Cumulants mixtes et arbres couvrants

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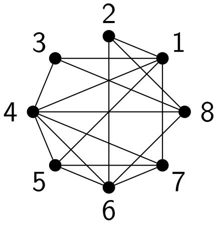
A problem in random graphs

Erdős-Rényi model of random graphs $G(n,p)$:

- G has n vertices labelled $1,\dots,n$;
- each possible edge (i,j) belongs to G independently with probability p;



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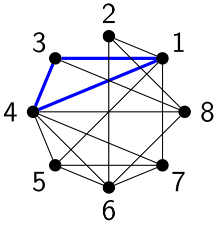

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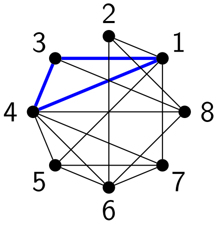

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Describe asymptotically the fluctuations of the number T_n of triangles.

Answer (Ruciński, 1988)

The fluctuations are asymptotically Gaussian.

A good tool for that: mixed cumulants

- the r-th mixed cumulant k_r of r random variables is a r-linear symmetric functional. It is a polynomial in joint moments. Examples:

$$\kappa_1(X) = \mathbb{E}(X), \quad \kappa_2(X,Y) = \mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$
$$\kappa_3(X,Y,Z) = \mathbb{E}(XYZ) - \mathbb{E}(XY)\mathbb{E}(Z) - \mathbb{E}(XZ)\mathbb{E}(Y) - \mathbb{E}(YZ)\mathbb{E}(X) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z).$$

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- if the variables can be split in two mutually independent sets, then the cumulant vanishes.

- if, for each $r \neq 2$, the sequence $\kappa_r(X_n) := \kappa_r(X_n, \dots, X_n)$ converges towards 0 and if $\mathbb{E}(X_n)$ and $\mathrm{Var}(X_n)$ have a limit, then X_n converges in distribution towards a Gaussian law.

Application to the number of triangles

$$T_n = \sum_{\Delta=\{i,j,k\}\subset[n]} B_\Delta, \text{ where } B_\Delta(G) = \begin{cases} 1 & \text{if } G \text{ contains the triangle } \Delta; \\ 0 & \text{otherwise.} \end{cases}$$

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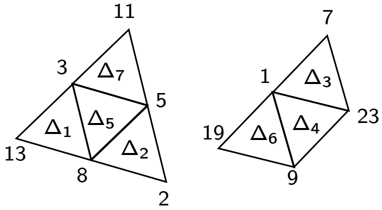

$\{\Delta_1, \Delta_2, \Delta_5, \Delta_7\}$ is independent from $\{\Delta_3, \Delta_4, \Delta_6\}$.
Reminder: different edges being in the graph are independent events.

$$\kappa_\ell(B_{\Delta_1}, \dots, B_{\Delta_7}) = 0.$$

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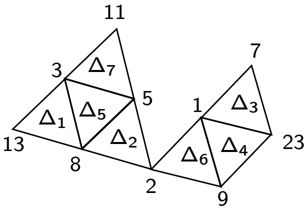

$\{\Delta_1,\Delta_2,\Delta_5,\Delta_7\}$ is independent from $\{\Delta_3,\Delta_4,\Delta_6\}$. Triangles need to share an edge to be dependent!

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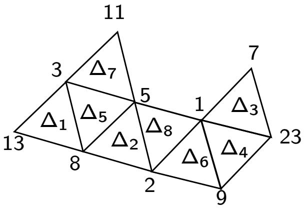

This configuration contributes to the sum. Note that it has only $\ell + 2$ vertices.

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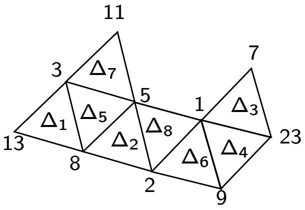

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$$|\kappa_\ell(T_n)| = O_\ell(n^{\ell+2})$$

The central limit theorem for triangles

Proposition (Leonov, Shirryaev, 1955)

If $X_1, \ldots, X_\ell$ can be split into two sets of mutually independent variables, then

$$\kappa_\ell(X_1, \cdots, X_\ell) = 0$$

Corollary (Janson, 1988 ?)

