Combinatorial specifications of permutation classes, via their decomposition trees

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talk based on joint works with
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Combinatorial specifications and trees

Combinatorial specifications and their byproducts

[Flajolet & Sedgewick 09]

A combinatorial specification describes (most of the time, recursively) a combinatorial class $\mathcal{C}$ (= a family of discrete objects) by ways of atoms and admissible constructions, like disjoint union, product, sequence, ...

Examples:

$\mathcal{D} = \varepsilon + u\mathcal{D}d\mathcal{D}$;

$$\begin{cases} \mathcal{T} &= \mathcal{U} + \mathcal{B} \\ \mathcal{U} &= \bullet + \begin{array}{c}\bullet \\ | \\ \mathcal{B}\end{array} \\ \mathcal{B} &= \circ + \begin{array}{c}\circ \\ / \quad \backslash \\ \mathcal{U} \quad \mathcal{U}\end{array} \end{cases}$$

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$$\begin{cases} \mathcal{A}_1 = \Phi_1(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p) \\ \mathcal{A}_2 = \Phi_2(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p) \\ \dots \\ \mathcal{A}_p = \Phi_p(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p) \end{cases}$$

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Systematic transcription of a specification into:

- System of equations for the generating function $C(z) = \sum c_n z^n$ [Flajolet & Sedgewick 09]
- Recursive [Flajolet, Zimmerman & Van Cutsem 94] and Boltzmann random samplers [Duchon, Flajolet, Louchard & Schaeffer 04]

Combinatorial specifications of trees

Consider classes of (unlabeled ordered) trees, with nodes from a (finite) set, possibly with some restrictions on the children of a node.

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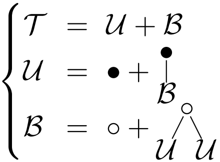

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"*Trees are the prototypical recursive structure*" [Flajolet & Sedgewick 09]

They are (one of) the most studied combinatorial objects, and a lot is known about them, both for specific classes of trees, but also for families of classes of trees.

Substitution decomposition and decomposition trees

Substitution decomposition of combinatorial objects

Combinatorial analogue of the decomposition of integers as products of primes. Applies to relations, graphs, posets, boolean functions, set systems, . . . and permutations [Möhring & Radermacher 84]

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Relies on:

- a principle for building objects (permutations, graphs) from smaller objects: the substitution
- some "basic objects" for this construction: **simple** permutations, **prime** graphs

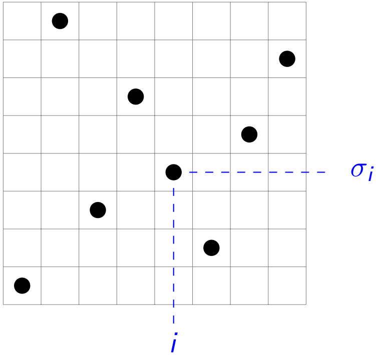
Required properties:

- every object can be (recursively) decomposed using only "basic objects"
- this decomposition is unique

Permutations

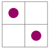
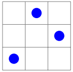
Permutation of size n = Bijection from $[1..n]$ to itself.
Set $\mathfrak{S}_n$, and $\mathfrak{S} = \cup_n \mathfrak{S}_n$.

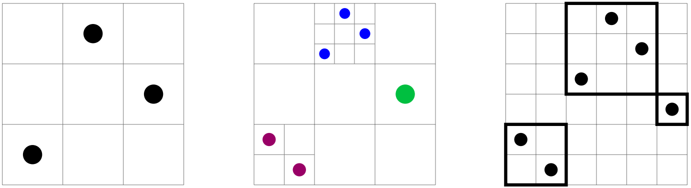
- Two lines notation:
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 4 & 2 & 5 & 7 \end{pmatrix}$$
- Linear notation:
$\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$
- Description as a product of cycles:
$\sigma = (1)\ (2\ 8\ 7\ 5\ 4\ 6)\ (3)$
- Graphical description, or diagram:



Substitution for permutations

Substitution or inflation : $\sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$.

