,1,i)=p^i+q^i.$$

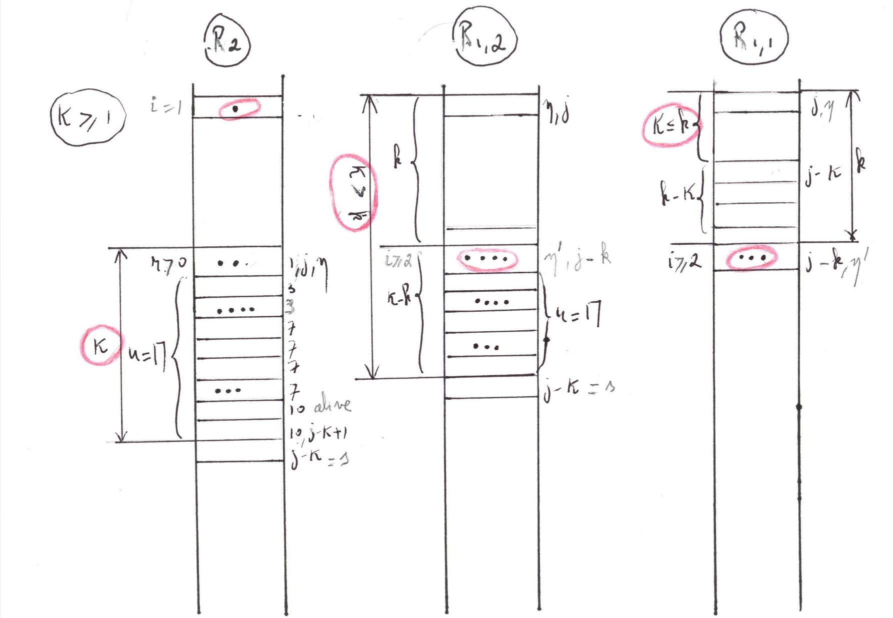
$$\sum_{\ell\geq 1} G(i,k,\ell)=P(i,J\geq k)=1-P(i,J<k).$$

$$\sum_{\ell\geq 1} G(i,1,\ell)=1.$$

$$\sum_{\ell=2}^{i} G(i,k-v,\ell)P(l,v)=P(i,k), \quad \forall k\geq v.$$

$$G(i,k,1)=P(i,k),$$

$$G(i,1,1)=P(i,1).$$


R2
R1,2
R1,1
K ≥ 1
i=1
K ≤ k
k
j-K
k-K
i≥2
j-k, η'
u=17
K
η'
u=17
10 alive
10, j-K+1
j-K=1

Let us first recall, from [15], the asymptotic distribution of $J(n)$. We will consider the maximal non-empty urn. Assume that it contains $i$ balls. Notice that there are no resurrections before.

- If $i \geq 2$, this means that we must restart the game with $i$ candidates. Assume that this costs $k \geq 1$ rounds. Denoting by $j-k$ the position of the maximal non-empty urn, we have

$$P_1(n,j) = \mathbb{P}(J(n)=j) \sim f_1(\eta),$$

where, with $\eta' = j-k-\log n^* = \eta - k$,

$$\begin{aligned} f_1(\eta) &= \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \Pi\left(\frac{q}{p}e^{-L\eta'},0\right)\Pi\left(e^{-L\eta'},i\right)P(i,k) \\ &= \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\left(-\frac{q}{p}e^{-L\eta'}\right)\exp\left(-e^{-L\eta'}\right)\frac{(e^{-L\eta'})^i}{i!}P(i,k) \\ &= \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\left(-\frac{1}{p}e^{Lk}e^{-L\eta}\right)\frac{e^{-L\eta i}e^{Lki}}{i!}P(i,k). \end{aligned}$$

*Justification:* this corresponds to $i \geq 2$ balls in urn $j-k$ (normalized as $\eta'$), no balls to the right of $j-k$ and the end of the game occurs at $j$ (normalized as $\eta$). We use asymptotic independence of urns and Poisson limiting random variables. We recognize the presence of the Gumbel distribution.
All sums should be finite, but, as proved in [14] for similar sums, asymptotically, we can use infinite sums.

- If $i = 1$, this means that the last alive candidate corresponds to some urn $j$, which is the last non-empty urn *before* the maximal non-empty urn. The distribution of balls *after* urn $j$ is asymptotically given by $\Pi(\frac{q}{p}e^{-L\eta}, u)$. So

$$P_2(n,j) = \mathbb{P}(J(n) = j) \sim f_2(\eta),$$

where

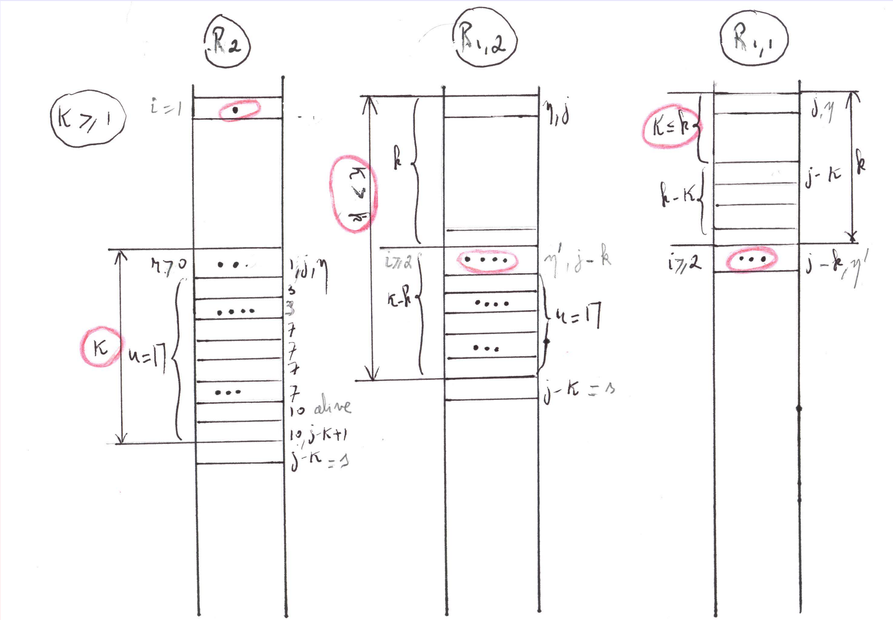
$$\begin{aligned} f_2(\eta) &= \Pi\left(\frac{q}{p}e^{-L\eta}, 1\right)\left(1 - \Pi\left(e^{-L\eta}, 0\right)\right) \\ &= \exp\left(-\frac{q}{p}e^{-L\eta}\right)\frac{q}{p}e^{-L\eta}\left(1 - \exp\left(-e^{-L\eta}\right)\right). \end{aligned}$$

*Justification:* this corresponds to a non-empty urn at $j$ (normalized as $\eta$), one ball to the right of $j$, and the end of the game occurs at $j$.

