Counting and optimizing with unimodular networks

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1 Introduction

These short lecture notes are devoted to the Benjamini-Schramm graph topology with a special emphasis on its applications to counting and combinatorial optimization problems. Section 2 defines this topology and introduces the main related concepts. Section 3 illustrates on the simple example of the minimal spanning tree how it allows to define a minimal spanning forest of a possibly infinite graph. In Section 4, on the example of graph matchings, we will study the convergence of partition functions. The material of these notes is largely borrowed from Aldous and Steele [5], Aldous and Lyons [4] and Bordenave, Lelarge and Salez [16].

2 Unimodular networks

2.1 Local weak topology

We first briefly introduce the theory of local weak convergence of graph sequences and the notion of unimodularity for random rooted graphs. It was introduced by Benjamini and Schramm [10] and has then become a popular topology for studying sparse graphs. Let us briefly introduce this topology, for details we refer to Aldous and Lyons [4] and Pete [33].

A graph $G = (V, E)$ is *locally finite* if for $v \in V$, the degree of v in G (number of incident edges), $\deg_G(v)$, is finite. A *rooted graph* (G, o) is a connected graph $G = (V, E)$ with a distinguished vertex $o \in V$, the root. Two rooted graphs $(G_i, o_i) = (V_i, E_i, o_i)$, $i \in \{1, 2\}$, are *isomorphic* if there exists a bijection $\sigma : V_1 \to V_2$ such that $\sigma(o_1) = o_2$ and $\sigma(G_1) = G_2$, where σ acts on E_1 through $\sigma(\{u, v\}) = \{\sigma(u), \sigma(v)\}$. We will denote this equivalence relation by $(G_1, o_1) \simeq (G_2, o_2)$. In graph theory terminology, an equivalence class of rooted graph is an *unlabeled rooted graph*. We denote by $\mathcal{G}^*$ the set of unlabeled rooted locally finite graphs.

The *local topology* is the smallest topology such that for any $g \in \mathcal{G}^*$ and integer $t \geq 1$, the $\mathcal{G}^* \to \{0, 1\}$ function $f(G, o) = \mathbf{1}((G, o)_t \simeq g)$ is continuous, where $(G, o)_t$ is the induced rooted graph spanned by the vertices at graph distance at most t from o. This topology is metrizable with

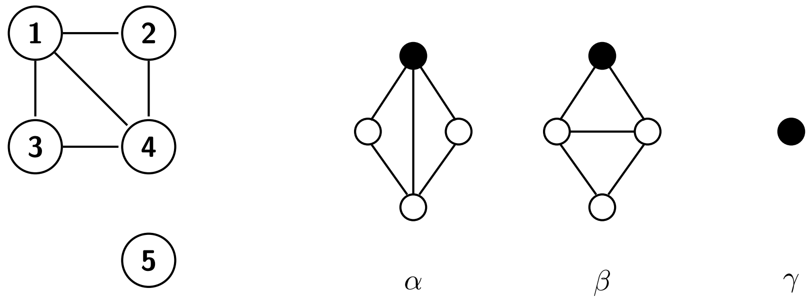

Figure 1: Example of a graph G and its empirical neighborhood distribution. Here $U(G) = \frac{1}{5}(2\delta_\alpha + 2\delta_\beta + \delta_\gamma)$, where $\alpha, \beta, \gamma \in \mathcal{G}^*$ are the unlabeled rooted graphs depicted above (the black vertex is the root), with $g(1) = g(4) = \alpha$, $g(2) = g(3) = \beta$, $g(5) = \gamma$.

the metric

$$d_{\mathrm{loc}}(g, h) = \sum_{t=1}^{\infty} 2^{-t}\mathbf{1}(g_t \neq h_t). \tag{1}$$

Moreover, it is not hard to check that the space $(\mathcal{G}^*, d_{\mathrm{loc}})$ is separable and complete metric space (or Polish space).

We now consider $\mathcal{P}(\mathcal{G}^*)$ the set of probability measures on $\mathcal{G}^*$. An element $\rho \in \mathcal{P}(\mathcal{G}^*)$ is the law of (G, o), a random rooted graph. Since $\mathcal{G}^*$ is a Polish space, we may safely consider the *local weak topology* on $\mathcal{P}(\mathcal{G}^*)$. Recall that it is the smallest topology such that for any continuous bounded function $f : \mathcal{G}^* \to \mathbb{R}$, the function $\rho \mapsto \mathbb{E}_\rho f(G, o)$ is continuous, where under $\mathbb{P}_\rho$, (G, o) has law ρ. It is well known that this weak convergence is metrizable by the Lévy-Prohorov distance which we will denote by d_{wloc} (the actual definition of the Lévy-Prohorov distance will not be used here). Then $(\mathcal{P}(\mathcal{G}^*), d_{\mathrm{wloc}})$ is also a separable and complete metric space.

For a finite graph $G = (V, E)$ and $v \in V$, one writes $G(v)$ for the connected component of G at v. One defines the probability measure $U(G) \in \mathcal{P}(\mathcal{G}^*)$ as the law of the equivalence class of the rooted graph $(G(o), o)$ where the root o is sampled uniformly on V :

$$U(G) = \frac{1}{|V|} \sum_{v \in V} \delta_{g(v)},$$

where $g(v)$ is the equivalence class of $(G(v), v)$. See Figure 1 for a concrete example. In the passage from G to $U(G)$ we have lost some information on the graph G, notably the labels of the vertices.

If $(G_n)_{n \geq 1}$, is a sequence of finite graphs, we shall say that G_n has *local weak limit* (or Benjamini-Schramm limit) $\rho \in \mathcal{P}(\mathcal{G}^*)$ if $U(G_n) \to \rho$ weakly in $\mathcal{P}(\mathcal{G}^*)$. A measure $\rho \in \mathcal{P}(\mathcal{G}^*)$ is called *sofic* if there exists a sequence of finite graphs $(G_n)_{n \geq 1}$, whose local weak limit is ρ. In other words, the set of sofic measures is the closure of the set $\{U(G) : G \text{ finite}\}$. The set of sofic measures will be denoted by $\mathcal{P}_{\mathrm{sof}}(\mathcal{G}^*)$.

2.2 Examples of local weak limits

Finite window approximation of a lattice : consider an integer $d \geq 1$, the graph of $\mathbb{Z}^d$ and $L_n = \mathbb{Z}^d \cap [1,n]^d$. Then, the local weak limit of L_n is the Dirac mass of the equivalence class of $(\mathbb{Z}^d, o)$. Indeed, if t is an integer $(L_n, v)_t \simeq (\mathbb{Z}^d, o)_t$ for all $v \in V(L_n)$ which are distance at least t from $\mathbb{Z}^d \backslash [1,n]^d$. It follows that $(L_n, v)_t \simeq (\mathbb{Z}^d, o)_t$ for all but $O(tn^{d-1}) = o(|V(L_n)|)$ vertices.

