Exit problem of a
two-dimensional risk process
from a cone: exact and large
deviations results

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1 One and two dimensional first passage problems

The classical and Sparre-Andersen reserves models of actuarial science.

\[
X_t = x + p t - S(t)
\]

\[
S(t) = \sum_{i=1}^{N(t)} \sigma_i
\]

\(x\) denotes the initial reserve of an insurance company, \(p\) is the premium rate per unit of time, the i.i.d. random variables \(\sigma_i\) with distribution function \(B(x)\) are the ”claims”, and \(S_t\) is compound Poisson. Alternatively, \(dX_t = p - dS_t\)

Figure 1:
The dual M/G/1 workload model. Quite similar to the classical reserves model is the M/G/1 workload model used in queueing theory, where only upward jumps are allowed. In fact, the stationary distribution of the M/G/1 waiting time/workload coincides with the perpetual ruin probability (4)\(^1\).

Levy processes. When \(N_t\) is Poisson, the reserves process (1) is an example of Levy process without upward jumps (spectrally negative), and in fact most of the results about it are also valid in this greater generality. Typically, results about Levy process maybe expressed in terms of the the moment generating function of the cumulant exponent \(\kappa(\theta)\):

\[
\kappa(\theta) := t^{-1} \log \left( E \left[ e^{\theta X(t)} \right] \right)
\]

While the general Levy case requires typically a Wiener-Hopf factorisation, the absence of upward jumps renders this unnecessary, simplifying considerably the analysis.

The ruin problem: analytic results. Let \(\tau\) denote the ruin time:

\[
\tau := \inf \{ t \geq 0 : X_t < 0 \}
\]

\(^1\)Let us note also that in queueing there is an alternative approach focusing on the integer number of customers waiting, a birth and death process which is skip-free in both directions.
Classical risk theory attempts to compute the finite-time ruin

\[ \psi(x, t) = P_x[\tau \leq t] \]  

or survival probabilities \( \bar{\psi}(x, t) = 1 - \psi(x, t) \), but this succeeds analytically only in a few cases, like that of exponential claims (see [2]) and of Brownian motion.

On the other hand, analytical results are sometimes available for the ”ultimate” or ”perpetual” ruin probability

\[ \psi(x) = P_x[\tau \leq \infty] \]  

For example, in the case of exponential claim sizes with intensity \( \mu \):

\[ \psi(x) = Ce^{-\gamma x} \text{ where } \gamma = \mu - \lambda/p > 0, \ C = \frac{\lambda}{\mu p} < 1. \]  

(5)

For \( X_t \) a Brownian motion with parameters \( m > 0, \sigma^2 > 0 \), the ultimate ruin probability is even simpler:

\[ \psi(x) = e^{-\frac{2m}{\sigma^2} x} \]  

(6)

**Killed ruin probabilities.** Similar formulas hold for the ”killed” probability of ruin before an
independent exponential horizon $E(q)$ of rate $q$:

$$\psi_q(u) = P_x[\tau \leq E(q)] = \int_0^{\infty} q e^{-qt} P_x[\tau \leq t] dt \quad (7)$$

$$= \int_0^{\infty} e^{-qt} \psi(x, dt) = E_x e^{-q\tau} \quad (8)$$

These ”killed ruin probabilities” admit analytic expressions generalizing those from the perpetual case (obtained when $q = 0$); at the same time, they allow recovering the finite time ruin probabilities numerically, by inversion of the Laplace transform.

**Asymptotics for ultimate ruin probabilities.** Under the Cramér assumption that there exists $\gamma > 0$ such that

$$\kappa(-\gamma) = 0 \quad (9)$$

exponential type asymptotics, known as the Cramér-Lundberg approximation, hold:

$$\lim_{x \to \infty} \frac{\psi(x)}{e^{-\gamma x}} = C$$

where $C$ is

$$C = -\kappa'(0)/\kappa'(-\gamma). \quad (10)$$

The saddle-point/large deviations approximation for finite time ruin probabilities. Sharp
asymptotics for $\psi(x, t)$ were first obtained by Arfwedson (1952) by the saddlepoint method. Later it turned out that this result has a convenient interpretation in terms of the exponential family of the original Levy process, via the so-called ”large deviations” machinery; we describe this now for completeness, following [3].

