Exercices

ALEA 2009, Francis Comets,

EXERCISE 1 (directed polymer on the regular tree)

Let $b \geq 2$ be an integer. The **rooted regular tree** with branching number b is the infinite tree where each vertex has the same number $b+1$ of neighbors except for one which has only b of neighbors. This particular vertex is called the root or origin of the tree[1]. We denote this tree by $\mathbb{T}_b$. It can be uncoded by words of lenghts $0, 1, \dots$ in the alphabet $\{1, 2, \dots, b\}$,

$$\mathbb{T}_b = \emptyset \cup \{1, 2, \dots, b\} \cup \{1, 2, \dots, b\}^2 \cdots \cup \{1, 2, \dots, b\}^n \cup \dots$$

Here, $\emptyset$ is the root, $\{1, 2, \dots, b\}^n$ is called the nth generation (or level), and the vertex $u = (u_1 u_2 \dots u_n) \in \{1, 2, \dots, b\}^n$ has for neighbors all its successors uv in $\{1, 2, \dots, b\}^{n+1}$ (with $v \in \{1, 2, \dots, b\}$) and its predecessor $(u_1 u_2 \dots u_{n-1}) \in \{1, 2, \dots, b\}^{n-1}$.

The simple random walk on $\mathbb{T}_b$ is obtained by starting from the root and by jumping at each time to one of the b successors of the current location, chosen uniformly at random. More precisely,

$$S_0 = \emptyset \quad \text{and} \qquad P(S_{n+1} = uv | S_n = u) = 1/b\,, \quad v \in \{1, 2, \dots, b\},\ n \geq 0\,.$$

In particular, S_n belongs to the nth generation. As before, we consider an i.i.d. field $\eta = (\eta(u), u \in \mathbb{T}_b \setminus \{\emptyset\})$, and set $\lambda(\beta) = \ln Q[e^{\beta \eta(u)}]$ that we assume to be finite for all β. For $u, u' \in \mathbb{T}_b$, we write

$$u \prec u' \quad \text{if} \quad \exists u'' \in \mathbb{T}_b : u = u'u''\,,$$

that is, if u' is a successor (in the wide sense) of u. For S a path of length n we define

$$H_n(S) = \sum_{\emptyset \neq u \prec S} \eta(u)$$

where the sum ranges over all the vertices, except the root, on the ray from the root to S_n; There are n such vertices, and H_n comprises n summands. This defines a polymer model, with partition function given by

$$Z_{n,\beta}^{\eta} = P[\exp \beta H_n(S)]$$

1. For $v \in \mathbb{T}_b$, let θ_v be the shift operator on the environment given by $(\theta_v \eta)(u) = \eta(vu)$. (Here, vu is the concatenation of v and u, it is a word of length equal to the sum of lengths of v and u). Check the recursion formula

$$Z_{n,\beta}^{\eta} = b^{-1} \sum_{i=1}^{b} \exp\{\beta \eta(i)\} \times Z_{n-1,\beta}^{\theta_i \eta}\,, \tag{0.1}$$

   and that $(Z_{n-1,\beta}^{\theta_i \eta}; i \leq b)$ is i.i.d.

[1] The tree is usually called the Bethe lattice with coordination number $b+1$ in the physics literature

2. In this section we **compute the free energy**. Let

$$\beta_c \text{ be the root } \beta \in (0,+\infty) \text{ of } \quad \beta\lambda'(\beta) - \lambda(\beta) = \ln b$$

if this equation has a solution, and $\beta_c = \infty$ otherwise. We will prove that, almost surely and in L^p-norm ($1 \le p < \infty$),

$$\lim_{n\to\infty} n^{-1} \ln Z^{\eta}_{n,\beta} = p(\beta) = \begin{cases} \lambda(\beta) & \text{if } 0 \le \beta \le \beta_c \\ \beta\lambda'(\beta_c) + \ln b & \text{if } \beta > \beta_c \end{cases} \tag{0.2}$$

Remark: The situation is *drastically different* on $\mathbb{Z}^d$, in which case the function p is *strictly* convex on $\mathbb{R}$.

(a) What is the annealed bound here? What is the result of proposition 1.2.6 here?

(b) Check that we can use the martingale approach developed in the notes: The sequence $W_n = Z^{\eta}_{n,\beta} e^{-n\lambda(\beta)}$ is still a positive martingale, it converges a.s. to some non-negative W_∞, which is either a.s. equal to 0, or a.s. positive. In the last case, we have $p(\beta) = \lambda(\beta)$.

(c) Prove that

$$\left(e^{(1-\alpha)\ln b - \alpha\lambda(\beta) + \lambda(\alpha\beta)} - 1\right) Q(W_n^{\alpha}) \le (1-\alpha) b^{1-\alpha} (b-1) Q(W_{n-1}^{\alpha/2})^2 \,.$$

[Hint: one can use the following inequality. There exists $\alpha_0 \in (0,1)$ such that, for all $\alpha \in (\alpha_0, 1)$,

$$\Big(\sum_{i=1}^{b} x_i\Big)^{\alpha} \ge \sum_{i=1}^{b} x_i^{\alpha} - 2(1-\alpha) \sum_{1\le i<j\le b} (x_i x_j)^{\alpha/2} \,, \qquad x_i > 0, i = 1, \dots, b. \tag{0.3}$$

(d) Deduce from this that $W_\infty > 0$ for $0 \le \beta < \beta_c$. [Hint: convergence in probability together with uniform integrability implies L^1-convergence.]

(e) Show that, for $\beta \ge \beta_c$, $\liminf_n p^{\eta}_{n,\beta} \ge \beta\lambda'(\beta_c) - \ln b$. [Hint: convexity and the above result for β_c]

(f) Show the reverse inequality for the lim sup, and conclude to the almost sure limit in (0.2). [Hint: show first, with $H_n^* = \max\{H_n(S); S \text{ path of length } n\}$, that $\limsup_{n\to\infty} n^{-1} H_n^* \le \lambda'(\beta_c)$ (convention $\lambda'(\beta_c) = \lim_{\beta\to\infty} \lambda'(\beta)$ when $\beta_c = \infty$).]

(g) Prove L^p-convergence.

3. Particular case of gaussian environment $\eta(u) \sim \mathcal{N}(0,1)$. Compute the values of β_c, of p and of

$$\beta_2 = \sup\{\beta \ge 0 : \sup_n QW_n^2 < \infty\}.$$

Conclude that the L^2-region is a strict subset of the weak disorder region.

4. (Extra question..) Prove (0.3).