So

$$P(n,j) = \mathbb{P}(J(n) = j) \sim f(\eta) = f_1(\eta) + f_2(\eta),$$

and $f_i(\eta) > 0$.


R2
R1,2
R1,1
K ≥ 1
i = 1
u = 17
alive
j - K = 1
K ≤ k
k - K
j - K
i ≥ 2

Now we turn to the main objective of this paper: the asymptotic distribution of the number of survivors at level $J(n) - \kappa$. The analysis is somewhat more complicated than the analysis of $J(n)$. If the maximal non-empty urn contains $i \geq 2$ balls, we must consider two subcases. We will use the following relation:

$$\begin{aligned}
&e^{-L(\eta'-1)} + e^{-L(\eta'-2)} + \ldots + e^{-L(\eta'-(\kappa-k-1))} \\
&= e^{-L\eta'}(Q + Q^2 + \ldots + Q^{\kappa-k-1}) \\
&= e^{-L\eta'}\frac{1}{p}\left(Q^{\kappa-k-1} - 1\right) = e^{-L\eta'}\varphi_1(\kappa, k), \text{ say.} \qquad (2)
\end{aligned}$$

- If $i \geq 2, \kappa \leq k$ then we have $\ell \geq 2$ survivors at position $j-\kappa$, with probability $G(i, k-\kappa, \ell)$ (with possible resurrections) and we finish the game with $\kappa$ steps, at position $j$, starting with $\ell$ balls. This leads to the following asymptotic joint distribution of $J(n)$ and $\mathcal{L}$

$$R_{1,1}(\eta, \ell) = \mathbb{P}(J(n) = j, \mathcal{L} = \ell)$$

$$\sim \sum_{k \geq \kappa} \sum_{i \geq \ell} G(i, k-\kappa, \ell) P(\ell, \kappa) \Pi\left(\frac{q}{p} e^{-L\eta'}, 0\right) \Pi\left(e^{-L\eta'}, i\right)$$

$$= \sum_{k \geq \kappa} \sum_{i \geq \ell} G(i, k-\kappa, \ell) P(\ell, \kappa) \exp\left(-\frac{1}{p} e^{-L\eta'}\right) \frac{e^{-Li\eta'}}{i!}$$

$$l \geq 2, \quad \eta' := \eta - k = j - k - \log(n).$$

Again the Gumbel distribution enters the picture.

- If $i \geq 2, \kappa > k$ then between steps $j-\kappa+1$ and $j-k-1$, we have $u$ dead candidates. The maximal non-empty urn at step $j-k$ contains $i$ balls and again, we finish the game with $k$ steps, at position $j$, starting with $i$ balls. The $u$ dead candidates asymptotically obey the Poisson distribution with parameter $e^{-L\eta'}\varphi_1(\kappa, k)$ (see (2)) and the total number of survivors at position $j-\kappa$ is given by $\ell = i + u$.

This leads to

$$R_{1,2}(\eta,\ell) = \mathbb{P}(J(n)=j, \mathcal{L}=\ell)$$

$$\sim \sum_{i+u=\ell, i\geq 2} \sum_{1\leq k<\kappa} \Pi\left(e^{-L\eta'}\varphi_1(\kappa,k), u\right) \Pi\left(\frac{q}{p}e^{-L\eta'}, 0\right) \times$$

$$\times \Pi\left(e^{-L\eta'}, i\right) P(i,k),$$

$\kappa \geq 2$,

$$= \sum_{i+u=\ell, i\geq 2} \sum_{1\leq k<\kappa} \exp\left(-e^{-L\eta'}\frac{Q^{\kappa-k-1}}{p}\right) \frac{e^{-Li\eta'}}{i!} \frac{\left(e^{-L\eta'}\varphi_1(\kappa,k)\right)^u}{u!} P(i,k)$$

$$= \sum_{i+u=\ell, i\geq 2} \sum_{1\leq k<\kappa} \exp\left(-e^{-L\eta}e^{Lk}\frac{Q^{\kappa-k-1}}{p}\right) \frac{e^{-Li\eta}e^{Lik}}{i!} \times$$

$$\times \frac{\left(e^{-L\eta}e^{Lk}\varphi_1(\kappa,k)\right)^u}{u!} P(i,k),$$

$$\varphi_1(\kappa,k) := \frac{1}{p}\left(Q^{\kappa-k-1}-1\right).$$

- Finally, if the maximal non-empty urn contains 1 ball, then we have $u$ dead candidates between $j-\kappa+1$ and $j-1$, $r$ dead players at position $j$ and exactly one dead player after $j$. Hence

$$R_2(\eta,\ell) = \mathbb{P}(J(n)=j, \mathcal{L}=\ell)$$

$$\sim \sum_{1+r+u=\ell, r>0, u\geq 0} \Pi\left(\frac{q}{p}e^{-L\eta},1\right)\Pi\left(e^{-L\eta},r\right)\Pi\left(e^{-L\eta}\varphi_2(\kappa),u\right)$$

$$= \Pi\left(\frac{q}{p}e^{-L\eta},1\right)\left[\Pi\left(e^{-L\eta}(1+\varphi_2(\kappa)),\ell-1\right)\right.$$

$$\left.-\Pi\left(e^{-L\eta},0)\Pi(e^{-L\eta}\varphi_2(\kappa)),\ell-1\right)\right]$$

$$= \exp\left(-e^{-L\eta}\frac{Q^{\kappa-1}}{p}\right)\frac{q}{p}e^{-L\eta}\frac{1}{(\ell-1)!}\times$$

$$\times\left[\left(e^{-L\eta}(1+\varphi_2(\kappa))\right)^{\ell-1}-\left(e^{-L\eta}\varphi_2(\kappa)\right)^{\ell-1}\right]$$

$$\varphi_2(\kappa) := \frac{1}{p}\left(Q^{\kappa-1}-1\right).$$

Following now the lines of [13], we proceed to the Laplace transforms w.r.t $\eta$. This leads to

$$\phi_{1,1}(\alpha,\ell) = \sum_{k\geq\kappa}\sum_{i\geq\ell} G(i,k-\kappa,\ell)P(\ell,\kappa)\frac{e^{Lk\tilde{\alpha}}}{Li!}(1/p)^{-i+\tilde{\alpha}}\Gamma(i-\tilde{\alpha}).$$

$$\begin{aligned}\phi_{1,2}(\alpha,\ell) &= \sum_{i+u=\ell, i\geq 2}\sum_{1\leq k<\kappa}\Gamma(i+u-\tilde{\alpha})\frac{1}{Li!u!}\left(e^{Lk}\frac{Q^{\kappa-k-1}}{p}\right)^{-i-u+\tilde{\alpha}}\times \\ &\times \varphi_1(\kappa,k)^u e^{Lki}e^{Lku}P(i,k) \\ &= \sum_{i+u=\ell, i\geq 2}\sum_{1\leq k<\kappa}\Gamma(i+u-\tilde{\alpha})\frac{1}{Li!u!}\left(\frac{Q^{\kappa-k-1}}{p}\right)^{-i-u+\tilde{\alpha}}\times \\ &\times e^{Lk\tilde{\alpha}}\varphi_1(\kappa,k)^u P(i,k).\end{aligned}$$