**Definition 1** For each $c$ such that $\kappa(c) < \infty$, the exponential family of $X$ is the family of measures $\{P^{(c)}\}$ with Radon-Nikodym derivatives given by the Wald martingales:

$$
\frac{dP^{(c)}}{dP}\bigg|_{\mathcal{F}_t} = \exp(cX_t - \kappa(c)t). \tag{11}
$$

The characteristic function of the process $X$ under the measure $P^{(c)}$ is given by

$$
\kappa_c(\theta) = \kappa(c + \theta) - \kappa(c). \tag{12}
$$

The basic idea is to approximate the finite time ruin probability by using a judiciously chosen member of the exponential family.

There are four conceptually distinct cases to consider for finite time ruin probabilities:

1. processes with sure eventual ruin, with $EX_1 = \kappa'(0) = p - \rho < 0$
2. the opposite case $EX_1 = \kappa'(0) = p - \rho > 0$, in which ruin is rare and it turns out that the asymptotics, conditional on ruin happening, depend on the shifted measure with parameter $-\gamma$ (under which eventual ruin is sure since $\kappa'(-\gamma) < 0$).

3. the case of ruin times which are shorter/longer than expected.

Despite the different interpretations, all these cases
have a common ”large deviations” formulation, expressible in terms of the cumulant exponent \( \kappa(\theta) \) of \( X \).

Suppose first \( \kappa'(0) < 0 \), in which case \( E\tau = \frac{x}{-\kappa'(0)} \), and a ”short” ruin time, i.e. \( t < E\tau \iff v > -\kappa'(0) \), where \( v = \frac{x}{t} \). The transformation formula

\[
\psi(x, t) = E^{(c)} \left[ e^{c(X_\tau - x) - \kappa(c)\tau}; \tau \leq t \right]
\]

suggests heuristically that in the limit \( x \to \infty \):

\[
\psi(x, t) \sim e^{-cx} E^{(c)} \left[ e^{-cZ} \right] E^{(c)} \left[ e^{-\kappa(c)\tau}; \tau \leq t \right]
\]

(13)

where \( Z \) is the asymptotic overshoot over the barrier, when \( x \to \infty \).

Choosing now \( c = -\alpha_v \), the unique shift for which \( E^{(c)}\tau = t \iff \kappa'(-\alpha_v) = -\frac{x}{t} \), we may approximate the third term in the RHS above via a Gaussian approximation:

\[
\tau \sim t + \sqrt{t} \sigma V
\]

with \( V \) standard normal, which yields:

\[
E^{(\alpha_v)} \left[ e^{-\kappa(\alpha_v)\tau}; \tau \leq t \right] \sim \frac{e^{-\kappa(\alpha_v)\frac{t}{\sqrt{2\pi t\kappa''(\alpha_v)}}} e^{-\kappa(\alpha_v) t}}{\kappa(\alpha_v) \sqrt{2\pi t\kappa''(\alpha_v)}} \]

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For the second term in the RHS of (13), we use the well-known expression of the asymptotic overshoot $E^{(-\gamma)} [e^{-\alpha Z}] = \frac{\gamma \kappa(\gamma-\alpha)}{-\alpha \kappa'(-\gamma)(\gamma-\alpha)}$, applied with $\alpha = \alpha_v$, $\gamma = \gamma_v = \alpha_v - \tilde{\alpha}_v$, with $\tilde{\alpha}_v$ being the "conjugate twist" satisfying $\kappa(\tilde{\alpha}_v) = \kappa(\alpha_v)$, $\tilde{\alpha}_v < \alpha_v$, yielding:

$$E^{(-\gamma)} [e^{-\alpha Z}] = \frac{\alpha_v - \tilde{\alpha}_v}{\alpha_v|\tilde{\alpha}_v|} \frac{\kappa(\gamma_v)}{v}$$

(14)