Example: Here, $\pi = 1\,3\,2$, and
$\alpha^{(1)} = 2\,1 =$ 
$\alpha^{(2)} = 1\,3\,2 =$ 
$\alpha^{(3)} = 1 =$  .



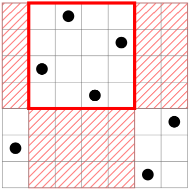
Hence $\sigma = 1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

Simple permutations

Interval (or block) = set of elements of σ whose positions **and** values form intervals of integers
Example: 5 7 4 6 is an interval of 2 5 7 4 6 1 3

Simple permutation = permutation with no interval, except the trivial ones: $1, 2, \ldots, n$ and σ
Example: 3 1 7 4 6 2 5 is simple

Not simple:



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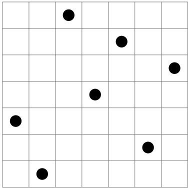

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The smallest simple permutations:
12, 21,   2413, 3142, 6 of size 5, . . .
Remark:
It is convenient to consider 12 and 21 **not** simple.

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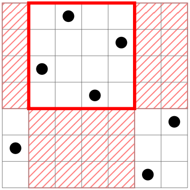

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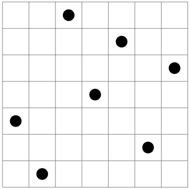

Substitution decomposition theorem(s) for permutations

Theorem: [Albert, Atkinson & Klazar 03]
Every σ ($\neq 1$) is uniquely decomposed as

- $12[\alpha^{(1)}, \alpha^{(2)}] = \oplus[\alpha^{(1)}, \alpha^{(2)}]$, where $\alpha^{(1)}$ is $\oplus$-indecomposable
- $21[\alpha^{(1)}, \alpha^{(2)}] = \ominus[\alpha^{(1)}, \alpha^{(2)}]$, where $\alpha^{(1)}$ is $\ominus$-indecomposable
- $\pi[\alpha^{(1)}, \dots, \alpha^{(k)}]$, where π is simple of size $k \geq 4$

Notations:

- $\oplus$-indecomposable: that cannot be written as $\oplus[\beta^{(1)}, \beta^{(2)}]$
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Observation: Equivalently, we may replace the first two items by

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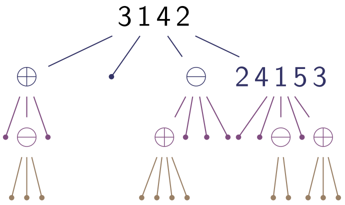
Decomposing recursively inside the $\alpha^{(i)}$ $\Rightarrow$ **decomposition tree**

Decomposition tree: witness of this decomposition

Example: Decomposition tree of

$\sigma =$ 10 13 12 11 14 1 18 19 20 21 17 16 15 4 8 3 2 9 5 6 7



Notations and properties:

- $\oplus = 12 \ldots k$, $\ominus = k \ldots 21$ $=$ linear nodes.
- π simple of size ≥ 4 $=$ prime node.
- No edge $\oplus - \oplus$ nor $\ominus - \ominus$.
- Rooted ordered trees.
- These conditions characterize decomposition trees.

$\sigma = 3\,1\,4\,2[\oplus[1, \ominus[1, 1, 1], 1], 1, \ominus[\oplus[1, 1, 1, 1], 1, 1, 1], 2\,4\,1\,5\,3[1, 1, \ominus[1, 1], 1, \oplus[1, 1, 1]]]$

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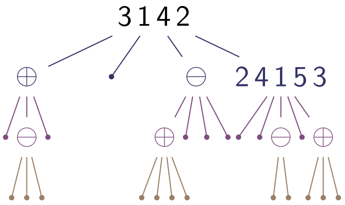

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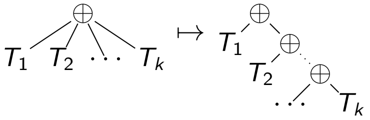
Observation: Adapts to binary case *via*



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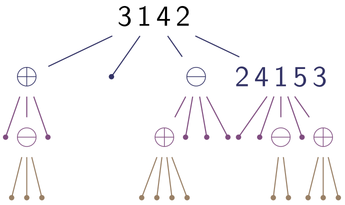

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Bijection between permutations and their decomposition trees.