The same argument will work for any amenable group along any Følner sequence (and any graph with a good notion of amenability).

Percolation on a lattice : Consider an integer $d \geq 1$ and the usual bond percolation on the graph of $\mathbb{Z}^d$ where each edge is kept with probability $p \in [0,1]$, we obtain a random subgraph G of $\mathbb{Z}^d$. Then, a.s. the local weak limit of $G_n = G \cap [1,n]^d$ is $\mathrm{perc}(\mathbb{Z}^d, p)$, the law of the equivalence class of $(G(o), o)$.

Unimodular Galton-Watson trees : Let $P \in \mathcal{P}(\mathbb{Z}_+)$ with positive and finite mean. The unimodular Galton-Watson tree with degree distribution P, denoted by $\mathrm{UGW}(P)$ (commonly known as size-biased Galton-Watson tree) is the law of the random rooted tree obtained as follows. The root has a number d of children sampled according to P, and, given d, the subtrees of the children of the root are independent Galton-Watson trees with offspring distribution

$$\widehat{P}(k) = \frac{(k+1)P(k+1)}{\sum_\ell \ell P(\ell)}. \tag{2}$$

These random trees appear naturally as a.s. local weak limits of uniform random graphs with a given degree distribution, see e.g. [22, 21, 12]. It is also well known that the Erdős-Rényi $G(n, c/n)$ has a.s. local weak limit the Galton-Watson tree with offspring distribution $\mathrm{Poi}(c)$. Note that if P is $\mathrm{Poi}(c)$ then $\widehat{P} = P$. The percolation on the hypercube $\{0,1\}^n$ with parameter c/n has the same a.s. local weak limit.

Skeleton tree : The infinite skeleton tree which consists of a semi-infinite line $\mathbb{Z}_+$ with i.i.d. critical Poisson Galton-Watson trees $\mathrm{Poi}(1)$ attached to each of the vertices of $\mathbb{Z}_+$. It is the a.s. local weak limit of the uniformly sampled spanning tree on n labeled vertices.

Exercise 2.1. *What is the local weak limit of a complete binary tree T_n of height n ?*

2.3 Estimable functions

The Benjamini-Schramm topology contains many compact sets (see below). There are however a surprisingly long list of interesting continuous functions invariant by isomorphism. More precisely, we will use the terminology of Abért, Csikvári, Frenkel and Kun [1].

Definition 2.2 (estimable functions)**.** *Let $\mathcal{G}$ be the set of finite unlabeled graphs and $\mathcal{H}$ a subset of $\mathcal{G}$. A function $f : \mathcal{G} \mapsto \mathbb{R}$ is estimable over $\mathcal{H}$ if for any sequence $(G_n), n \geq 1$, in $\mathcal{H}$ and any $\rho \in \mathcal{P}_{\text{sof}}(\mathcal{G}^*)$ such that $U(G_n)$ converges weakly to ρ, we have that $f(G_n)$ converges f_ρ, where f_ρ depends only on ρ.*

Let us give a toy example. Fix g a vertex transitive unlabeled graph with p vertices (for example a triangle). Define $f(G)$ as the density of distinct subgraphs of f isomorphic to g. More precisely, if $G = (V, E)$,

$$f(G) = \frac{1}{|V|} \sum_S \mathbf{1}(G \cap S \simeq g), \tag{3}$$

where the sum is over all subsets $S \subset V$ of cardinal p. It is possible to rewrite f as a function on rooted graphs :

$$f(G) = \frac{1}{p|V|} \sum_{v \in V} \varphi(G(v), v) = \mathbb{E}_{U(G)} \frac{\varphi(G, o)}{p},$$

where $\varphi(G, o)$ is the number of distinct subgraphs, rooted at o, rooted isomorphic to g^*, the rooted version of g. If G has diameter t then $\varphi(G, o)$ depends only on $(G, o)_t$ and it follows easily that φ is continuous on $\mathcal{G}^*$ rooted graph. Hence, if G_n has local weak limit $\rho \in \mathcal{P}(\mathcal{G}^*)$ and φ is uniformly integrable over the probability measures $U(G_n)$ then $f(G_n)$ converges to $\mathbb{E}_\rho \varphi(G, o)/p$. It implies for example that f is estimable on the space $\mathcal{G}_d$ of finite graphs with degrees bounded by d.

The above example is not very surprising. More complicated functions $f(G)$ are estimable in the above sense. For example, the following function is estimable on $\mathcal{G}_d$,

$$f(G) = \frac{1}{|V|} \log c_k(G),$$

where $c_k(G)$ is the number of proper k-coloring and $k > 2d$, Borgs, Chayes, Kahn and Lovász [17]. Similarly, if $t(G)$ is the number of spanning trees of G, the function

$$f(G) = \frac{1}{|V|} \log t(G)$$

is estimable over connected graphs of $\mathcal{G}$, Lyons [29, 30]. We will see other examples in the next sections.

Exercise 2.3. *Assume in the example (3) that g is not vertex-transitive. What is the analog of $\mathbb{E}_{U(G)} \varphi(G, o)/p$?*

2.4 Compact sets of graphs

Explicit compact subsets of $\mathcal{G}^*$ are easy to find. If $g \in \mathcal{G}^*$, we define $|g| = \sum_v \deg(v)$, i.e. twice the total number of edges in g.

Lemma 2.4. *Let $t_0 \geq 0$ and $\varphi : \mathbb{N} \to \mathbb{R}_+$ be a non-negative function. Then*

$$K = \{ g \in \mathcal{G}^* : \forall t \geq t_0,\, |g_t| \leq \varphi(t) \},$$

is a compact subset of $\mathcal{G}^*$ for the local topology.

Proof. For each $t \geq t_0$, there is a finite number of elements in $\mathcal{G}^*$, say $f_{t,1}, \cdots, f_{t,n_t}$, such that $|g| \leq \varphi(t)$ and for any vertex the distance to the root is at most t. Therefore, the collection $A_{t,1}, \cdots, A_{t,n_t}$ where $A_{t,k} = \{g \in \mathcal{G}^* : g_t = f_{t,k}\}$ is a finite covering of K of radius 2^{-t}. □

Let G_n be a sequence of finite graphs. We now give a condition which guarantees that the sequence $U(G_n)$ is tight for the local weak topology. The next lemma is a sufficient condition for tightness in $\mathcal{P}(\mathcal{G}^*)$, for a proof see [13, 9].