**Theorem 1 (Arfwedson-Asmussen)** Let $X_t$ be a Levy process satisfying Cramer’s condition. Then,

$$\psi(x, t) \sim \begin{cases} C_v e^{-t\kappa^*(v)} & \text{if } \kappa'(0) < 0, \ t < \frac{x}{-\kappa'(0)} \\ \psi(x) - C_v e^{-t\kappa^*(v)} & \text{if } \kappa'(0) < 0, \ t > \frac{x}{-\kappa'(0)} \end{cases}$$

(15)

and

$$\psi(x, t) \sim \begin{cases} C_v e^{-t\kappa^*(v)} & \text{if } \kappa'(0) > 0, \ t < \frac{x}{-\kappa'(-\gamma)} \\ \psi(x) - C_v e^{-t\kappa^*(v)} & \text{if } \kappa'(0) > 0, \ t > \frac{x}{-\kappa'(-\gamma)} \end{cases}$$

(16)

where $v = \frac{x}{t}$, $\alpha_v$, $\tilde{\alpha}_v$ are defined by $\kappa'(\alpha_v) = v$, $\kappa(\tilde{\alpha}_v) = \kappa(\alpha_v)$, $\tilde{\alpha}_v < \alpha_v$, $C_v = \frac{\alpha_v - \tilde{\alpha}_v}{\alpha_v|\tilde{\alpha}_v| \sqrt{2\pi t \kappa'''(\alpha_v)}}$ and $\kappa^*(v) = v\alpha_v - \kappa(\alpha_v)$ is the conjugate dual of the cumulant exponent.
Two-dimensional model and problem. Two companies split the amount they pay out of each claim in proportions $\delta_1$ and $\delta_2$ where $\delta_1 + \delta_2 = 1$, and receive premiums at rates $c_1$ and $c_2$, respectively. Let $U_i$ denote the risk process of the $i$’th company

$$U_i(t) := -\delta_i S(t) + c_i t + u_i, \quad i = 1, 2,$$

where $u_i$ denotes the initial reserve and

$$S(t) = \sum_{i=1}^{N(t)} \sigma_i$$

and $N(t)$ is a renewal process with i.i.d. inter-arrival times $\tau_i$, and the claims $\sigma_i$ are i.i.d. random variables independent of $N(t)$, with distribution function $F(x)$. We shall denote by $\lambda$ and $\mu$ the reciprocals of the means of $\tau_i$ and $\sigma_i$, respectively. We shall assume that the second company, to be called reinsurer, receives less premium per amount paid out, i.e.:

$$p_1 := \frac{c_1}{\delta_1} > \frac{c_2}{\delta_2} := p_2$$

On the other hand, the reinsurer needs to have larger reserves than the insurer.

As usual in risk theory, we assume that $p_i > \rho := \frac{\lambda}{\mu} = E\sigma/E\tau \implies EU_i(t) \to \infty$. Several ruin problems may be of interest. We consider here:
1. the first time \( \tau = \tau(u_1, u_2) \) when (at least) one insurance company is ruined, i.e. the exit time of \((U_1(t), U_2(t))\) from the positive quadrant:

\[
\tau_{\text{or}}(u_1, u_2) := \inf\{t \geq 0 : U_1(t) < 0 \text{ or } U_2(t) < 0\}
\]

\[
\psi_{\text{or}}(u_1, u_2) = P[\tau_{\text{or}}(u_1, u_2) < \infty].
\]

2. the first time \( \tau = \tau(u_1, u_2) \) when both insurance companies are ruined, i.e. the entrance time of \((U_1(t), U_2(t))\) into the negative quadrant:

\[
\tau_{\text{and}}(u_1, u_2) := \inf\{t \geq 0 : U_1(t) < 0 \& U_2(t) < 0\}
\]

\[
\psi_{\text{and}}(u_1, u_2) = P[\tau_{\text{and}}(u_1, u_2) < \infty].
\]

3. \( \psi_{\text{all}}(u_1, u_2) = P_{(u_1, u_2)}[(\tau_1(u_1) < \infty) \& (\tau_2(u_2) < \infty)] \)

where \( \tau_i(u_i) = \inf\{t \geq 0 : U_i(t) < 0\}, i = 1, 2. \)