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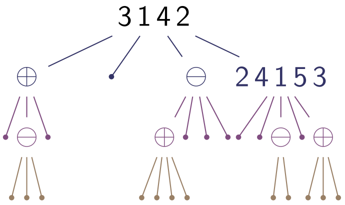

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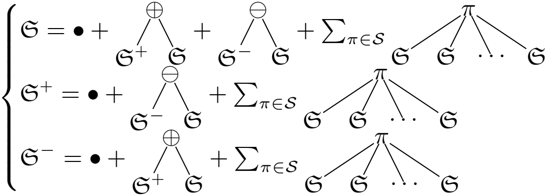
Bijection between permutations and their decomposition trees.

Computation: Linear time algorithm [Uno & Yagiura 00] [Bui Xuan, Habib & Paul 05] [Bergeron, Chauve, Montgolfier & Raffinot 08]

A tree grammar for permutations

$\mathcal{S}$ denotes the set of simple permutations



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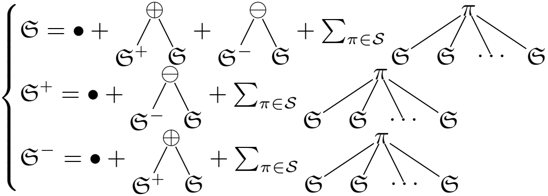

Allows to relate the (ordinary) generating function for simples with that of all permutations ($F(z) = \sum n! z^n$) [Albert, Atkinson & Klazar 03]:

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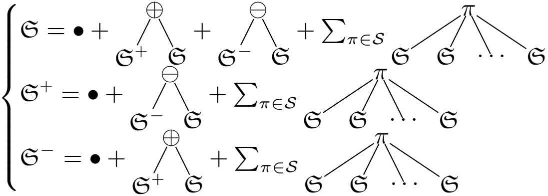

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Consequences for the enumeration of simple permutations:

- Asymptotically $\frac{n!}{e^2}$, but no exact enumeration.
- The generating function is not D-finite.

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$$\begin{cases} \mathfrak{S} = \bullet + \oplus(\mathfrak{S}^+, \mathfrak{S}) + \ominus(\mathfrak{S}^-, \mathfrak{S}) + \sum_{\pi \in \mathcal{S}} \pi(\mathfrak{S}, \mathfrak{S}, \dots, \mathfrak{S}) \\ \mathfrak{S}^+ = \bullet + \ominus(\mathfrak{S}^-, \mathfrak{S}) + \sum_{\pi \in \mathcal{S}} \pi(\mathfrak{S}, \mathfrak{S}, \dots, \mathfrak{S}) \\ \mathfrak{S}^- = \bullet + \oplus(\mathfrak{S}^+, \mathfrak{S}) + \sum_{\pi \in \mathcal{S}} \pi(\mathfrak{S}, \mathfrak{S}, \dots, \mathfrak{S}) \end{cases}$$

Can we specialize this tree grammar to subsets of $\mathfrak{S}$, and in particular to permutation classes $\mathcal{C}$?

Can we do it automatically? even algorithmically?

Yes, when the number of simple permutations in $\mathcal{C}$ is finite.

Permutation patterns
and permutation classes

Permutation patterns

Pattern relation $\preccurlyeq$:
$\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists\ 1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is in the same relative order ($\equiv$) as π.

Notation: $\pi \preccurlyeq \sigma$.

Equivalently:
The normalization of $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields π.