$$
\begin{aligned}
\phi_2(\alpha,\ell) &= \Gamma(1+(\ell-1)-\tilde{\alpha})\left(\frac{Q^{\kappa-1}}{p}\right)^{-1-(\ell-1)+\tilde{\alpha}}\frac{q}{pL}\frac{1}{(\ell-1)!}\times \\
&\times\left[(1+\varphi_2(\kappa))^{\ell-1}-\varphi_2(\kappa)^{\ell-1}\right] \\
&= \Gamma(\ell-\tilde{\alpha})\left(\frac{Q^{\kappa-1}}{p}\right)^{-\ell+\tilde{\alpha}}\frac{q}{pL}\frac{1}{(\ell-1)!}\times \\
&\times\left[(1+\varphi_2(\kappa))^{\ell-1}-\varphi_2(\kappa)^{\ell-1}\right].
\end{aligned}
$$

This leads to the dominant (non-periodic) part of $G(n, J(n)-\kappa, \ell)$ namely the probability that number of survivors at step $J(n)-\kappa$ is given by $\ell$, starting with $n$ candidates

## Theorem 3.1

*The dominant (non-periodic) part of the probability $G(n, J(n)-\kappa, \ell)$ that $\mathcal{L}$: the number of survivors at step $J(n)-\kappa$, is given by $\ell$, (starting with n candidates), is asymptotically given by*

$$G(n, J(n)-\kappa, \ell) \sim R_{1,1}(\ell) + R_{1,2}(\ell) + R_2(\ell),$$

*with*

$$R_{1,1}(\ell) = \phi_{1,1}(0,\ell) = \sum_{k\geq\kappa}\sum_{i\geq\ell} G(i, k-\kappa, \ell)P(\ell,\kappa)\frac{p^i}{Li}, \quad \ell \geq 2,$$

$$R_{1,2}(\ell) = \phi_{1,2}(0,\ell)$$

$$= \sum_{i+u=\ell, i\geq 2} \sum_{1\leq k<\kappa} \Gamma(i+u)\frac{1}{Li!u!}\left(\frac{Q^{\kappa-k-1}}{p}\right)^{-i-u} \varphi_1(\kappa,k)^u P(i,k)$$

$$= \sum_{i=2}^{\ell} \sum_{1\leq k<\kappa} \Gamma(\ell)\frac{1}{Li!(\ell-i)!}\left(\frac{Q^{\kappa-k-1}}{p}\right)^{-\ell} P(i,k)\varphi_1(\kappa,k)^{\ell-i}, \quad \ell \geq 2,$$

$$R_2(\ell) = \phi_2(0,\ell) = \left(\frac{Q^{\kappa-1}}{p}\right)^{-\ell} \frac{q}{pL}\left[(1+\varphi_2(\kappa))^{\ell-1} - \varphi_2(\kappa)^{\ell-1}\right], \quad \ell \geq 2$$

The presence of $\frac{p^i}{Li}$ in $R_{1,1}(\ell)$ is not surprising: this is explained as follows. The asymptotic probability of a position $v$ of maximal non-empty urn and a number $i$ of players in this urn, is given, with $\eta = j - \log n^*$, by

$$\Pi\left(\frac{q}{p}e^{-L\eta}, 0\right)\Pi\left(e^{-L\eta}, i\right) = \exp\left(-\frac{1}{p}e^{-L\eta}\right)\frac{e^{-L\eta i}}{i!}.$$

This is the classical asymptotic for the Binomial distribution, and is easily checked as follows:

$$\begin{aligned}
&\binom{n}{i}\left(1-q^{j-1}\right)^{n-i} q^{(j-1)i}p^i \\
&= \binom{n}{i}\left(1-e^{-L(j-1)}\right)^{n-i} e^{-L(j-1)i}p^i \\
&= \binom{n}{i}\left(1-\frac{e^{-L\eta}}{pn}\right)^{n-i}\frac{e^{-L\eta i}}{n^i p^i}p^i \\
&\sim \exp\left(-e^{-L\eta}/p\right)\frac{e^{-L\eta i}}{i!}.
\end{aligned}$$

The probability of a number $i$ of balls *independently* of the position $j$, is given by the asymptotic distribution

$$\sum_{j=1}^{\infty} \exp\left(-\frac{1}{p}e^{-L\eta}\right)\frac{e^{-L\eta i}}{i!}.$$

This is an harmonic sum, and with

$$\phi_3(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} \exp\left(-\frac{1}{p}e^{-L\eta}\right)\frac{e^{-L\eta i}}{i!}d\eta = \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i-\tilde{\alpha}),$$

the asymptotic probability of $i$ balls is given by

$$\phi_3(0) = \frac{p^i}{iL}. \quad (3)$$

This is an honest probability measure as

$$\sum_{i=1}^{\infty} \frac{p^i}{iL} = 1.$$

We have no periodic component here. Indeed, it should be given by

$$\sum_{t\neq 0}\sum_{i=1}^{\infty}\phi_3(-L\chi_t)e^{-2t\pi\mathbf{i}\log n^*}=\sum_{t\neq 0}\sum_{i=1}^{\infty}\frac{(1/p)^{-i-\chi_t}}{Li!}\Gamma(i+\chi_t)e^{-2t\pi\mathbf{i}\{\log n^*\}}.$$

We note that replacing $\log n^*$ by $\log n + M + 1$ allows the elimination of $p^{\chi_t}$. We are left with a term

$$\sum_{i=1}^{\infty}\frac{p^i}{Li!}\Gamma(i+\chi_t),$$

but this is null by [15], Appendix A.