Lemma 2.5. *Let $\varphi : \mathbb{N} \to \mathbb{R}_+$ be a non-decreasing function such that $\varphi(x)/x \to \infty$ as $x \to \infty$. There exists a compact set $\Pi = \Pi(\varphi) \subset \mathcal{P}_{\mathrm{sof}}(\mathcal{G}^*)$ such that if a finite graph $G = (V, E)$ satisfies*

$$\frac{1}{|V|} \sum_{v \in V} \varphi(\deg_G(v)) \leq 1 \tag{4}$$

then $U(G) \in \Pi$.

Considering a sequence $U(G_n), n \geq 1$, condition (4) amounts to the uniform integrability of the degree sequences of the graphs $(G_n), n \geq 1$. It may seem quite paradoxical that a sole condition on the degrees implies the tightness of the whole graph sequence. However, the soficity of $U(G)$ yields enough uniformity for this result to hold. Lemma 2.5 shows that most graph sequences with typical degrees of order 1 are converging along subsequences in the Benjamini-Schramm sense.

2.5 Unimodularity

We may define similarly locally finite connected graphs with two roots (G, o, o') and extend the notion of isomorphisms to such structures. We define $\mathcal{G}^{**}$ as the set of equivalence classes of graphs (G, o, o') with two roots and associate its natural local topology. A function f on $\mathcal{G}^{**}$ can be extended to a function on connected graphs with two roots (G, o, o') through the isomorphism classes. Then, a measure $\rho \in \mathcal{P}(\mathcal{G}^*)$ is called *unimodular* if for any measurable function $f : \mathcal{G}^{**} \to \mathbb{R}_+$, we have

$$\mathbb{E}_\rho \sum_{v \in V} f(G, o, v) = \mathbb{E}_\rho \sum_{v \in V} f(G, v, o), \tag{5}$$

where under $\mathbb{P}_\rho$, (G, o) has law ρ. Unimodularity appears as a mass transport principle if $f(G, u, v)$ is thought as a quantity of mass sent from u to v in G : the average mass sent by the root is equal to the average mass it receives.

It is immediate to check that if G is finite then $U(G)$ is unimodular: indeed, if u and v are in the same connected component then by definition, $G(u) = G(v)$. It follows that

$$\begin{aligned}\mathbb{E}_{U(G)} \sum_{v \in V} f(G, o, v) &= \frac{1}{|V|} \sum_{u \in V} \sum_{v \in V(G(u))} f(G(u), u, v) \\ &= \frac{1}{|V|} \sum_{v \in V} \sum_{u \in V(G(v))} f(G(v), u, v) \\ &= \mathbb{E}_{U(G)} \sum_{v \in V} f(G, v, o).\end{aligned}$$

We will denote by $\mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ the set of unimodular measures. With a standard abuse of notation, we shall say that a random rooted graph (G, o) is unimodular if its law is unimodular.

Lemma 2.6. *The set $\mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ is closed for the local weak topology.*

Proof. We follow [10]. Let $\rho_n \to \rho$ and $f : \mathcal{G}^{**} \to \mathbb{R}_+$. Let $t > 0$ and $g \in \mathcal{G}_*$ with radius from the root at most t, observe that by dominated convergence (5) holds for $f_{t,g}(G, u, v) = t \wedge f(G, u, v)\mathbf{1}(d_G(u, v) \leq t)\mathbf{1}((G, u)_t \simeq g)$. Then, summing over all countably many g, it holds for $f_t(G, u, v) = t \wedge f(G, u, v)\mathbf{1}(d_G(u, v) \leq t)$. By monotone convergence, it also holds for f. □

In particular, the above lemma implies that all sofic measures are unimodular, the converse is open, for a discussion see [4]. It is however known that all unimodular probability measures supported on rooted trees are sofic, see Elek [23], Bowen [18], and for alternative proofs [9, 13]. In this last reference, the asymptotic number of graphs G with n vertices and m edges such that $U(G)$ is close to a given $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ is computed when ρ is supported on rooted trees.

The next lemma is a useful tool to check that a given measure is unimodular. For a proof see [4, Proposition 2.2].

Lemma 2.7 (involution invariance)**.** *Let $\rho \in \mathcal{P}(\mathcal{G}^*)$ such that (5) holds for all functions $f : \mathcal{G}^{**} \to \mathbb{R}_+$ such that $f(G, u, v) = 0$ unless $\{u, v\} \in E(G)$. Then ρ is unimodular.*

Exercise 2.8. *Using Lemma 2.7, show directly from their definitions that the unimodular Galton-Watson tree with degree distribution P and the skeleton tree are unimodular.*

The next lemma illustrates that for unimodular measures, the graph seen from root contains global information on the whole graph.

Lemma 2.9 (everything shows at the root)**.** *Let $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ and $f : \mathcal{G}^* \to \{0, 1\}$ be a measurable function such that ρ-a.s. $f(G, o) = 1$, then ρ-a.s. for all $v \in V$, $f(G, v) = 1$.*

Proof. Define $h(G, u, v) = \mathbf{1}(f(G, u) \neq 1)$ and apply (5). We obtain $0 = \mathbb{E}_\rho \sum_{v \in V} \mathbf{1}(f(G, v) \neq 1)$. □

Exercise 2.10. *Is the infinite 2-ary tree a unimodular graph ?*

2.6 Cayley graphs

Let Γ be a countable transitive group and $S \subset \Gamma$ a generating set such that $S^{-1} \subset S$ and the unit of Γ, say o, is not in S. The Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ associated to S has vertex set Γ and edge set $E = \{\{u,v\}, vu^{-1} \in S\}$. It is not hard to check that the graph G is vertex transitive. Also, the counting measure on Γ, $\nu = \sum_{v\in\Gamma} \delta_v$ is unimodular in the group theoretic sense (invariant by left and right multiplication). In particular, any function $f : \Gamma \times \Gamma \to \mathbb{R}_+$ invariant by right multiplication (i.e. such that $f(u,v) = f(u\gamma, v\gamma)$ for all $\gamma \in \Gamma$) will satisfy

$$\sum_{v\in\Gamma} f(o,v) = \sum_{v\in\Gamma} f(o,v^{-1}) = \sum_{v\in\Gamma} f(v,o).$$

It implies that if we define the measure $\rho \in \mathcal{P}(\mathcal{G}^*)$ which puts a Dirac mass at the equivalence class of (G,o), then ρ is unimodular.