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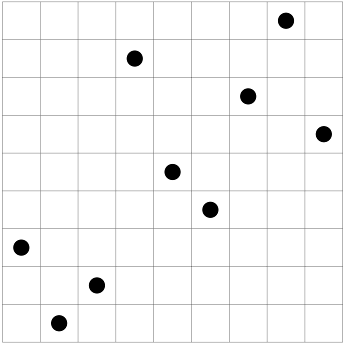

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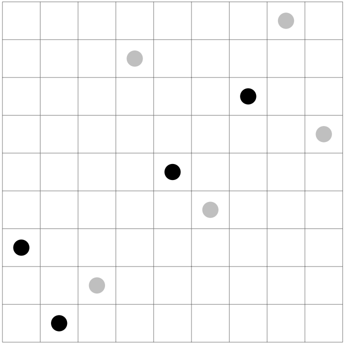

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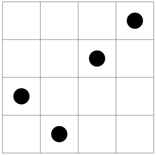

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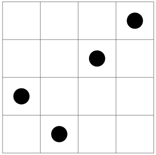

Observation: $\preccurlyeq$ is a partial order on $\mathfrak{S} = \bigcup_n \mathfrak{S}_n$.

This is the key to defining permutation classes.

Permutation classes

• A **permutation class** is a set $\mathcal{C}$ of permutations that is downward closed for $\preccurlyeq$, i.e. whenever $\pi \preccurlyeq \sigma$ and $\sigma \in \mathcal{C}$, then $\pi \in \mathcal{C}$.

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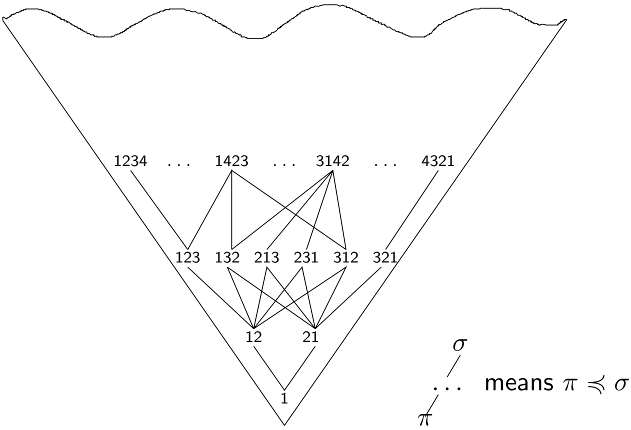

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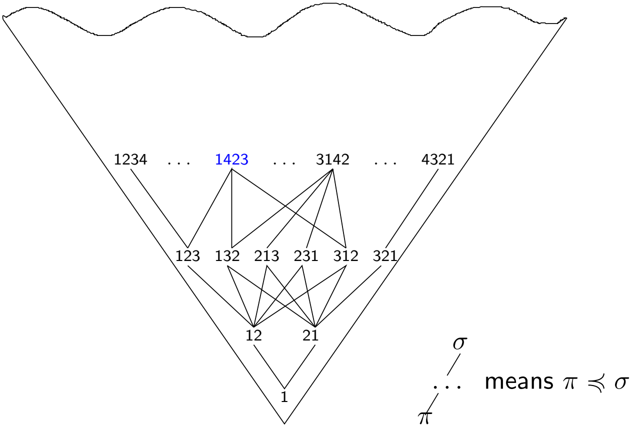
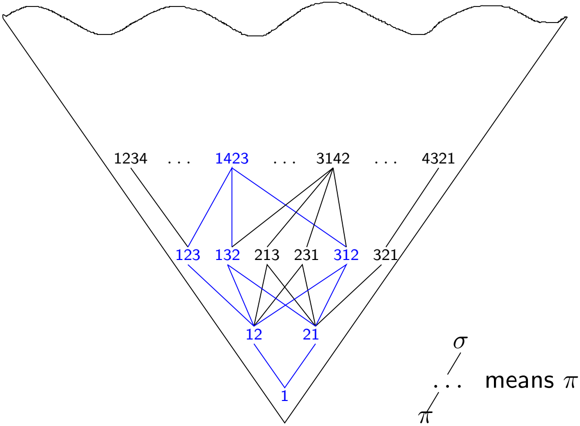


Permutation classes

• A **permutation class** is a set $\mathcal{C}$ of permutations that is downward closed for $\preccurlyeq$, i.e. whenever $\pi \preccurlyeq \sigma$ and $\sigma \in \mathcal{C}$, then $\pi \in \mathcal{C}$.