Let us check that our theorem also leads to an honest probability measure. Set

$$S_{1,1} = \sum_{\ell \geq 2} R_{1,1}(\ell) = \sum_{k \geq \kappa} \sum_{i \geq 2} \sum_{\ell=2}^{i} G(i, k-\kappa, \ell) P(\ell, \kappa) \frac{p^i}{Li}$$

$$= \sum_{k \geq \kappa} \sum_{i \geq 2} \frac{p^i}{Li} P(i, k).$$

$$
\begin{aligned}
S_{1,2} &= \sum_{\ell\geq 2} R_{1,2}(\ell) = \sum_{\ell\geq 2}\sum_{i=2}^{\ell}\sum_{1\leq k<\kappa}\Gamma(\ell)\frac{1}{Li!(\ell-i)!}\times \\
&\times\left(\frac{Q^{\kappa-k-1}}{p}\right)^{-\ell} P(i,k)\varphi_1(\kappa,k)^{\ell-i} \\
&= \sum_{1\leq k<\kappa}\sum_{i\geq 2}\varphi_1(\kappa,k)^{-i}P(i,k)\frac{1}{Li!}\sum_{\ell\geq i}\Gamma(\ell)\frac{1}{(\ell-i)!}\times \\
&\times\left(\frac{Q^{\kappa-k-1}}{p}\right)^{-\ell}\varphi_1(\kappa,k)^{\ell} \\
&= \sum_{1\leq k<\kappa}\sum_{i\geq 2}\varphi_1(\kappa,k)^{-i}P(i,k)\frac{1}{Li!}\Gamma(i)\left(Q^{\kappa-k-1}-1\right)^{i} \\
&= \sum_{1\leq k<\kappa}\sum_{i\geq 2}\frac{p^i}{Li}P(i,k).
\end{aligned}
$$

$$S_2 = \sum_{\ell \geq 2} R_2(\ell) = \frac{q}{pL} \sum_{\ell \geq 2} \left( \frac{Q^{\kappa-1}}{p} \right)^{-\ell} \left[ (1 + \varphi_2(\kappa))^{\ell-1} - \varphi_2(\kappa)^{\ell-1} \right]$$

$$= \frac{q}{pL} . \frac{p^2}{q} = \frac{p}{L}.$$

Now we have

$$S_{1,1} + S_{1,2} + S_2 = \sum_{i \geq 2} \frac{p^i}{Li} + \frac{p}{L} = 1,$$

as it should.

We now turn to the periodic component of our asymptotics. Following again the techniques detailed in [13], this leads to (again, we have the elimination of $p^{\chi_t}$.)

### Theorem 3.2

*The periodic component of the probability $G(n, J(n) - \kappa, \ell)$ that $\mathcal{L}$: the number of survivors at step $J(n) - \kappa$, is given by $\ell$, (starting with n candidates), is asymptotically given by*

$$w(n, J(n) - \kappa, \ell) \sim w_{1,1}(\ell) + w_{1,2}(\ell) + w_2(\ell),$$

with

$$
\begin{aligned}
w_{1,1}(\ell) &= \sum_{t\neq 0} \phi_{1,1}(\alpha,\ell)|_{\alpha=-L\chi_t}\, e^{-2t\pi\mathbf{i}\{\log n^*\}} \\
&= \sum_{t\neq 0}\sum_{k\geq\kappa}\sum_{i\geq\ell} G(i,k-\kappa,\ell)P(\ell,\kappa)\frac{e^{-k\chi_t}}{Li!}p^i\Gamma(i+\chi_t)e^{-2t\pi\mathbf{i}\{\log n\}}, \\
w_{1,2}(\ell) &= \sum_{t\neq 0} \phi_{1,2}(\alpha,\ell)|_{\alpha=-L\chi_t}\, e^{-2t\pi\mathbf{i}\{\log n^*\}} \\
&= \sum_{t\neq 0}\sum_{i+u=\ell, i\geq 2}\sum_{1\leq k<\kappa} \Gamma(i+u+\chi_t)\frac{1}{Li!u!}p^{i+u}Q^{(\kappa-k-1)(-i-u-\chi_t)}\times \\
&\times e^{-k\chi_t}\varphi_1(\kappa,k)^u P(i,k)e^{-2t\pi\mathbf{i}\{\log n\}},
\end{aligned}
$$

$$\begin{aligned} w_2(\ell) &= \sum_{t\neq 0} \phi_2(\alpha,\ell)|_{\alpha=-L\chi_t}\, e^{-2t\pi \mathbf{i}\{\log n^*\}} \\ &= \sum_{t\neq 0} \Gamma(\ell+\chi_t) p^\ell Q^{(\kappa-1)(-\ell-\chi_t)} \frac{q}{pL} \frac{1}{(\ell-1)!} \times \\ &\times \left[(1+\varphi_2(\kappa))^{\ell-1} - \varphi_2(\kappa)^{\ell-1}\right]. \end{aligned}$$

# Asymptotics for $p \to 0$

We will only deal with first order asymptotics. We first let $n \to \infty$ and then $p \to 0$.
Let, with $\varepsilon = o(1)$,

$$p = \varepsilon, \quad q = 1 - \varepsilon,$$
$$Q = \frac{1}{1-\varepsilon} \sim 1 + \varepsilon,$$
$$L = \ln(1-\varepsilon) \sim \varepsilon,$$
$$\log n^* \sim \frac{\ln n}{\varepsilon} + \frac{\ln \varepsilon}{\varepsilon}.$$

As the resurrection would lead here to negligible contribution, (1) leads to

$$P(i,k) = (1 - i\varepsilon)P(i,k-1) + i\varepsilon P(i-1,k-1). \qquad (4)$$

Indeed the probability that all $i$ players die is given by $\varepsilon^i$ which is negligible.

We have the asymptotic

$$P(i,k) \sim i(i-1)\left(1-e^{-(k-1)\varepsilon}\right)^{i-2} e^{-2(k-1)\varepsilon}\varepsilon.$$

This is justified as follows: we first have 2 survivors at position $k-1$, without resurrection, with probability $\frac{i(i-1)}{2}\left(1-e^{-(k-1)\varepsilon}\right)^{i-2} e^{-2(k-1)\varepsilon}$, and with probability $2\varepsilon$, one survive at position $k$. The event that $u>2$ players survive at position $k-1$ entails a $\mathcal{O}(\varepsilon^{u-1})$ contribution at position $k$, negligible w.r.t. $\varepsilon$.

Let us check that this asymptotic for $P(i,k)$ satisfies (4). Set

$$\begin{aligned}\Delta &:= i(i-1)\left(1-e^{-(k-1)\varepsilon}\right)^{i-2} e^{-2(k-1)\varepsilon}\varepsilon \\ &- \Big[(1-i\varepsilon)i(i-1)\left(1-e^{-(k-2)\varepsilon}\right)^{i-2} e^{-2(k-2)\varepsilon}\varepsilon \\ &+ i\varepsilon(i-1)(i-2)\left(1-e^{-(k-2)\varepsilon}\right)^{i-3} e^{-2(k-2)\varepsilon}\varepsilon\Big],\end{aligned}$$

and, with $\alpha := e^{-(k-2)\varepsilon}$,

$$\Delta \sim i(i-1)\left[(1-\alpha)^{i-3} + (i-2)\varepsilon\alpha\,(1-\alpha)^{i-3}\right]\left[\alpha^2(1-2\varepsilon)\right]\varepsilon$$
$$-\Big[(1-i\varepsilon)i(i-1)\,(1-\alpha)^{i-3}\,(1-\alpha)\alpha^2\varepsilon$$
$$+i\varepsilon(i-1)(i-2)\,(1-\alpha)^{i-3}\,\alpha^2\varepsilon\Big]$$
$$= -2i\varepsilon^3\,(1-\alpha)^{i-3}\,\alpha^3(i-1)(i-2),$$

which is null by first order asymptotics. Similarly, we have

$$G(i,k,\ell) \sim \binom{i}{\ell}\left(1-e^{-k\varepsilon}\right)^{i-\ell} e^{-\ell k\varepsilon},$$

and $G^*$ is asymptotically equivalent to $G$.
Note that $J(n)$ has a mean $\sim \frac{\ln n}{\varepsilon}$ and is spread with a standard deviation $= \mathcal{O}\left(\frac{1}{\varepsilon}\right)$.