2.7 Extension to weighted graphs

Let Ω be a Polish space whose distance is denoted by δ. A weighted graph (G,ω) is a graph $\bar{G} = (V,E)$ equipped with a weight function $\omega : V^2 \to \Omega$ such that $\omega(u,v) = 0$ if $u \neq v$ and $\{u,v\} \notin E$. The weight function is edge-symmetric if $\omega(u,v) = \omega(v,u)$ and $\omega(u,u) = 0$. A weighted graph is locally finite for any $v \in V$,

$$\sum_{u\in V} |\omega(u,v)| \vee |\omega(v,u)| \vee \mathbf{1}(\{u,v\} \in E) < \infty^{1}.$$

We define $\mathcal{G}^*_\Omega$ has the set of unlabeled locally finite weighted rooted graphs with weights in Ω. It is straightforward to extend the local topology to $\mathcal{G}^*_\Omega$, it suffices to replace in the definition of d_{loc} in (1) the indicator $\mathbf{1}(g_t \neq h_t)$ by

$$\mathbf{1}\big(\bar{g}_t \neq \bar{h}_t \text{ or for any isomorphism of } \bar{g}_t \text{ and } \bar{h}_t,\ \exists (u,v) \in V(\bar{g}_t)^2,\ \delta(\omega_g(u,v), \omega_h(u,v)) \leq 2^{-t}\big),$$

where the unlabeled rooted weighted graph g is written as $g = (\bar{g}, \omega_g)$. Then we may define similarly the local weak topology for random weighted graphs. The definition of unimodularity carries over naturally with the natural definition of $\mathcal{G}^{**}_\Omega$ (see the definition of unimodular network in [4]).

2.8 Stability of unimodularity

In the sequel, we will use a few times that unimodularity is stable by weights mappings, global conditioning and invariant percolation. More precisely, let (G,o) be a unimodular random weighted rooted graph with distribution ρ. The weights on G are denoted by $\omega : V^2 \to \Omega$. For ease of notation, we set $\mathcal{G}^*_\Omega = \mathcal{G}^*$ and $\mathcal{G}^{**}_\Omega = \mathcal{G}^{**}$. The following trivially holds :

[1] It is also possible to remove the term $\vee \mathbf{1}(\{u,v\} \in E)$ in the definition of locally finite weighted graphs, see [5]

Weight mapping : let $\psi : \mathcal{G}^* \to \Omega$ and $\phi : \mathcal{G}^{**} \to \Omega$ be two measurable functions. We define $\bar{G}$ as the weighted graph with weights $\bar{\omega}$, obtained from G by setting for $u \in V$, $\omega(u,u) = \psi(G,u)$ and for $u, v \in V^2$ with $\{u,v\} \in E(G)$, $\omega(u,v) = \phi(G,u,v)$. The random rooted weighted graph $(\bar{G}, o)$ is unimodular. Indeed, the $\mathcal{G}^* \to \mathcal{G}^*$ map $G \mapsto \bar{G}$ is measurable and we can apply (5) to $f(G,u,v) = h(\bar{G},u,v)$ for any measurable $h : \mathcal{G}^{**} \to \mathbb{R}_+$.

Global conditioning : let A be a measurable event on $\mathcal{G}^*$ which is invariant by re-rooting: i.e. for any (G,o) and (G',o) in $\mathcal{G}^*$ such that G and G' are isomorphic, we have $(G,o) \in A$ iif $(G',o) \in A$. Then, if $\rho(A) > 0$, the random rooted weighted graph (G,o) conditioned on $(G,o) \in A$ is also unimodular (apply (5) to $f(G,u,v) = \mathbf{1}((G,u) \in A))h(G,u,v)$ for any measurable $h : \mathcal{G}^{**} \to \mathbb{R}_+$).

Invariant percolation : let $B \subset \Omega$. We may define a random weighted graph $\hat{G}$ with edge set $E(\hat{G}) \subset E(G)$ by putting the edge $\{u,v\} \in E(G)$ in $E(\hat{G})$ if both $\omega(u,v)$ and $\omega(v,u)$ are in B. We leave the remaining weights unchanged. Then the random weighted rooted graph $(\hat{G}(o), o)$ is also unimodular (apply (5) to $f(G,u,v) = h(\hat{G}(u),u,v)$ for any measurable $h : \mathcal{G}^{**} \to \mathbb{R}_+$).

3 Minimal Spanning Trees

In some instances of continuous length combinatorial optimization problems, the local weak graph topology may help to define relevant infinite versions of these problems. In this section, we will illustrate this one of the simpler example, the minimal spanning tree. Notably, more advanced constructions have successfully been used for minimal weight matching and traveling salesperson tour on the weighted complete graph, see [3, 6, 35, 36]. We will adapt the presentation in [5].

3.1 Definition and basic properties

Let $\bar{G} = (V,E)$ be a finite connected graph, $\omega : V^2 \to \mathbb{R}_+$ an edge-symmetric weight function and $G = (\bar{G}, w)$ the associated weighted graph. We assume that for any subsets $A \neq B$ of E,

$$\sum_{e \in A} \omega(e) \neq \sum_{e \in B} \omega(e). \tag{6}$$

The *minimal spanning tree* $\mathrm{MST}(G)$ is the unique minimizer of

$$\tau(G) = \min_{T \in \mathrm{ST}(G)} \sum_{e \in E(T)} \omega(e), \tag{7}$$

where $\mathrm{ST}(G)$ is the set of spanning trees of $\bar{G}$ (uniqueness is a consequence of (6)). For $t > 0$, let $G(t) = (V, E(t))$, where $E(t) = \{e \in E : \omega(e) < t\}$ and for $v \in V$, let $G(t,v)$ be the connected component of v in $G(t)$. The following standard lemma gives a criterion to build to the minimal spanning tree.

Lemma 3.1. *An edge $e = \{u, v\} \in E$ belongs to* $\mathrm{MST}(G)$ *if and only if* $G(\omega(e), u) \cap G(\omega(e), v) = \emptyset$.

Lemma 3.1 can be used as a criterion to define the minimal spanning forest of an infinite graph.

Definition 3.2 (Minimal spanning forest)**.** *Let $G = (\bar{G}, \omega)$ be a locally finite weighted graph with $\omega : V^2 \to \mathbb{R}_+$ edge-symmetric such that (6) holds for all finite subsets $A \neq B$ of edges in E. The minimal spanning forest of G, $\mathrm{MSF}(G)$ is the graph with vertex set V and edges, the set of $e \in E$ such that (i) $G(\omega(e), u) \cap G(\omega(e), v) = \emptyset$ and (ii) $G(\omega(e), u)$ and $G(\omega(e), v)$ are not both infinite.*

It is easy to check that $\mathrm{MSF}(G)$ is indeed a forest (no cycles). Also, by Lemma 3.1, if G is finite, our definition is consistent and $\mathrm{MSF}(G) = \mathrm{MST}(G)$. Finally, if G is an infinite graph, then each connected component of $\mathrm{MSF}(G)$ is infinite.

3.2 Aldous-Steele continuity theorem

The following theorem illustrates that the Benjamini-Schramm topology has interesting continuous functions.