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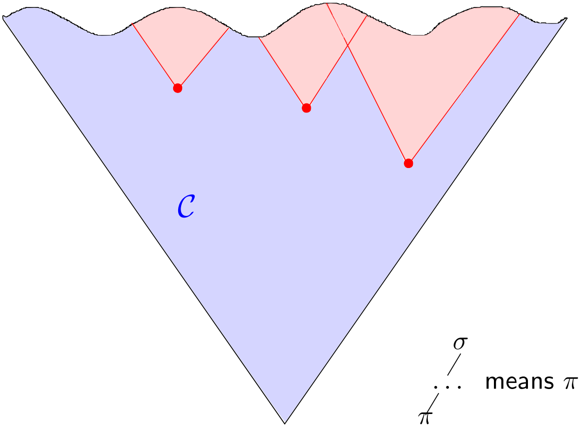

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- Fact: For every permutation class $\mathcal{C}$, $\mathcal{C} = Av(B)$ for
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- Observations:
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  - In this talk, we focus on classes with finite basis.

Main steps of an algorithm to compute a specification

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Remark: Substitution-closed classes are a special (and easier) case.

One more definition: substitution-closed classes

Def.: A permutation class $\mathcal{C}$ is **substitution-closed** when $\pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}] \in \mathcal{C}$ for all $\pi, \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)} \in \mathcal{C}$.

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From the finite basis of $\mathcal{C}$ to the simple permutations in $\mathcal{C}$

Characterizing when a class contains finitely many simples

Theorem [Brignall, Huczynska & Vatter 08]:
$\mathcal{C} = \mathsf{Av}(B)$ contains finitely many simple permutations iff $\mathcal{C}$ contains:

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Goal: Give an efficient algorithm instead.

Testing whether $\mathcal{C} = \mathrm{Av}(B)$ contains finitely many simples

Easy part: testing whether $\mathcal{C}$ contains finitely many parallel alternations and finitely many wedge simple permutations

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$\hookrightarrow$ Improvement from effective and recursive construction of deterministic and complete automata

- in $\mathcal{O}(n + s^{2k}) \;= \mathcal{O}(n + 2^{k \cdot 2 \log s})$ [Bassino, Bouvel, Pierrot & Rossin 14+]
- in $\mathcal{O}(n)$ if $\mathcal{C}$ is substitution-closed [Bassino, Bouvel, Pierrot & Rossin 10]

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Computing the set $\mathcal{S}_\mathcal{C}$ of simple permutations in $\mathcal{C}$ ...

(... assuming that $\mathcal{S}_\mathcal{C}$ is finite.)

Basic idea: Compute $\mathcal{S}_{\mathcal{C},n} = \mathcal{S}_\mathcal{C} \cap \mathfrak{S}_n$, for increasing n.
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Algorithm to compute $\mathcal{S}_\mathcal{C}$:

- Naive algorithm: $\mathcal{O}(\sum_{j=1..\ell+2} j! j^{p+1} \cdot |B|)$
- Improved algorithm for substitution-closed classes: $\mathcal{O}(N \cdot \ell^4)$
  Using properties of $\preccurlyeq$ on simple permutations [Pierrot & Rossin 14+]
- Adaptation to non substitution-closed classes: $\mathcal{O}(N \cdot \ell^{p+2} \cdot |B|)$

where $N = |\mathcal{S}_\mathcal{C}|$, $p = \max_{\beta \in B} |\beta|$, $\ell = \max_{\pi \in \mathcal{S}_\mathcal{C}} |\pi|$.