Clearly:

\[
\psi_{\text{and}}(u_1, u_2) \leq \psi_{\text{all}}(u_1, u_2) = \psi_1(u_1) + \psi_2(u_2) - \psi_{\text{or}}(u_1, u_2)
\]

where

\[
\psi_i(u) := P(\tau_i(u) < \infty)
\]

(19)

denotes the ruin probability of \( U_i \) when \( U_i(0) = u. \)
Figure 3: Geometrical considerations

Geometrical considerations and solution in the lower cone $C$. The process $(U_1, U_2)$ will be subject to a ”and/or” ruin precisely at the first crossing time $\tau$ the $u_i$ axis. Thus, in the domain $C$ ”and/or” ruin occurs iff there is ruin in the one-dimensional problem corresponding to the risk process $U_i$, and the solutions coincide with the ultimate ruin probabilities $\psi_i(x_i)$ of the classical risk processes $U_i(t)$

$$
\begin{align*}
\psi_{or}(u_1, u_2) &= \psi_2(u_2), \\
\psi_{and}(u_1, u_2) &= \psi_1(u_1)
\end{align*}
$$

(20)

In the opposite case $u_2 > (\delta_2/\delta_1)u_1$ the solution is more complicated; note that this is precisely the case of interest for reinsurance.
Solution in the upper cone $C^c$: piecewise linear barriers. Recalling that $U_i(t) := -\delta_i S(t) + c_i t + u_i$, $i = 1, 2$, a key observation is that the "or" and "and" ruin times $\tau$ in (19), (19) are:

$$\tau(u_1, u_2) = \inf\{t \geq 0 : S(t) > b(t)\},$$

where

$$b(t) = \min\{(u_1 + c_1 t)/\delta_1, (u_2 + c_2 t)/\delta_2\}$$
$$b(t) = \max\{(u_1 + c_1 t)/\delta_1, (u_2 + c_2 t)/\delta_2\}$$

in the "or"/ "and" cases, respectively.

Figure 4:

Note dependence on initial reserves.

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Scaling. The process

\[ X_i(t) := U_i(t)/\delta_i \]

with initial capitals \( x_i = u_i/\delta_i \), \( i = 1, 2 \) has the same ruin probability as the original process. Thus, it suffices to suppose \( \delta_1 = \delta_2 = 1 \).

The probability of crossing piecewise linear barriers. crossing at \( T = T(x_1, x_2) = \frac{x_2-x_1}{p_1-p_2} \)

\[ b_{\min}(t) = \min_{i=1,2} \{x_i + p_it\} \quad b_{\max}(t) = \max_{i=1,2} \{x_i + p_it\} \]

![Diagram](image)

Figure 5:

In the “or” case for example, this requires staying below the barrier \( x_1 + p_1t \) between the times 0 and \( T \) and subsequently staying below the barrier \( x_2 + p_2t \)
after time $T$. By conditioning at time $T$:

$$\bar{\psi}_{or}(x_1, x_2) = \int_0^\infty \tilde{P}_{x_1, p_1}(dz, T)\bar{\psi}_2(x_2 + p_2T - z)$$

where

$$\tilde{P}_{x, p}(dz, T) = P_0(S(t) \leq x + pt, \forall t \in [0, T], S(T) \in dz)$$

(21)

is the density at time $T$ of the paths $S(T)$ which "survive" the upper barrier $x + pt$ and where we used the fact that $x_1 + p_1T = x_2 + p_2T$. We find it convenient to reformulate this result in terms of the two coordinates of our reserves process

$$X_i(t) := x_i + p_i t - S(t), i = 1, 2$$

their infima

$$I_i(t) = \inf_{0 \leq s \leq t} X_i(s) \land 0$$

and the coordinate-wise densities of the "non-ruined" paths

$$\bar{\psi}_i(dz, T|x_i) := P_{x_i}(I_i(T) \geq 0, X_i(T) \in dz) = \tilde{P}_{x_i, p_i}(dz, T)$$

(22)