Theorem 3.3 (Aldous and Steele)**.** *Let $G_n = (\bar{G}_n, \omega_n)$ be a sequence of finite weighted connected graphs with edge-symmetric weights in $\mathbb{R}_+$ such that $U(G_n)$ converges weakly to $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*_{\mathbb{R}_+})$. Let $(\bar{G}, \omega, o)$ be a random weighted rooted graph with law ρ. We assume that condition (6) holds for G_n and ρ-a.s. for $G = (\bar{G}, \omega)$. Let $G'_n = (\bar{G}_n, \omega'_n)$ and $G' = (\bar{G}, \omega')$ be the weighted graphs with edge-symmetric weights in $\Omega = \mathbb{R}_+ \times \{0, 1\}$,*

$$\omega'_n(e) = (\omega_n(e), \mathbf{1}(e \in \mathrm{MST}(G_n))) \quad \text{and} \quad \omega'(e) = (\omega(e), \mathbf{1}(e \in \mathrm{MSF}(G))).$$

Then $U(G'_n)$ converges weakly to $\rho' \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*_{\Omega})$, the law of (G', o).

Under uniform integrability of the weights, Theorem 3.3 readily implies the convergence of the length of the minimal spanning tree defined in (7).

Corollary 3.4. *With the notation of Theorem 3.3, assume further that $\ell_n(o) = \sum_{v \in V_n} \omega_n(o, v)$ is uniformly integrable under $U(G_n)$, that is for some non-increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(x)/x \to \infty$ as $x \to \infty$,*

$$\frac{1}{|V_n|} \sum_{v \in V} \varphi(\ell_n(v)) \leq 1.$$

Then,

$$\lim_{n \to \infty} \frac{\tau(G_n)}{|V_n|} = \frac{1}{2} \mathbb{E}_\rho \sum_{v \in V} \omega(o, v) \mathbf{1}(\{o, v\} \in \mathrm{MSF}(G)).$$

In particular, $\tau(G)/|V|$ is estimable on the set of weighted graphs with bounded degrees and bounded weights.

Proof. Define the function on $\mathcal{G}^*_{\mathbb{R}_+}$,

$$L(G,o) = \frac{1}{2}\sum_{v\in V} \omega(o,v)\mathbf{1}(\{o,v\} \in \mathrm{MSF}(G)).$$

By the hand-shaking lemma, we have

$$\mathbb{E}_{U(G_n)} L = \frac{\tau(G_n)}{|V_n|}.$$

Theorem 3.3 implies that for any $t > 0$, $\mathbb{E}_{U(G_n)} L \wedge t$ converges to $\mathbb{E}_\rho L \wedge t$. The uniform integrability assumption allows to take safely the limit $t \to \infty$. □

The key lemma in the proof of Theorem 3.3 is the following consequence of unimodularity. It relates infinite graphs and mean degree of the root.

Lemma 3.5. *Let $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$ be supported on infinite rooted graphs. Then $\mathbb{E}_\rho \deg_G(o) \geq 2$.*

Proof. If $e = \{u,v\} \in E$ is an edge of $G = (V,E)$, we define the (non-symmetric) weight $\ell(u,v)$ to be equal to (f,f), (∞,f), (f,∞) or (∞,∞) depending on whether the connected components of u and v in $G\backslash\{e\}$ are finite or infinite. For $a,b \in \{f,\infty\}$, we denote by D_{ab} the number of edges (o,v) such that $\ell(o,v) = (a,b)$. If G is infinite then $D_{ff} = 0$. Moreover, we have

$$2D_{f\infty} + D_{\infty\infty} \geq 2. \tag{8}$$

Indeed, either there is an edge $\{o,v\}$ such that $\ell(o,v) = (f,\infty)$ or there are at least two edges such that $\ell(o,v) = (\infty,\infty)$ (there cannot be just one in this case).

Applying unimodularity (5) to $f(G,u,v) = \mathbf{1}(\ell(u,v) = (f,\infty))$ gives

$$\mathbb{E}_\rho D_{f\infty} = \mathbb{E}_\rho \sum_{v\in V} \mathbf{1}(\ell(o,v) = (f,\infty)) = \mathbb{E}_\rho \sum_{v\in V} \mathbf{1}(\ell(v,o) = (f,\infty)) = \mathbb{E}_\rho D_{\infty f},$$

(if $\ell(u,v) = (a,b)$ then if $\ell(v,u) = (b,a)$). We deduce from (8) that

$$\mathbb{E}_\rho(D_{f\infty} + D_{\infty f} + D_{\infty\infty}) \geq 2.$$

We conclude by observing that $\deg_G(o) = D_{f\infty} + D_{\infty f} + D_{\infty\infty}$. □

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. Since $U(G_n)$ converges to $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*_{\mathbb{R}_+})$ and $\{0,1\}$ is a finite set, $U(G'_n)$ is tight in $\mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*_\Omega)$. Up to considering subsequences, we may assume that $U(G'_n)$ converges to $\tilde\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*_\Omega)$, the law of $(\bar G, \tilde\omega, o)$ where $\tilde\omega(e) = (\omega(e), \mathbf{1}(e \in S))$ and S is a subset of edges of $\bar G$. Identifying a graph with its edges, we should prove that $\tilde\rho$-a.s. $S = \mathrm{MSF}(G)$.

We start with the inclusion $\mathrm{MSF}(G) \subset S$. Let o_n be uniformly distributed on the vertices of G_n. From Skorohod's representation theorem, we may assume that on a common probability space, with probability one, $(\bar{G}_n, \tilde{\omega}_n, o_n)$ converges to $(\bar{G}, \tilde{\omega}, o)$ for the distance d_{loc}. Let $e = \{u, v\} \in \text{MSF}(G)$. We may assume that $G(\omega(e), u)$ is finite. We may take t large enough such that $(G, o)_t$ contains $G(\omega(e), u)$ and $G(\omega(e) - 2^{-t}, u) = G(\omega(e), u)$ (from (6)). From the definition of d_{loc}, for all n large enough, $(\bar{G}_n, o_n)_{t+1}$ is rooted isomorphic to $(\bar{G}, o)_{t+1}$ and all weights in $(G_n, o_n)_{t+1}$ are within distance 2^{-t-1} from their corresponding weight in $(G, o)_{t+1}$. If u_n, v_n and $e_n = \{u_n, v_n\}$ denote the vertices and edge associated with u, v and e by the isomorphism, then there can be no path in G_n from u_n to v_n that has all of its edges shorter than $\omega_n(e)$. Thus, we find that for all n large enough, the edge e_n is in $\text{MST}(G_n)$. Since $\text{MST}(G_n)$ converges to S, we get the inclusion $\text{MSF}(G) \subset S$.