Now we turn to asymptotics for $R_{1,1}(\ell), R_{1,2}(\ell), R_2(\ell)$. From Theorem 3.1, we derive

$$R_{1,1}(\ell) \sim \sum_{k\geq\kappa}\sum_{i\geq\ell}\binom{i}{\ell}\left(1-e^{-(k-\kappa)\varepsilon}\right)^{i-\ell}e^{-\ell(k-\kappa)\varepsilon}\ell(\ell-1)\times$$

$$\times\left(1-e^{-(\kappa-1)\varepsilon}\right)^{\ell-2}e^{-2(\kappa-1)\varepsilon}\varepsilon\frac{\varepsilon^i}{Li},$$

and with $u := (k-\kappa)\varepsilon$,

$$R_{1,1}(\ell) \sim \sum_{k\geq\kappa}\frac{\varepsilon^\ell}{\ell\left[1-(1-e^{-u})\varepsilon\right]^\ell}e^{-\ell u}\ell(\ell-1)\left((\kappa-1)\varepsilon\right)^{\ell-2}\varepsilon\frac{1}{L},$$

and with Euler-MacLaurin, (the error terms are negligible),

$$R_{1,1}(\ell) \sim \frac{\varepsilon^\ell}{\ell}\ell(\ell-1)\left((\kappa-1)\varepsilon\right)^{\ell-2}\varepsilon\frac{1}{L}\int_0^\infty e^{-\ell u}\left[(1+\ell(1-e^{-u})\varepsilon\right]\frac{du}{\varepsilon}$$

$$\sim \frac{\varepsilon^\ell}{\ell}\ell(\ell-1)\left((\kappa-1)\varepsilon\right)^{\ell-2}\frac{1}{L}\frac{1}{\ell}$$

$$\sim \frac{\varepsilon^{\ell-1}}{\ell}(\ell-1)\left((\kappa-1)\varepsilon\right)^{\ell-2} = \frac{\varepsilon^{2\ell-3}}{\ell}(\ell-1)\left((\kappa-1)\right)^{\ell-2}.$$

$$
\begin{aligned}
R_{1,2}(\ell) &\sim \sum_{i=2}^{\ell} \sum_{1 \le k < \kappa} \Gamma(\ell) \frac{1}{L i!(\ell-i)!} \varepsilon^{\ell} (1-\varepsilon)^{(\kappa-k-1)\ell} i(i-1) \times \\
&\quad \times \left(1 - e^{-(k-1)\varepsilon}\right)^{i-2} e^{-2(k-1)\varepsilon} \varepsilon \left[\frac{1}{\varepsilon}((1+\varepsilon)^{\kappa-k-1} - 1)\right]^{\ell-i} \\
&\sim \sum_{1 \le k < \kappa} \varepsilon^{\ell} (1-\varepsilon)^{(\kappa-k-1)\ell} \varepsilon \left[1 - e^{-(k-1)\varepsilon} + \frac{1}{\varepsilon}((1+\varepsilon)^{\kappa-k-1} - 1)\right. \\
&\sim \sum_{1 \le k < \kappa} \varepsilon^{\ell} (\ell-1)(\kappa-k-1)^{\ell-2}. \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
R_2(\ell) &\sim \frac{1}{\varepsilon L}\varepsilon^{\ell}(1-\varepsilon)^{(\kappa-1)\ell}\left[\left[1+\frac{1}{\varepsilon}((1+\varepsilon)^{\kappa-1}-1)\right]^{\ell-1}\right.\\
&\quad\left.-\left[\frac{1}{\varepsilon}((1+\varepsilon)^{\kappa-1}-1)\right]^{\ell-1}\right]\\
&\sim \varepsilon^{\ell-2}\left[\kappa^{\ell-1}-(\kappa-1)^{\ell-1}\right]. \qquad (7)
\end{aligned}
$$

This leads to the following theorem

## Theorem 4.1

*If $p = \varepsilon = o(1)$, the dominant (non-periodic) part of the probability $G(n, J(n) - \kappa, \ell)$ that $\mathcal{L}$: the number of survivors at step $J(n) - \kappa$, is given by $\ell$, (starting with n candidates), is asymptotically given by*

$$G(n, J(n) - \kappa, \ell) \sim R_{1,1}(\ell) + R_{1,2}(\ell) + R_2(\ell),$$

*with*

$$R_{1,1}(\ell) \sim \frac{\varepsilon^{2\ell-3}}{\ell}(\ell - 1)\left((\kappa - 1)\right)^{\ell-2},$$

$$R_{1,2}(\ell) \sim \sum_{1 \le k < \kappa} \varepsilon^{\ell}(\ell - 1)(\kappa - k - 1)^{\ell-2},$$

$$R_2(\ell) \sim \varepsilon^{\ell-2}\left[\kappa^{\ell-1} - (\kappa - 1)^{\ell-1}\right].$$

So $R_2(\ell)$ is dominant.

For $\ell = 2$, this leads respectively to

$$R_{1,1}(2) \sim \frac{\varepsilon}{2},$$
$$R_{1,2}(2) \sim \kappa\varepsilon^2,$$
$$R_2(2) \sim 1.$$

So, asymptotically, only 2 survivors remain at step $J(n) - \kappa$, independently of $\kappa$. We can check that, to first order, $\sum_2^\infty (R_{1,1}(\ell) + R_{1,2}(\ell) + R_2(\ell)) \sim 1$. Indeed, we obtain (here we need more precision in the asymptotics, we omit the details)

$$\sum_2^\infty R_{1,1}(\ell) \sim \frac{\varepsilon}{2},$$

$$\sum_2^\infty R_{1,2}(\ell) \sim \mathcal{O}(\kappa\varepsilon^2),$$

$$\sum_2^\infty R_2(\ell) \sim 1 - \frac{\varepsilon}{2} + \mathcal{O}(\kappa\varepsilon^2).$$