We arrive thus at the following result, which relates the survival probability of the two dimensional process to the one dimensional survival characteristics of its coordinates:
Theorem 2 Let $X_t$ be a two-dimensional Lévy process (1) with equal cumulative claims $S(t) = S_1(t) = S_2(t)$ given by an arbitrary Levy process. If $x_2 > x_1, p_2 < p_1$, then the two-dimensional survival probabilities associated to the or/and ruin problems (??), (??), are given by:

$$
\overline{\psi}_{or}(x_1, x_2) = \int_0^\infty \overline{\psi}_1(dz, T|x_1) \overline{\psi}_2(z)
$$

$$
\overline{\psi}_{and}(x_1, x_2) = \int_0^\infty \overline{\psi}_2(dz, T|x_2) \overline{\psi}_1(z)
$$

where $T$ is given in (??), $\overline{\psi}_i(dz, T|x_i)$ in (22) and $\overline{\psi}_i(z) = P_z(I_i(\infty) \geq 0)$ are perpetual one-dimensional survival probabilities.

2 Exact ultimate ruin probabilities

In this section we propose methods for computing the ruin probabilities $\psi_{or}(x_1, x_2)$ and $\psi_{and}(x_1, x_2)$ of the model with equal claims in the case when $(x_1, x_2) \in C^c$ and the claims sizes $\sigma_i$ follow a phase-type distribution $(\beta, B)$, i.e. $P[\sigma > x] = \beta e^{Bx}1$, the solution simplifies. In this case the one-dimensional
ruin probability may be written in a simple matrix exponential form:

\[ \psi_i(x_i) = \eta_i e^{Q_i x_i} 1 \]  

(23)

with \( Q_i = B + b\eta_i \) and \( \eta_i = \frac{\lambda}{p_i} \beta(-B)^{-1} \) (see for example (4) in [4]). Combining this explicit formula (23) with Theorem 2 yields the following result:

**Corollary 1** Suppose \( S \) is a compound Poisson process with phase-type jumps \((\beta, B)\). If \( x_2 x_1 \), it holds that

\[
\psi_{or}(x_1, x_2) = P_{x_1}(I_1(T) < 0) + \eta_2 \int_0^\infty e^{Q_2 z} \tilde{P}_{x_1, T}(dz)
\]

\[
\psi_{and}(x_1, x_2) = P_{x_2}(I_2(T) < 0) + \eta_1 \int_0^\infty e^{Q_1 z} \tilde{P}_{x_2, T}(dz)
\]

where \( Q_i = B + b\eta_i \) and \( \eta_i = \frac{\lambda}{p_i} \beta(-B)^{-1} \).

In the special case of exponential claims \( \sigma_i \) with rate \( \mu \) equation (24) can be developed further by employing the technique of change of measure and by applying the Markov property of \( X_i \). Indeed, as a particular case of the phase-type relation (24), we see that

\[
\psi_{or}(x_1, x_2) = P_{x_1}(I_1(T) < 0) + C_2 E_{x_1}[e^{-\gamma X_1(T)} \mathbf{1}_{\{I_1(T)\geq 0\}}]
\]
By a change of measure and using that $-\gamma_2 x_1 + \kappa_1(-\gamma_2)T = -\gamma_2 x_2$, we find that the second term in (2) is equal to

$$e^{-\gamma_2 x_1 + \kappa_1(-\gamma_2)T}E_{x_1}[M_T \mathbf{1}_{\{I_1(T) \geq 0\}}]$$

$$= e^{-\gamma_2 x_2} P_x^{(-\gamma_2)}(I_1(T) \geq 0).$$

In conclusion, the original two-dimensional ruin problems $\psi_{or}/\psi_{and}$ have been reduced to one-dimensional finite time ruin problems $\psi_i^{(c)}(x, t) := P_x^{(c)}[I_i(t) < 0]$, as follows:

**Corollary 2** Suppose $S$ is a compound Poisson process with exponential jumps. Then it holds that

$$\psi_{and}(x_1, x_2) = \psi_2(x_2, T) + \psi_1(x_1)\bar{\psi}_2^{(-\gamma_1)}(x_2, T)$$

$$\psi_{or}(x_1, x_2) = \psi_1(x_1, T) + \psi_2(x_2)\bar{\psi}_1^{(-\gamma_2)}(x_1, T)$$

$$\psi_{all}(x_1, x_2) = w_1(x_1, T) + \psi_2(x_2)\bar{\psi}_1^{(-\gamma_2)}(x_1, T)$$

where $w_i(x, t) = w_{\lambda, \mu, p_i}(x, t)$ is given by

$$w_{\lambda, \mu, p_i}(x, t) = P_x[I_i(\infty) < 0, I_1(t) \geq 0] = \psi_i(x) - \psi_i(x, t).$$
3 Cramer type sharp large deviations asymptotics in the upper cone $x_2 \geq x_1$.

Light-tailed case. We assume throughout that the Cramér conditions for $X_1$ and $X_2$ are satisfied, that is, it is assumed that there exists $\gamma_i > 0$ $(i = 1, 2)$ such that

$$\kappa_i(-\gamma_i) = 0. \quad (24)$$

We turn now to asymptotics in the case that the initial reserves tend to infinity along a ray $x_1/x_2 = a$ with $a < 1$. While this may be achieved by classical one-dimensional convergence results in renewal theory, our example is also an interesting illustration of two-dimensional large deviations theory, which derives asymptotic results for exponentially rare events for a process $X$ using two principles:

- viewed from far away, the paths along which rare events are realized concentrate typically along a finite number of locally dominating paths, which are the solutions of a deterministic variational problem.

- these paths maybe viewed as the expected drifts
of some measures belonging to the exponential family of the process.

Three possible types of hitting trajectories

![Diagram of hitting trajectories]

Figure 6: Typical paths hitting the negative quadrant

Informally, we say that rare events happen due to one of the (most likely) change/shift of measure. The change of measure may be useful for deriving asymptotic correction terms: precise large deviations.
The “dominant directions” may be obtained, as in the onedimensional case, from the ”cumulant exponent” or ”symbol”, which for a degenerate reserves process is:

\[
\Lambda(\theta) := p \cdot \theta + \lambda(\mathbb{E}e^{-(\theta \cdot \delta)\sigma} - 1)
\]

where \(\delta = (1, 1)\).

**Definition 2** The mean function is:

\[
\mu(\theta) := \Lambda'(\theta)
\]

This vector represents the expected drift of the measure \(P(\theta)\).

**Definition 3** The Cramer set is defined as the set of all \(\theta\) satisfying

\[
\Lambda(\theta) \leq 0
\]

The ”most probable drifts” in many large deviations estimates are associated to changes of measure \(\theta\) belonging to the boundary of the Cramer set.

For the quadrant hitting problem, the most important shifts are the intersections different from the origin of the Cramer set with the axes, \(\theta^{(2)} = (0, -\gamma_2)\) and \(\theta^{(1)} = (-\gamma_1, 0)\).
Figure 7: The Cramer set, the direct twists and the dominant points for hitting the negative quadrant.

**Remark 1** Putting \( \kappa_i(\theta) = p_i \theta + \lambda (\mathbb{E}e^{-\theta \sigma} - 1) \), we have

\[
\Lambda(\theta) = \kappa_1(\theta_1 + \theta_2) - (p_1 - p_2)\theta_2 = \kappa_2(\theta_1 + \theta_2) + (p_1 - p_2)\theta_1
\]

On the axis \( x_i = 0 \), the symbol equation reduces thus to the one dimensional Cramer-Lundberg equation \( \kappa_i(-\gamma_i) = 0 \) and the intersections of the Cramer set with the axes are indeed provided by the roots.
$-\gamma_i$ of the corresponding one-dimensional Cramer-Lundberg equations for $X_i, i = 1, 2$.