For the converse inclusion $S \subset \text{MSF}(G)$, we may assume without loss of generality that ρ is supported on infinite graphs (on the event that G is a finite graph, the equality of S and $\text{MST}(G) = \text{MSF}(G)$ is trivial and conditioning on G infinite preserves unimodularity, see §2.8). First, we have

$$\mathbb{E}_{U(G_n)} \deg_{\text{MSF}(G)}(o) = \frac{1}{|V_n|} \sum_{v \in V_n} \deg_{\text{MST}(G_n)}(v) = 2 \frac{|V_n| - 1}{|V_n|} \leq 2.$$

Hence, since $\text{MST}(G_n)$ converges to S, from Fatou's Lemma, we find

$$\mathbb{E}_{\tilde{\rho}} \deg_S(o) \leq 2.$$

However, since G is an infinite graph, the connected component of the root in $\text{MSF}(G)$, $\text{MSF}(G)(o)$ is an infinite tree. By Lemma 3.5, we get

$$\mathbb{E}_{\tilde{\rho}} \deg_{\text{MSF}(G)}(o) \geq 2.$$

From the first inclusion $\text{MSF}(G) \subset S$, we deduce that $\mathbb{E}_{\tilde{\rho}} \deg_{\text{MSF}(G)}(o) = 2$ and

$$\mathbb{E}_{\tilde{\rho}} \sum_{v \in V} \mathbf{1}(\{o, v\} \in S \backslash \text{MSF}(G)) = \mathbb{E}_{\tilde{\rho}} \deg_S(o) - \mathbb{E}_{\tilde{\rho}} \deg_{\text{MSF}(G)}(o) = 0.$$

In particular, $\tilde{\rho}$-a.s. no neighbor of the root is in $S \backslash \text{MSF}(G)$. It remains to apply Lemma 2.9. □

Exercise 3.6. *We define an* end *of a rooted infinite tree (T, o) as a semi-infinite self-avoiding path on T starting from o. Let $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ be supported on infinite rooted graph. Show that if $\mathbb{E}_\rho \deg_G(o) = 2$ then ρ-a.s. (G, o) is a tree and it has one or two ends. (Hint : refine the proof of Lemma 3.5).*

3.3 Example of an explicit computation

The unimodularity can often be used more or less implicitly to derive close form interesting formulas. We will illustrate this on the minimal spanning forest of a random weighted tree $T = (\bar{T}, \omega)$, where $(\bar{T}, o)$ has distribution $\text{UGW}(P)$, where P is a probability measure on $\mathbb{N}$ such that

$$\sum_{k \geq 0} k^2 P(k) < \infty$$

and

$$P(0) = P(1) = 0. \tag{9}$$

For $x \in [0,1]$, the generating functions of P and $\widehat{P}$ defined by (2) are $\varphi(x) = \sum_{k\geq 0} P(k)x^k$ and $\widehat{\varphi}(x) = \varphi'(x)/\varphi'(1)$. Given $\bar{T}$, we consider a collection $(\omega(e))_{e\in E}$ of iid variables uniform on $[0,1]$. It builds a weighted tree $T = (\bar{T}, \omega)$ and we may then define the associated minimal spanning forest $\mathrm{MSF}(T)$.

Lemma 3.7. *Let ρ be the law of the above weighted random rooted tree (T, o). We have*

$$\tau = \frac{1}{2}\mathbb{E}_\rho \sum_{v\in V} w(o,v)\mathbf{1}(\{o,v\} \in \mathrm{MSF}(T)) = -\varphi'(1)\int_0^1 (1-x)\ln\left(1-\widehat{\varphi}(x)\right)dx.$$

Lemma 3.7 can be used in conjunction with Corollary 3.4. For example, if $\bar{T}$ is the infinite d-regular tree, we find

$$\tau = -d\int_0^1 (1-x)\ln\left(1-x^{d-1}\right)dx.$$

Proof of Lemma 3.7. For $t > 0$, let $S(t) = T(t, o)$ be the connected component of the root in the forest spanned by weights less than t. If P_t is the law of $\deg_{S(t)}(o)$, it is a simple exercise to check that

$$\varphi_t(x) = \mathbb{E}[x^{\deg_{S(t)}(o)}] = \varphi(1-t+tx),$$

and $(S(t), o)$ has distribution $\mathrm{UGW}(P_t)$. We denote by $\widehat{S}(t)$ a Galton-Watson tree with offspring distribution $\widehat{P}_t$ and by $(1, \ldots, N)$, the neighbors of the root o in T. Using the conditional independence of $((T\backslash\{o,1\})(o), o)$ and $((T\backslash\{o,1\})(1), 1)$ given N, we have

$$\begin{aligned}
2\tau &= \sum_{k\geq 1} kP(k)\int_0^1 t\mathbb{P}(\{o,1\} \in \mathrm{MSF}(T)|\omega(o,1) = t, N = k)dt \\
&= \sum_{k\geq 1} kP(k)\int_0^1 t\Big(1 - \mathbb{P}\Big(|\widehat{S}(t)| = \infty\Big)\mathbb{P}(|S(t)| = \infty|N = k-1)\Big)dt.
\end{aligned}$$

Now, from (2), if $(\widehat{T}, o)$ is weighted Galton-Watson tree with offspring distribution $\widehat{P}$ then, for any measurable function $h : \mathcal{G}^*_{[0,1]} \to \mathbb{R}_+$,

$$\sum_{k\geq 1} kP(k)\mathbb{E}[h(T,o)|N = k-1] = \varphi'(1)\mathbb{E}h(\widehat{T}, o)^{2}. \tag{10}$$

Applied to $h = \mathbf{1}(|S(t)| = \infty)$, we deduce that

$$2\tau = \varphi'(1)\int_0^1 t\left(1 - \mathbb{P}\left(|\widehat{S}(t)| = \infty\right)^2\right)dt = \varphi'(1)\int_0^1 t\left(1 - q(t)^2\right)dt,$$

[2]Writing the left hand side of (10) as $\mathbb{E}\sum_{v\in V}\mathbf{1}(\{o,v\} \in E)h((T\backslash\{o,v\})(o), o))$, we see that (10) is the unimodularity equation (5) applied to $f(G,u,v) = \mathbf{1}(\{u,v\} \in E)h((G\backslash\{u,v\})(u), u))$

where $\rho(t) = 1 - q(t)$ is the smallest solution in $[0,1]$ of $x = \widehat{\varphi}_t(x)$. Now, from (9), $\widehat{\varphi}$ and q are C^1 invertible maps from $[0,1]$ to $[0,1]$. We also observe that $1 - q(t) = \widehat{\varphi}(1 - tq(t))$, hence

$$t = \frac{1 - \widehat{\varphi}^{-1}(1 - q(t))}{q(t)}$$

By integration by parts and change of variable $s = 1 - q(t)$, it follows that

$$\int_0^1 t(1 - q(t)^2)dt = \int_0^1 t^2 q'(t)q(t)dt = \int_0^1 \frac{(1 - \widehat{\varphi}^{-1}(s))^2}{1 - s} ds.$$

We now repeat the same procedure, by integration by parts and change of variable $x = \widehat{\varphi}^{-1}(s)$, we get

$$-\int_0^1 \frac{(1 - \widehat{\varphi}^{-1}(s))^2}{1 - s} ds = 2\int_0^1 (\widehat{\varphi}^{-1})'(s)(1 - \widehat{\varphi}^{-1}(s))\ln(1 - s)ds = 2\int_0^1 (1 - x)\ln(1 - \widehat{\varphi}(x))dx.$$

$\square$

Exercise 3.8. *We consider the skeleton tree and we put iid uniform* $[0,1]$ *weights on its edges. What is its minimal spanning forest ?*

4 Graph Matchings

We will now give a new illustration of the local weak topology in the context of counting problems and convergence of Boltzmann-Gibbs measure. We will restrict ourselves to the case of graph matchings. Many other cases have been treated both in the mathematical and physics literature, to a cite a few [32, 17, 29, 21, 34]. The content of this section is essentially contained in [16] and use important ingredients from [26, 25, 37].