From the basis of $\mathcal{C}$ and the simples in $\mathcal{C}$ to a combinatorial specification for $\mathcal{C}$

From a constructive proof to an algorithm

Theorem:
If $\mathcal{C}$ contains a finite number of simple permutations, then it has a finite basis and an algebraic generating function $C(z)$. [Albert, Atkinson 2005]

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- Specialize the substitution decomposition theorem to $\mathcal{C}$.
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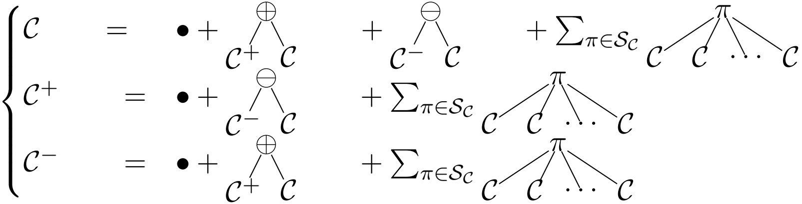
Remark (on the *finite basis* part of the theorem): The real restriction is not having a finite basis, but rather containing finitely many simples.



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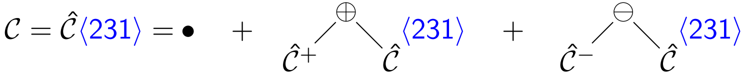
Pushing restrictions in the subtrees

Example: $\mathcal{C} = \text{Av}(231)$.
We have $\mathcal{S}_{\mathcal{C}} = \mathcal{S}_{\hat{\mathcal{C}}} = \emptyset$, and $\mathcal{C} = \hat{\mathcal{C}}\langle 231 \rangle$.



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$$= \bullet$$

Pushing restrictions in the subtrees

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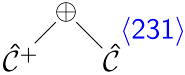
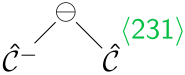
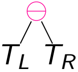
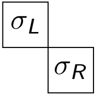
$\mathcal{C} = \hat{\mathcal{C}}\langle 231\rangle = \bullet \quad +$  $\quad +$ 

$= \bullet$

Claim:  $=$  $\in \mathrm{Av}(231) \Leftrightarrow \sigma_L \in \mathrm{Av}(12)$ and $\sigma_R \in \mathrm{Av}(231)$

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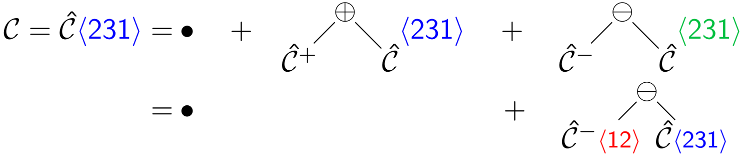
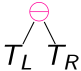
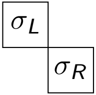


Claim: 

$=$ 

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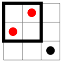
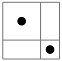
Because of an embedding of 231 into $21 = \ominus$:  $= 231 \hookrightarrow 21 =$ 

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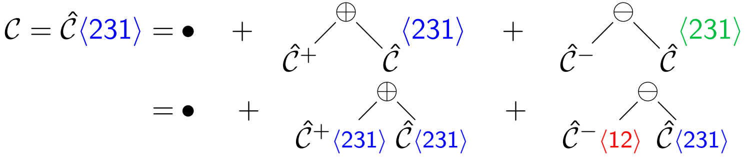
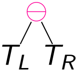
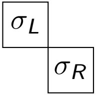
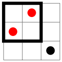
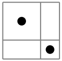


Claim:  =  $\in \text{Av}(231) \Leftrightarrow \sigma_L \in \text{Av}(12)$ and $\sigma_R \in \text{Av}(231)$

Because of an embedding of 231 into $21 = \ominus$:  $= 231 \hookrightarrow 21 =$ 

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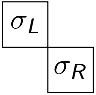
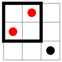
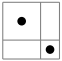
Claim: $\ominus(T_L, T_R) =$  $\in \mathrm{Av}(231) \Leftrightarrow \sigma_L \in \mathrm{Av}(12)$ and $\sigma_R \in \mathrm{Av}(231)$

Because of an embedding of 231 into $21 = \ominus$:  $= 231 \hookrightarrow 21 =$ 

Need of a new equation for $\hat{\mathcal{C}}^{-}\langle 12 \rangle$ ... And keep going

Why the grammar may be ambiguous

Pattern avoidance constraints in the subtrees come from embeddings of $\beta \in B^{\star}$ into $\pi \in \mathcal{S}_{\mathcal{C}} \cup \{12, 21\}$.