The associated dominating directions are:

$$\mu^{(i)} := \mu(\theta^{(i)}) = \Lambda'(\theta^{(i)}), i = 1, 2.$$  

**Remark 2** From the remark 1, we find

$$\Lambda'(\theta) = (\kappa'_1(\theta_1 + \theta_2), \kappa'_2(\theta_1 + \theta_2))$$  

(25)

In particular,

$$\mu^{(1)} = (\kappa'_1(-\gamma_1), \kappa'_2(-\gamma_1)) \quad \mu^{(2)} = (\kappa'_1(-\gamma_2), \kappa'_2(-\gamma_2))$$

Inspired by this remarks, we introduce now the dominating directions via the definition:

**Definition 4** The dominating directions for hitting the $x_i = 0$ axes of a degenerate process are the vectors $\mu^{(i)}$:

$$\mu^{(1)} := (\kappa'_1(-\gamma_1), \kappa'_2(-\gamma_1)) := -(\kappa_{1,1}, \kappa_{2,1})$$
$$\mu^{(2)} := (\kappa'_1(-\gamma_2), \kappa'_2(-\gamma_2)) := -(\kappa_{1,2}, \kappa_{2,2})$$

**Note:** The dominating directions involve the expected drifts of the coordinates $X_i$ under the shifted measures $P(-\gamma_j)$, which figured prominently in the exact solution of the exponential jumps case.
Figure 8: The Cramer set, the direct twists and the dominant points for hitting the negative quadrant

3.1 Three terms asymptotics

The asymptotic results are given in terms of the roots $\gamma_i$ and the convex conjugates $\kappa_i^*$ of $\kappa_i$,

$$\kappa_i^*(v) = \sup_{p \in \mathbb{R}} [vp - \kappa_i(-p)].$$
They identify boundary influence regions where the dominant terms corresponds to the ruin of $X_i, i = 1, 2$, as well as a third term, corresponding to ”simultaneous ruin of both companies”. Each term is feasible only within a cone determined by the geometry.

**Theorem 3** Suppose that $v^2_a := v^1_a/a < \Sigma_2$ with $v^1_a := (p_1 - p_2)a/(1 - a)$ and let $\kappa'_1(-\gamma_2) < 0$. Suppose that $1/\phi(-\theta v^a_i)$ belongs to the interior of the domain of $\chi^{-1}$. Then there exist constants $\gamma = \gamma(a) = a\kappa^*_1(v^1_a)/v^1_a = \kappa^*_2(v^2_a)/v^2_a$, and $D_1, D_1^{(-\gamma_1)}, D_2, D_2^{(-\gamma_1)}$ such that, as $K \to \infty$,

$$
\psi_{or}(aK, K) \sim \begin{cases} 
C_2e^{-\gamma_2 K} + D_1K^{-1/2}e^{-\gamma(a)K} \\
C_1e^{-\gamma_1 a K} + C_2e^{-\gamma_2 K} \\
C_1e^{-\gamma_1 a K} + C_2D_1^{(-\gamma_2)}K^{-1/2}e^{-\gamma(a)K}
\end{cases}
$$

$s_1 < s_2$ $0 < s_1 < s_2$

$$
\psi_{and}(aK, K) \sim \begin{cases} 
C_1e^{-\gamma_1 a K} + D_2K^{-1/2}e^{-\gamma(a)K} \\
(D_2 + C_1D_2^{(-\gamma_1)})K^{-1/2}e^{-\gamma(a)K} \\
C_2e^{-\gamma_2 K} + C_1D_2^{(-\gamma_1)}K^{-1/2}e^{-\gamma(a)K}
\end{cases}
$$

$s_1 < s_2$ $0 < s_1 < s_2$

$$
\psi_{all}(aK, K) \sim \begin{cases} 
C_1e^{-\gamma_1 a K} + C_2D_1^{(-\gamma_2)}K^{-1/2}e^{-\gamma(a)K} \\
(D_1 + C_2D_1^{(-\gamma_2)})K^{-1/2}e^{-\gamma(a)K} \\
C_2e^{-\gamma_2 K} + D_1K^{-1/2}e^{-\gamma(a)K}
\end{cases}
$$

$s_1 < s_2$ $0 < s_1 < s_2$
Three possible types of hitting trajectories

Figure 9: Typical paths hitting the negative quadrant

References


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