4.1 Definition and first properties

A *matching* on a finite graph $G = (V, E)$ is a subset of pairwise non-adjacent edges $M \subseteq E$. The $|V| - 2|M|$ isolated vertices of (V, M) are said to be *exposed* by M. We let $\mathbb{M}(G)$ denote the set of all possible matchings on G. The *matching number* of G is defined as

$$\nu(G) = \max_{M \in \mathbb{M}(G)} |M|, \tag{11}$$

and those M which achieve this maximum – or equivalently, have the fewest exposed vertices – are called *maximum matchings*.

The partition function of matchings is encoded into the *matching polynomial*,

$$P_G(z) = \sum_{M \in \mathbb{M}(G)} z^{|V| - 2|M|}.$$

Note that $P_G(1) = |\mathbb{M}(G)|$ is the number of matchings in G.

We now consider a natural family of probability distributions on the set of matchings $\mathbb{M}(G)$, parametrized by a single parameter $z > 0$ called the *temperature* (note that the standard temperature T in physics would correspond to $z = e^{-1/T}$ but this will not be important here): for any $M \in \mathbb{M}(G)$,

$$\mu_G^z(M) = \frac{z^{|V|-2|M|}}{P_G(z)}, \tag{12}$$

In statistical physics, this is called the *monomer-dimer model* at temperature z on G (see Heilmann and Lieb [26] for a complete treatment). At temperature $z = 1$, μ_G^1 is the uniform distribution on matchings. Note also that the lowest degree coefficient of P_G is precisely the number of maximal matchings on G. Therefore, μ_G^z converges to the uniform distribution on the maximum matching as the temperature z tends to zero. The *free entropy* of μ_G^z is defined as,

$$\log P_G(z).$$

In particular, the free entropy of μ_G^1 is $\log |\mathbb{M}(G)|$.

In the sequel, we will consider a sequence of finite graphs G_n converging to a local weak limit $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$. As we shall see the asymptotic of $\nu(G_n)$, $\mu_{G_n}^z$ and $\log P_{G_n}(z)$ can be computed directly on the local weak limit ρ. To perform this, following Zdeborová and Mézard [37], we introduce the main order parameter, the *cavity ratio* of rooted graph (G, o) defined as

$$R_z(G, o) = \frac{P_{G-o}(z)}{P_G(z)}, \tag{13}$$

where $G - o$ be the graph obtained from G by removing o. The basic identities of statistical physics imply that the Boltzmann-Gibbs measure μ_G^z can be recovered from the cavity ratios.

Lemma 4.1. *For any finite graph G and $o \in V$, we have*

$$\mu_G^z\,(o \textit{ is exposed}) = zR_z(G, o).$$

If M has distribution μ_G^z, and $m \in \mathbb{M}(G)$, the cylinder-event marginals satisfy

$$\mu_G^z(m \subseteq M) = \prod_{k=1}^{2|m|} R_z(G - \{v_1, \ldots, v_{k-1}\}, v_k),$$

they are consistent and independent of the ordering $v_1, \ldots v_{2|m|}$ of the vertices spanned by m. Finally

$$\log P_G(z) = |V| \log z + \sum_{v \in V} \int_z^\infty \frac{1 - xR_x(G, v)}{x} dx.$$

Proof. The matching which expose o are precisely the matchings of $G-o$. It gives the first formula. Similarly, for the second formula, matchings which contain m are the machings of form $m \cup m'$ where $m' \in \mathbb{M}(G - \{v_1, \ldots, v_{2|m|}\})$. We get

$$\mu_G^z(m \subseteq M) = \frac{P_{G-\{v_1,\ldots,v_{2|m|}\}}(z)}{P_G(z)} = \prod_{k=1}^{2|m|} \frac{P_{G-\{v_1,\ldots,v_k\}}(z)}{P_{G-\{v_1,\ldots,v_{k-1}\}}(z)}.$$

For the last formula, we first observe that

$$\begin{aligned} P'_G(z) &= \sum_{M\in\mathbb{M}(G)} (|V| - 2|M|) z^{|V|-1-2|M|} \\ &= \sum_{M\in\mathbb{M}(G)} \sum_{v\in V} \mathbf{1}(v \text{ exposed in } M) z^{|V|-1-2|M|} \\ &= \sum_{v\in V} P_{G-v}(z). \end{aligned}$$

We deduce that the logarithmic derivative of P_G satisfies,

$$(\log P_G(z))' = \sum_{v\in V} R_z(G,v). \tag{14}$$

Integrating for any $z < y$,

$$\begin{aligned} \log P_G(z) &= \log P_G(y) - \sum_{v\in V} \int_z^y R_s(G,v)ds \\ &= |V| \log z + \log\left(y^{-|V|} P_G(y)\right) + \sum_{v\in V} \int_z^y \left(\frac{1}{x} - R_x(G,v)\right)dx \end{aligned}$$

Since P_G is a monic polynomial of degree $|V|$, as $y \to \infty$, $y^{-|V|}P_G(y)$ goes to 1. Also, since $R_y(G,v)$ is a ratio of two monic polynomials of degree $|V|-1$ and $|V|$, we have, as $y \to \infty$, $R_y(G,v) = 1/y + O(1/y^2)$ (where the $O(\cdot)$ depends on G). We may thus take the limit as $y \to \infty$ in the above expression. □

Dividing the matchings on whether or not they expose the root, we find the recursion,

$$P_G(z) = zP_{G-o}(z) + \sum_{v\sim o} P_{G-o-z}(z). \tag{15}$$

Dividing by $P_{G-o}(z)$,

$$\frac{1}{R_z(G,o)} = z + \sum_{v\sim o} R_z(G-o,v).$$

We thus arrive at the fundamental local recursion :

$$R_z(G,o) = \left(z + \sum_{v\sim o} R_z(G-o,v)\right)^{-1}. \tag{16}$$

The recursion (16) determines uniquely the functional R_z on the class of finite rooted graphs, and may thus be viewed as an inductive definition of the cavity ratio. We will use this alternative characterization to define a continuous extension to infinite graphs with bounded degree, even though the above recursion never ends. The key argument will be a development around the following remarkable Lee-Yang type theorem.