It is interesting to check our asymptotics by a direct more probabilistic approach. For $R_{1,1}(\ell)$, the probabilistic interpretation, by (3) is obvious. For $R_{1,2}(\ell)$, we have $\ell$ survivors at step $s = j - \kappa$ (no resurrections). Again, with no resurrection, frome these $\ell$ players, $i$ survive at position $j - k - 1$ and they are all killed at time $j - k$. The game ends at step $j$ in $k$ steps. Hence

$$R_{1,2}(\ell) \sim \sum_{s=1}^{\infty} \sum_{i=2}^{\ell} \sum_{1 \le k < \kappa} \binom{n}{\ell} \left(1 - e^{-s\varepsilon}\right)^{n-\ell} e^{-\ell s \varepsilon} \times$$

$$\times\, G(\ell, \kappa - k - 1, i)\varepsilon^i P(i, k),$$

and with $s\varepsilon = \eta + \ln(n - \ell)$,

$$R_{1,2}(\ell) \sim \sum_{s=1}^{\infty} \sum_{i=2}^{\ell} \sum_{1 \le k < \kappa} \binom{n}{\ell} \left(1 - \frac{e^{-\eta}}{n-\ell}\right)^{n-\ell} \frac{e^{-\ell\eta}}{(n-\ell)^{\ell}} \times$$

$$\times\, G(\ell, \kappa - k - 1, i)\varepsilon^i P(i, k),$$

and with Euler-MacLaurin,

$$R_{1,2}(\ell) \sim \int_{\eta=-\infty}^{\infty} \sum_{i=2}^{\ell} \sum_{1 \le k < \kappa} \frac{1}{\ell!} \exp(-e^{-\eta}) e^{-\ell\eta} \frac{d\eta}{\varepsilon} \times$$

$$\times\, G(\ell, \kappa - k - 1, i) \varepsilon^i P(i, k)$$

$$\sim \frac{1}{\varepsilon\ell} \sum_{i=2}^{\ell} \sum_{1 \le k < \kappa} \binom{\ell}{i} \left(1 - e^{-(\kappa-k-1)\varepsilon}\right)^{\ell-i} e^{-i(\kappa-k-1)\varepsilon} \varepsilon^i i(i-1) \times$$

$$\times \left(1 - e^{-(k-1)\varepsilon}\right)^{i-2} e^{-2(k-1)\varepsilon} \varepsilon$$

$$\sim \sum_{1 \le k < \kappa} \frac{1}{\varepsilon} (\ell - 1) \left[(\kappa - k - 1)\varepsilon\right]^{\ell-2} \varepsilon^3$$ which is identical to (6).

Concerning $R_2(\ell)$, we have $\ell$ survivors at step $s = j - \kappa$ (no resurrections). Again, with no resurrection, from these $\ell$ players, $v$ survive at position $j-1$ and they are all killed but 1 at time $j$. This leads to

$$R_2(\ell) \sim \sum_{s=1}^{\infty}\sum_{v=2}^{\ell}\binom{n}{\ell}\left(1-e^{-s\varepsilon}\right)^{n-\ell}e^{-\ell s\varepsilon}G(\ell,\kappa-1,v)v\varepsilon^{v-1}(1-\varepsilon)$$

$$\sim \frac{1}{\varepsilon\ell}\sum_{v=2}^{\ell}\binom{\ell}{v}\left(1-e^{-(\kappa-1)\varepsilon}\right)^{\ell-v}e^{-v(\kappa-1)\varepsilon}v\varepsilon^{v-1}(1-\varepsilon)$$

$$\sim \frac{1}{\varepsilon}\left[(\kappa-1)\varepsilon\right]^{\ell-2}\varepsilon\left[\kappa\left(\frac{\kappa}{\kappa-1}\right)^{\ell-1}+1-\kappa\right]$$

$$= \varepsilon^{\ell-2}\left[\kappa^{\ell-1}-(\kappa-1)^{\ell-1}\right] \text{ which is identical to (7).}$$

# Asymptotics for $p \to 1$

Here, we have

$$\begin{aligned} p &= 1 - \varepsilon, \\ q &= \varepsilon, \\ Q &= \frac{1}{\varepsilon}, \\ L &= -\ln(\varepsilon), \\ \log n^* &\sim \frac{\ln n}{-\ln(\varepsilon)} + \frac{\varepsilon}{\ln \varepsilon}. \end{aligned}$$

The distribution of $J(n)$ is concentrated at $\log n$.
Now we must distinguish between $G$ and $G^*$.

$$P(i,k) \sim (1-\varepsilon)^{i(k-1)} \binom{i}{1} (1-\varepsilon)^{i-1} \varepsilon \sim i\varepsilon e^{-\varepsilon ik}.$$

*Explanation*: the $i$ players are asymptotically all killed and resurrected during the first $k-1$ steps. At step $k$, only one survive.

$$G(i,k,\ell) \sim (1-\varepsilon)^{i(k-1)}\binom{i}{\ell}(1-\varepsilon)^{i-\ell}\varepsilon^{\ell} \sim e^{-\varepsilon ik}\varepsilon^{\ell}\binom{i}{\ell}e^{\varepsilon\ell},$$

$$G^*(i,k,\ell) \sim \binom{i}{\ell}(1-\varepsilon^k)^{i-\ell}\varepsilon^{k\ell} \sim e^{-\varepsilon^k(i-\ell)}.$$

Concerning $G$, the justification is similar to the one for $P$. Concerning $G^*$, there are $\ell$ survivors during $k$ steps (no resurrection), which leads to a Binomial distribution.

Now we turn to asymptotics for $R_{1,1}(\ell), R_{1,2}(\ell), R_2(\ell)$. From Theorem 3.1, we derive

$$R_{1,1}(\ell) \sim \sum_{k\geq\kappa}\sum_{i\geq\ell}\binom{i}{\ell}e^{-\varepsilon i(k-\kappa)}e^{\varepsilon\ell}\varepsilon^{\ell}e^{-\ell\kappa\varepsilon}\ell\varepsilon\frac{(1-\varepsilon)^i}{Li}$$

$$\sim \sum_{k\geq\kappa}e^{-\varepsilon\ell(k-\kappa)}\left[e^{\varepsilon(\kappa-k)}(\varepsilon-1)+1\right]^{-\ell}\varepsilon^{\ell}e^{-\ell\kappa\varepsilon}\frac{\varepsilon}{L},$$

and with $k=\kappa+u$,

$$R_{1,1}(\ell) \sim \int_0^\infty e^{-\varepsilon\ell u}\left[e^{-\varepsilon u}(\varepsilon-1)+1\right]^{-\ell}\frac{du}{\varepsilon}\varepsilon^{\ell}e^{-\ell\kappa\varepsilon}\frac{\varepsilon}{L}$$

$$\sim \varepsilon^{\ell}e^{-\ell\kappa\varepsilon}\frac{1}{L}(I_1+I_2),$$

$$I_1 \sim \int_0^{1/\varepsilon} (1 - \varepsilon \ell u) \left[(1+u)\varepsilon\right]^{-\ell} du$$
$$\sim \frac{\varepsilon^{-\ell}}{\ell - 1},$$
$$I_2 \sim \int_{1/\varepsilon}^{\infty} e^{-\varepsilon \ell u} \exp(e^{-\varepsilon u} \ell) du$$
$$\sim \int_0^{\exp(-1)} y^{\ell} e^{\ell y} \frac{dy}{y \varepsilon}$$
$$\sim \frac{C(\ell)}{\varepsilon}, \text{ with a rapidly decreasing function}$$
$$C(\ell) := (-\ell)^{-\ell}(\Gamma(\ell) - \Gamma(\ell, -\ell e^{-\ell})),$$