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Corollary (Ruciński, 1988)

$$\frac{T_n - \mathbb{E}(T_n)}{\sqrt{\mathrm{Var}(T_n)}} \to \mathcal{N}(0,1)$$

Proof: $\mathrm{Var}(T_n) \approx n^4$ and $\kappa_\ell(T_n/n^2) = n^{2-\ell} = o_\ell(1)$ for $\ell > 2$.

Our work

Theorem (F., Méliot, Nighekbali, 2014)

Let $X_1, \ldots, X_\ell$ be random variables with finite moments of order ℓ,

$$|\kappa_\ell(X_1, \cdots, X_\ell)| \leq 2^{\ell-1} \|X_1\|_\ell \cdots \|X_\ell\|_\ell \cdot \mathrm{ST}\left(G_{\mathrm{dep}}(X_1, \cdots, X_\ell)\right),$$

where $\mathrm{ST}\left(G_{\mathrm{dep}}(X_1, \cdots, X_\ell)\right)$ is the number of spanning trees of the dependency graph of $X_1, \cdots, X_\ell$.

Dependency graphs for a list $(B_{\Delta_1}, \cdots, B_{\Delta_\ell})$:

$$B_{\Delta_i} \sim B_{\Delta_j} \Leftrightarrow \Delta_i \text{ and } \Delta_j \text{ share an edge}$$

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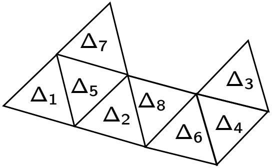

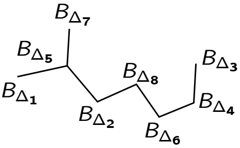

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Naive bound:

$$\ell^\ell (\ell-1)! \|X_1\|_\ell \cdots \|X_\ell\|_\ell$$

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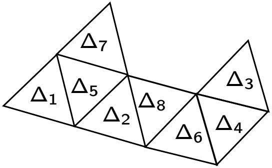

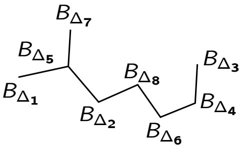

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Corollary (FMN, 2014)

There exists an absolute constant C such that

$$|\kappa_\ell(T_n)| = (C\ell)^\ell n^{\ell+2}$$

Bound in Janson's proof: $(C\ell)^{3\ell} n^{\ell+2}$ (Döring, Eichelsbacher, 2012)

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Corollary (FMN, 2014)

Very precise extension of the central limit theorem: if $1 << v << n^{1/2}$,

$$\mathbb{P}\left[T_n \geq \binom{n}{3} p^3 + v \cdot n^2\right] \sim \frac{1}{\sqrt{\pi\, p^5 (1-p) v^2}} \exp\left(-\frac{v^2}{p^5\,(1-p)} + \frac{(7-8p)\,v^3}{n\sqrt{p(1-p)/2}}\right)$$

Moment-cumulant relation

Mixed cumulants can be expressed in terms of mixed moments:

$$\kappa(X_1, \dots, X_r) = \sum_{\pi} \mu(\pi) M_\pi,$$

where

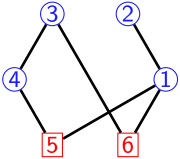
- π runs over set-partitions of $[\ell]$,
- $\mu(\pi) = \mu(\pi, \{[\ell]\})$ is the Möbius function of the set-partition poset (it is explicit!),
- $M_\pi = \prod_{B \in \pi} \mathbb{E}\left[\prod_{i \in B} X_i\right]$.