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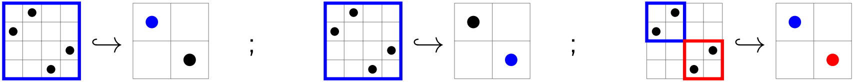
Example with $\beta = 3412$ and $\pi = 21$. Three embeddings of β into π:



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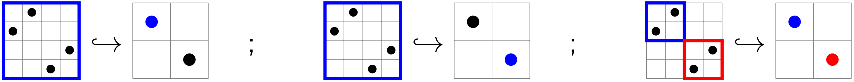

Hence:

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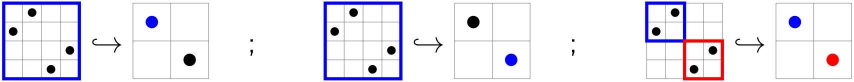

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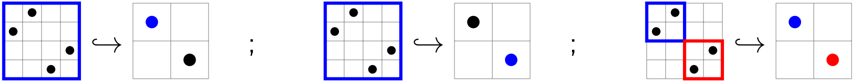

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Observation: The new excluded patterns are some $\alpha \preccurlyeq \beta \in B^\star$

Need of introducing pattern *containment* constraints

Example: Disambiguation of $\ominus\left(\hat{\mathcal{C}}^{-}\langle 12\rangle,\ \hat{\mathcal{C}}\langle 3412\rangle\right) \cup \ominus\left(\hat{\mathcal{C}}^{-}\langle 3412\rangle,\ \hat{\mathcal{C}}\langle 12\rangle\right)$.

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- $A \cup B = A \cap B \ \uplus\ \overline{A} \cap B \ \uplus\ A \cap \overline{B}$

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Notice that the terms $\ominus[\hat{\mathcal{C}}^-\langle 3412\rangle, \hat{\mathcal{C}}_{3412}\langle 12\rangle]$ and $\ominus[\hat{\mathcal{C}}^-_{3412}\langle 12\rangle, \hat{\mathcal{C}}\langle 3412\rangle]$ are empty, and have been deleted.

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$\Rightarrow$ Need to propagate avoidance and containment constraints:

$$\hat{\mathcal{C}}^{\varepsilon}_{\gamma_1,\dots,\gamma_p}\langle \alpha_1, \dots, \alpha_k\rangle \text{ with } \varepsilon \in \{\ \ , +, -\}$$

Observation: γ_i and α_j are all patterns of some $\beta \in B^{\star}$.

A first specification for $\mathcal{C}$

Find a specification for all

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with $\{\gamma_1,\dots,\gamma_p\} \subseteq \widetilde{B^\star}$ and $\{\alpha_1,\dots,\alpha_k\} \subseteq \widetilde{B^\star}$,
where $\widetilde{B^\star} = \{\alpha \preccurlyeq \beta \mid \beta \in B^\star\}$ = set of patterns of some $\beta \in B^\star$.

How to:
For $\alpha \in \widetilde{B^\star}$ and $\sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(k)}]$,
considering embeddings of α in π,
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Approach reminiscent of the query-complete sets of [Brignall, Huczynska & Vatter 08].

Computing only the necessary restrictions

Algorithm: [Bassino, Bouvel, Pierrot, Pivoteau & Rossin, 14+]

- Start from $\mathcal{C} = \hat{\mathcal{C}}\langle B^\star \rangle$, $\mathcal{C}^+$ and $\mathcal{C}^-$, and propagate the pattern avoidance constraints in the subtrees.
- Disambiguate the equations, introducing pattern containment constraints.
- For each term $\hat{\mathcal{C}}^{\varepsilon}_{\gamma_1,\ldots,\gamma_p}\langle \alpha_1, \ldots, \alpha_k \rangle$ that appears on the RHS, repeat this process, recursively.