$I_1$ is dominant, hence

$$R_{1,1}(\ell) \sim \frac{e^{-\ell \kappa \varepsilon}}{L(\ell - 1)}.$$

$$R_{1,2}(\ell) \sim \sum_{i=2}^{\ell} \sum_{1 \le k < \kappa} \Gamma(\ell) \frac{1}{L i! (\ell - i)!} \varepsilon^{(\kappa-k-1)\ell} (1-\varepsilon)^{\ell} e^{-ik\varepsilon} i \varepsilon A^{\ell-i}, \tag{8}$$

with $A := \varepsilon^{k-\kappa+1} - 1$,

$$R_{1,2}(\ell) \sim \sum_{1 \le k < \kappa} \frac{1}{L} \varepsilon^{(\kappa-k-1)\ell} e^{-\varepsilon\ell} \varepsilon A^{\ell} B \frac{1}{A^2 e^{2k\varepsilon}},$$

with $B := (1 + e^{k\varepsilon} A) \left( \frac{1 + e^{k\varepsilon} A}{e^{k\varepsilon} A} \right)^{\ell-2} - e^{k\varepsilon} A,$

$$B \sim (1 + (1+k\varepsilon)A) \left( \frac{1 + (1+k\varepsilon)A}{(1+k\varepsilon)A} \right)^{\ell-2} - (1+k\varepsilon)A.$$

for $k = \kappa - 1, A = 0, \quad A^{\ell} B / A^2 \to 1,$ for $A \to 0,$

for $k < \kappa - 1, A \sim \varepsilon^{k-\kappa+1}, B \sim \ell - 1.$

$$
\begin{aligned}
R_{1,2}(\ell) &\sim S_1 + S_2, \\
S_1 &\sim \frac{1}{L} e^{-\varepsilon\ell} \varepsilon \varepsilon^{-2(\kappa-1)\varepsilon}, \\
S_2 &\sim \sum_{1 \le k < \kappa-1} \frac{1}{L} \varepsilon^{(\kappa-k-1)\ell} e^{-\varepsilon\ell} \varepsilon \varepsilon^{(k-\kappa+1)\ell} (\ell-1) \varepsilon^{-2(k-\kappa+1)} e^{-2k\varepsilon} \\
&\sim \frac{1}{L} e^{-\varepsilon\ell} (\ell-1) \varepsilon^3. \\
&S_2 \text{ is negligible wrt } S_1, \text{ hence} \\
R_{1,2}(\ell) &\sim \frac{1}{L} e^{-\varepsilon\ell} \varepsilon \varepsilon^{-2(\kappa-1)\varepsilon}.
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
R_2(\ell) &\sim (1-\varepsilon)^{\ell}\varepsilon^{(\kappa-1)\ell}\frac{\varepsilon}{(1-\varepsilon)L}\times \\
&\times\left[\left(1+\frac{1}{1-\varepsilon}(C-1)\right)^{\ell-1}-\left(\frac{1}{1-\varepsilon}(C-1)\right)^{\ell-1}\right], \quad (10) \\
&\text{with } C := \varepsilon^{1-\kappa}, \\
R_2(\ell) &\sim (1-\varepsilon)^{\ell}\varepsilon^{(\kappa-1)\ell}\frac{\varepsilon}{(1-\varepsilon)L}(\ell-1)C^{\ell-2} \\
&\sim \frac{1}{L}\varepsilon^{2\kappa-1}(\ell-1)e^{-(\ell-1)\varepsilon}.
\end{aligned}
$$

This leads to the following theorem

## Theorem 5.1

*If $p = 1 - \varepsilon, \varepsilon = o(1)$, the dominant (non-periodic) part of the probability $G(n, J(n) - \kappa, \ell)$ that $\mathcal{L}$: the number of survivors at step $J(n) - \kappa$, is given by $\ell$, (starting with n candidates), is asymptotically given by*

$$G(n, J(n) - \kappa, \ell) \sim R_{1,1}(\ell) + R_{1,2}(\ell) + R_2(\ell),$$

*with*

$$R_{1,1}(\ell) \sim \frac{e^{-\ell\kappa\varepsilon}}{L(\ell - 1)},$$
$$R_{1,2}(\ell) \sim \frac{1}{L} e^{-\varepsilon\ell} \varepsilon\varepsilon^{-2(\kappa-1)\varepsilon},$$
$$R_2(\ell) \sim \frac{1}{L} \varepsilon^{2\kappa-1} (\ell - 1) e^{-(\ell-1)\varepsilon}.$$

So $R_{1,1}(\ell)$ is dominant.

We can check that, to first order, $\sum_2^\infty (R_{1,1}(\ell) + R_{1,2}(\ell) + R_2(\ell)) \sim 1$. Indeed, we obtain

$$\sum_2^\infty R_{1,1}(\ell) \sim \frac{-\ln(\varepsilon\kappa)}{L} = 1 + \mathcal{O}(\frac{1}{L}),$$

$$\sum_2^\infty R_{1,2}(\ell) \sim \mathcal{O}(\frac{1}{L}),$$

$$\text{if } \kappa > 1, \sum_2^\infty R_2(\ell) \sim \frac{1}{L}\varepsilon^{2\kappa-3},$$

$$\text{if } \kappa = 1, \sum_2^\infty R_2(\ell) \sim \frac{1}{L}.$$

It is again interesting to check our asymptotics by a direct more probabilistic approach. For $R_{1,1}(\ell)$, the probabilistic interpretation, by (3) is again obvious. For $R_{1,2}(\ell)$, we need the following observation. For $i = \mathcal{O}(1)$, the probability that there are $i$ survivors from $n$ players, at time $s$, without resurrection, is given by $G^*(n, s, i)$, and

$$\sum_s G^*(n,s,i) = \sum_s \binom{n}{i}(1-\varepsilon^s)^{n-i}\varepsilon^{si}$$
$$= \sum_s \binom{n}{i}(1-e^{-Ls})^{n-i}e^{-Lsi},$$

and with $Ls = \ln(n-i) + \eta$,

$$\sum_s G^*(n,s,i) = \sum_s \binom{n}{i}\left(1 - \frac{e^{-\eta}}{n-i}\right)^{n-i} \frac{e^{-i\eta}}{(n-i)^i}$$

$$\sim \sum_s \exp(-e^{-\eta}) \frac{e^{-i\eta}}{i!}$$

$$\sim \int_{-\infty}^{\infty} \exp(-e^{-\eta}) \frac{e^{-i\eta}}{i!} \frac{d\eta}{L} = \frac{1}{iL}. \qquad (11)$$