Example:

$$\kappa_3(X, Y, Z) = \mathbb{E}(XYZ) - \mathbb{E}(XY)\mathbb{E}(Z) - \mathbb{E}(XZ)\mathbb{E}(Y) - \mathbb{E}(YZ)\mathbb{E}(X) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z).$$

Using independence to simplify M_π

Example: $\pi = \big\{\{1,2,3,4\},\{5,6\}\big\}$ and

$H := G_{\text{dep}}(X_1,\ldots,X_6) =$ 

$$\text{Then } M_\pi := \mathbb{E}(X_1X_2X_3X_4)\mathbb{E}(X_5X_6)$$
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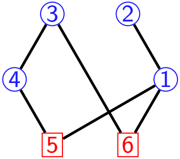

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In general, $M_\pi = M_{\phi_H(\pi)}$, with the following definition of $\phi_H(\pi)$.

replace each part π_i of π by the connected components of the induced graph $H[\pi_i]$.

Rewriting the summation

$$\kappa(X_1, \ldots, X_r) = \sum_{\pi} \mu(\pi) M_\pi = \sum_{\pi} \mu(\pi) M_{\phi_H(\pi)}$$
$$= \sum_{\pi'} M_{\pi'} \left(\sum_{\substack{\pi \text{ s.t.} \\ \phi_H(\pi) = \pi'}} \mu(\pi) \right)$$

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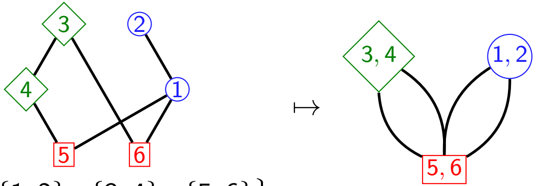
- $\phi_H(\pi) = \pi' \;\Rightarrow$ for all part π'_i of π', the induced graph $H[\pi'_i]$ is connected.
- if so, we have to compute

$$\alpha_H^{\pi'} := \sum_{\substack{\pi \text{ s.t.} \\ \phi_H(\pi)=\pi'}} \mu(\pi).$$

$\alpha_H^{\pi'}$ and Tutte polynomial

Consider the contracted graph H/π. Example:



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It is a multigraph.

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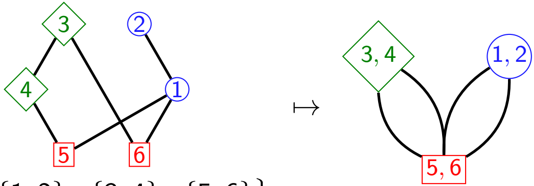

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Lemma

$$\alpha_H^{\pi'} = \sum_{E \subset E(H/\pi')} (-1)^{|E|},$$

where the sum runs over spanning connected subgraphs of H/π'.

If H/π' is connected, $\left|\alpha_H^{\pi'}\right|$ is Tutte polynomial evaluated at $(1,0)$.

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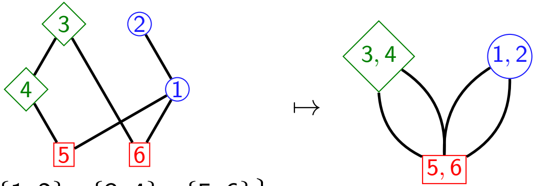

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Corollary: $\left|\alpha_H^{\pi'}\right| \leq \mathrm{ST}(H/\pi')$.

Bounding everything

Reminder:

$$\kappa(X_1, \dots, X_\ell) = \sum_{\pi'} M_{\pi'} \alpha_H^{\pi'}$$

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We have the following inequalities

$$\begin{aligned} |M_\pi| &\leq \|X_1\|_\ell \cdots \|X_\ell\|_\ell \quad \text{(Hölder inequality)}; \\ \left|\alpha_H^{\pi'}\right| &\leq \mathsf{ST}(H/\pi'); \\ \mathbf{1}_{H[\pi'_i] \text{ connected}} &\leq \mathsf{ST}(H[\pi'_i]) \end{aligned}$$