Properties:

- This algorithm terminates and produces a specification for $\mathcal{C}$.

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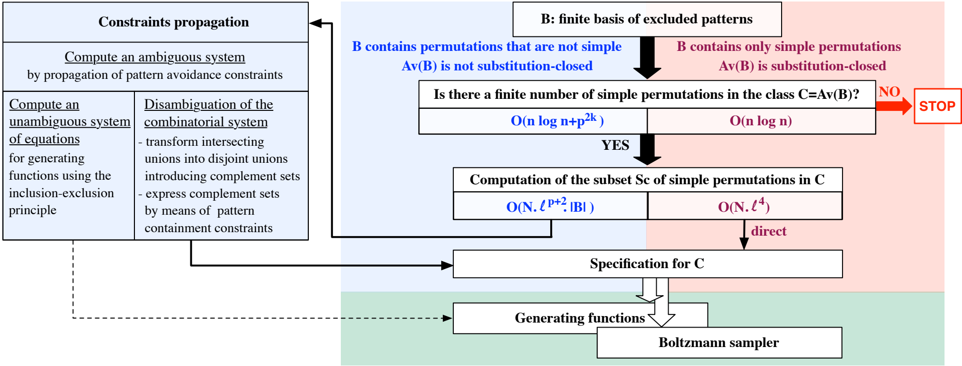
Questions:

- What is the complexity?
- What is the size of the specification produced?

$\hookrightarrow$ It can be exponential in $|B|$. But how big can it be?

Summary: From the basis to the specification

Algorithmic chain from B finite to a specification for $\mathcal{C} = \text{Av}(B)$.



where $n = \sum_{\beta \in B} |\beta|$, $p = \max_{\beta \in B} |\beta|$, $k = |B|$, $N = |\mathcal{S}_{\mathcal{C}}|$, $\ell = \max_{\pi \in \mathcal{S}_{\mathcal{C}}} |\pi|$.

Remark: It succeeds only when $\mathcal{C}$ contains finitely many simples (and this condition is tested algorithmically).

Byproducts of specifications and perspectives

A specification for $\mathcal{C}$ gives access to...

- A polynomial system defining $C(z)$ (implicitly)

  [Flajolet & Sedgewick 09]

$\hookrightarrow$ Can it be used to obtain information on the dominant singularity of $C(z)$, or equivalently the growth rate of $\mathcal{C}$?

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↪ Can it be used to obtain information on the dominant singularity of $C(z)$, or equivalently the growth rate of $\mathcal{C}$?

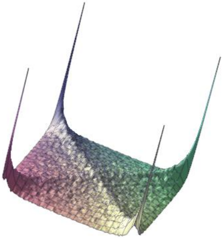
- Random samplers of permutations in $\mathcal{C}$:
  - by the recursive method [Flajolet, Zimmerman & Van Cutsem 94]
  - by the Boltzmann method [Duchon, Flajolet, Louchard & Schaeffer 04]

↪ Implementation (in progress) to observe random permutations in permutation classes.

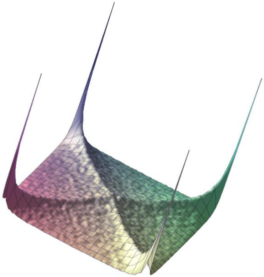
↪ Can we describe the "average shape" or average properties of random permutations in permutation classes?
For some given classes, or for families of classes?

Random permutations in permutation classes

• $\mathcal{C}_1 = \mathrm{Av}(2413, 3142) =$ separables.
Substitution-closed with no simples.
10000 permutations of size 100 in $\mathcal{C}_1$.



• Substitution-closed class $\mathcal{C}_2$,
with simples 2413, 3142 and 24153.
10000 permutations of size 500 in $\mathcal{C}_2$.



• $\mathcal{C}_3 = \mathrm{Av}(2413, 1243, 2341, 531642, 41352)$.
Not substitution-closed.
Almost 30000 permutations of size 500 in $\mathcal{C}_3$.