For $R_{1,2}(\ell)$, we have $\ell$ survivors at step $s = j - \kappa$ (no resurrections). Again, with no resurrection, from these $\ell$ players, $i$ survive during the next $\kappa - k - 1$ steps and they are all killed at time $j - k$. The game ends at step $j$ in $k$ steps. Hence, by (11),

$$R_{1,2}(\ell) \sim \sum_{i=2}^{\ell} \sum_{1 \leq k < \kappa} \frac{1}{L\ell} G^*(\ell, \kappa - k - 1, i)(1-\varepsilon)^i P(i,k)$$

$$\sim \sum_{i=2}^{\ell} \sum_{1 \leq k < \kappa} \frac{1}{L\ell} \binom{\ell}{i} \varepsilon^{(\kappa-k-1)i} \left(1 - \varepsilon^{\kappa-k-1}\right)^{\ell-i} (1-\varepsilon)^i i\varepsilon e^{-\varepsilon ik},$$

which is equivalent to (8), after some algebra.

Similarly, for $R_2(\ell)$, we have $\ell$ survivors at step $s = j - \kappa$ (no resurrections). Again, with no resurrection, from these $\ell$ players, $v$ survive during the next $\kappa - 1$ steps and they are all killed but 1 at time $j$. This leads to

$$
\begin{aligned}
R_2(\ell) &\sim \frac{1}{\ell L}\sum_{v=2}^{\ell} G^*(\ell,\kappa-1,v)v(1-\varepsilon)^{v-1}\varepsilon \\
&\sim \frac{1}{\ell L}\sum_{v=2}^{\ell}\binom{\ell}{v}\varepsilon^{(\kappa-1)v}\left(1-\varepsilon^{\kappa-1}\right)^{\ell-v} v(1-\varepsilon)^{v-1}\varepsilon \\
&\sim \frac{\varepsilon}{L}D((D-D\varepsilon+E)^{\ell-1}-E^{\ell-1}), \\
&\text{with } D := \varepsilon^{\kappa-1}, \quad E := 1-\varepsilon^{\kappa-1},
\end{aligned}
$$

$$(1-\varepsilon)^{\ell-1}\varepsilon^{\kappa-1}\left[\left(\varepsilon^{\kappa-1}+\frac{1}{1-\varepsilon}(1-\varepsilon^{\kappa-1})\right)^{\ell-1}-\left(\frac{1}{1-\varepsilon}(1-\varepsilon^{\kappa-1})\right)^{\ell-1}\right.$$

$$=\varepsilon^{\kappa-1}\left[\left(\varepsilon^{\kappa-1}(1-\varepsilon)+(1-\varepsilon^{\kappa-1})\right)^{\ell-1}-\left(1-\varepsilon^{\kappa-1}\right)^{\ell-1}\right].$$

which is equivalent to (10), after some algebra.

# Conclusion

This paper is devoted to another application of some tools related to urn model, Mellin-Laplace technique for Harmonic sums and Gumbel distribution. The asymptotic distributions of the number of survivors near the end of the classical leader election algorithm can be derived in an almost mechanical way. The analysis of some other election algorithms will be the object of future work.

J.A. Fill, H. Mahmoud, and W. Szpankowski.
On the distribution for the duration of a randomized leader election algorithm.
*Annals of Applied Probability*, 6:1260–1283, 1996.

P. Flajolet, X. Gourdon, and P. Dumas.
Mellin transforms and asymptotics: Harmonic sums.
*Theoretical Computer Science*, 144:3–58, 1995.

P. Hitczenko and G. Louchard.
Distinctness of compositions of an integer: a probabilistic analysis.
*Random Structures and Algorithms*, 19(3,4):407–437, 2001.

S. Janson, C. Lavault, and G. Louchard.
Convergence of some leader election algorithms.
*Discrete Mathematics and Theoretical Computer Science*, 10,3:171–196, 2008.

S. Janson and W. Szpankowski.

Analysis of an asymmetric leader election algorithm.
*Electronic Journal of Combinatorics*, 4(R17):1–16, 1997.

R. Kalpathy.
Perpetuities in fair leader election algorithms. ph.d. dissertation.
Technical report, Department of Statistics, The George Washington University, Washington, D.C., 2013.

R. Kalpathy and H. Mahmoud.
Perpetuities in fair leader election algorithms.
*Advances in Applied Probability*, 46, in press, 2014.

R. Kalpathy, H. Mahmoud, and W. Rosenkrantz.
Survivors in leader election algorithms.
*Statistics and Probability Letters*, to appear, 2014.

R. Kalpathy, H.M. Mahmoud, and M.D. Ward.
Asymptotic properties of a leader election algorithm.
*Journal of Applied Probability*, 48:569–575, 2011.

C. Knessl.
Asymptotics and numerical studies of the leader election algorithm.
*European Journal of Applied Mathematics*, 12:645–664, 2001.

C. Lavault and G. Louchard.
Asymptotic analysis of a leader election algorithms.
*Theoretical Computer Science*, 359,1,3:239–254, 2006.

G. Louchard, C. Martinez, and H. Prodinger.
The swedish leader election protocol: Analysis and variations.
In *Proceedings ANALCO 2011*, pages 127–134, 2011.

G. Louchard and H. Prodinger.
Asymptotics of the moments of extreme-value related distribution functions.
*Algorithmica*, 46:431–467, 2006.
Long version:
http://www.ulb.ac.be/di/mcs/louchard/moml.ps.

G. Louchard and H. Prodinger.
On gaps and unoccupied urns in sequences of geometrically distributed random variables.
*Discrete Mathematics*, 308,9:1538–1562, 2008.
Long version:
http://www.ulb.ac.be/di/mcs/louchard/gaps18.ps.

G. Louchard and H. Prodinger.
The asymmetric leader election algorithm: Another approach.
*Annals of Combinatorics*, 12:449–478, 2009.

G. Louchard and H. Prodinger.
The asymmetric leader election algorithm with swedish stopping: A probabilistic analysis.
*Discrete Mathematics and Theoretical Computer Science*, 14,2:91–128, 2012.

G. Louchard, H. Prodinger, and M. D. Ward.
Number of survivors in the presence of a demon.

*Periodica Mathematica Hungarica*, 64,1:101–117, 2012.

G. Louchard, H. Prodinger, and M.D. Ward.
The number of distinct values of some multiplicity in sequences of geometrically distributed random variables.
*Discrete Mathematics and Theoretical Computer Science*, AD:231–256, 2005.
2005 International Conference on Analysis of Algorithms.

H. Prodinger.
How to select a loser.
*Discrete Mathematics*, 120:149–159, 1993.