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Thus

$$|\kappa(X_1, \dots, X_\ell)| \leq \|X_1\|_\ell \cdots \|X_\ell\|_\ell \left[\sum_{\pi'} \mathrm{ST}(H/\pi') \left(\prod_i \mathrm{ST}(H[\pi'_i]) \right) \right]$$

A combinatorial identity

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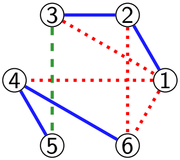

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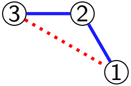

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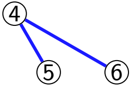

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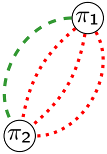

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A precise bound on cumulants of T_n

Recall that $\kappa_\ell(T_n) = \sum_{\Delta_1,\dots,\Delta_\ell} \kappa_\ell(B_{\Delta_1}, \dots, B_{\Delta_\ell})$.

Thus

$$|\kappa_\ell(T_n)| \leq \sum_{\Delta_1,\dots,\Delta_\ell} 2^{\ell-1} \cdot \mathrm{ST}\left(G_{\mathrm{dep}}(B_{\Delta_1,\dots,\Delta_\ell})\right).$$

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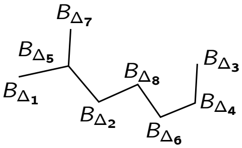

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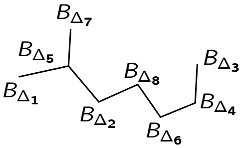

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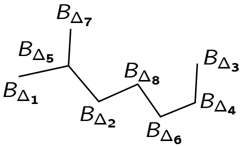

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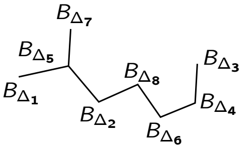

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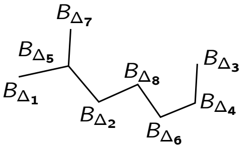

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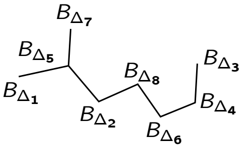

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- ... $\quad |\kappa_\ell(T_n)| \le \ell^{\ell-2}\binom{n}{3}(6n-12)^{\ell-1} \le (6\ell)^\ell n^{\ell+2}$

Moderate deviations

Let $X_n = (T_n - \mathbb{E}(T_n))/n^{5/3}$, then

$$\begin{aligned}\log \mathbb{E}(\exp(zX_n)) &= \sum_{\ell \geq 2} \kappa^{(\ell)}(X_n) z^\ell / \ell! \\ &= n^{2/3}\sigma^2 z^2/2 + L z^3/6 + \underbrace{\sum_{\ell \geq 4} n^{5/3}\, \kappa^{(\ell)}(T_n)\, z^\ell/\ell!}_{\text{call it } R(z)} + O(n^{-1/3})\end{aligned}$$

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Then use $\mathbb{P}[X \geq x] = \lim_{R \to \infty} \left(\frac{1}{2\pi} \int_{-R}^{R} \frac{\exp(-x(h+\mathrm{i}u))}{h+\mathrm{i}u}\, \mathbb{E}\big(\exp((h+\mathrm{i}u)X)\big)\, du \right)$.

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$\rightarrow$ looks like, but is stronger than the hypotheses in Hwang's quasi-power theorem (convergence on $\mathbb{C}$!) $\Rightarrow$ stronger results.

Conclusion

- very general bound on mixed cumulants, with a strong combinatorial flavor ;
- implies a good uniform bound on cumulants of sums of partially dependent random variables (number of copies of subgraphs, character of a random irreducible representation, . . .) ;
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Questions:

- Large deviations $\mathbb{P}(T_n \geq \mathbb{E}(T_n) + v\, n^3) \sim$? ;
- other models: $p_n \to 0$, $G(n, M)$ (fixed number of edges $\Rightarrow$ almost-independence).