Theorem 4.2 (Heilmann-Lieb [26]). *The roots of $P_G(z)$ are pure imaginary.*

Sketch of proof. The polynomial $Q_G(z) = \sum_{M \in \mathbb{M}(G)} (-1)^{|M|} z^{|V|-2|M|} = (-i)^{|V|} P_G(iz)$ satisfies from (15) the recursion $Q_G(z) = z Q_{G-o}(z) - \sum_{v \sim o} Q_{G-o-z}(z)$. As in [26, Theorem 4.2], we may then check by recursion on the size of the graph that the roots of Q_G are real. In the forthcoming Lemma 4.3, we will give an alternative proof using instead the recursion (16). □

A consequence of Theorem 4.2 is that the domain of analycity of $z \mapsto R_z(G, o)$ contains the right complex half-plane

$$\mathbb{H}_+ = \{z \in \mathbb{C} : \Re(z) > 0\}$$

(see [20, 19] for generalizations of Heilmann-Lieb's Theorem and related Lee-Yang type theorems in combinatorics).

4.2 The monomer-dimer model on infinite graphs

4.2.1 Continuity of the monomer-dimer model

We let $\mathcal{H}$ denote the space of analytic functions on $\mathbb{H}_+$, equipped with its usual topology of uniform convergence on compact sets. To be precise, let $\{K_j\}_{j \geq 1}$ be an exhaustion of $\mathbb{H}_+$ by compact sets, that is K_{j+1} is contained in the interior of K_j and any compact K of $\mathbb{H}_+$ is contained in some K_j. We introduce the distance on $\mathcal{H}$,

$$d(f, g) = \sum_{j \geq 1} 2^{-j} \frac{\|f - g\|_{L^\infty(K_j)}}{1 + \|f - g\|_{L^\infty(K_j)}},$$

where, $\|g\|_{L^\infty(K)} = \sup_{z \in K} |g(z)|$. Then $(\mathcal{H}, d)$ is a complete separable metric space. We fix an integer d and define $\mathcal{G}^*_d \subset \mathcal{G}^*$ as the set of unlabeled rooted graphs with degrees bounded by d. The next central lemma can be seen as an extension of Heilmann-Lieb's Theorem 4.2.

Lemma 4.3 (Local recursion). *The following holds,*

(i) For every fixed $z \in \mathbb{H}_+$, the local recursion (16) determines a unique $R_z : \mathcal{G}^*_d \to \mathbb{H}_+$.

(ii) For every fixed $g \in \mathcal{G}^*_d$, $R(g) : z \mapsto R_z(g)$ is in $\mathcal{H}$.

(iii) The $\mathcal{G}^*_d \to \mathcal{H}$ mapping $g \mapsto R(g)$ is continuous.

This above result has strong implications for the monomer-dimer model, which we now list. The first one is the existence of an infinite volume limit for the Gibbs-Boltzmann distribution.

Theorem 4.4 (Monomer-dimer model on infinite graphs)**.** *Consider a graph G with degrees bounded by d and a temperature $z > 0$. As m ranges over all finite matchings of G, the cylinder-event marginals*

$$\mu_G^z(m \subseteq M) = \prod_{k=1}^{2|m|} R_z(G - \{v_1, \dots, v_{k-1}\}, v_k),$$

are consistent and independent of the ordering $v_1, \dots v_{2|m|}$ of the vertices spanned by m. They thus determine a unique probability distribution μ_G^z over the matchings of G. It coincides with the former definition in the case where G is finite, and extends it continuously in the following sense : for any $o \in V$ and for any sequence (G_n, o_n) of finite rooted graphs in $\mathcal{G}_d^*$ converging for the local topology to (G, o), we have

$$(G_n, o_n, M_n) \xrightarrow[n\to\infty]{d} (G, o, M),$$

in the local weak sense for random weighted graphs, where M_n has law $\mu_{G_n}^z$ and M has law μ_G^z.

We may also deduce from the continuity of the *free entropy* in the monomer-dimer model.

Corollary 4.5. *Let $G_n = (V_n, E_n)$ be a sequence of finite graphs such that $U(G_n)$ converges weakly to $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ and $|E_n| = O(|V_n|)$. There exists an analytic function $s(z)$ on $\mathbb{H}_+$ which depends only on ρ such that for any $z > 0$,*

$$\lim_{n\to\infty} \frac{1}{|V_n|} \log P_{G_n}(z) = s(z).$$

Finally, if ρ-a.s. $\deg_G(o) \le d$ for some integer $d \ge 1$ then $s(z) = \log z + \int_z^\infty \frac{1 - x\mathbb{E}_\rho R_x(G,o)}{x} dx$, where under $\mathbb{E}_\rho$, (G, o) has distribution ρ, the function $z \mapsto \log z$ is the branch of the logarithm analytic on $\mathbb{C}\backslash\mathbb{R}_-$ and the integral $\int_z^\infty$ is the limit of the integral from $z \in \mathbb{H}_+$ to $y \in \mathbb{R}_+$, $y \to +\infty$.

A similar result was established in [26] for the lattice case and in [8, 7, 34, 1] under various assumptions. It implies notably that the free entropy for matchings is estimable over all bounded degree graphs. It is possible to extend Lemma 4.3 and Theorem 4.4 to non-uniformly bounded graphs to graphs whose path trees have finite branching number (for the definition of paths trees see [25] and branching number see [31]).

Without entering into details, it can be proved that $g_z(G, o) = iR_{-iz}(G, o)$ is the Cauchy-Stieltjes transform of a symmetric probability measure on $\mathbb{R}$, the *matching measure*, $\mu_{(G,o)}$. That is, for any $z \in \mathbb{C}\backslash\mathbb{R}$,

$$g_z(G, o) = iR_{-iz}(G, o) = \int \frac{d\mu_{(G,o)}(\lambda)}{\lambda - z}. \tag{17}$$

Then, for real $x > 0$, $R_x(G, o) = \int x/(\lambda^2 + x^2) d\mu_{(G,o)}(\lambda)$ and

$$s(z) = \mathbb{E} \int \log |iz + \lambda| d\mu_{(G,o)}(\lambda),$$

see [1, 2]. Also, it follows from [25] that $\mu_{(G,o)}$ is the spectral measure at the root of the adjacency operator of the path-tree of (G, o). This illustrates interesting links between spectral graph theory and matching theory, see the monographs [25, 28].

4.2.2 Proof of Lemma 4.3

The local recursion (16) involves mappings of the form :

$$\phi_{z,d} \colon (x_1, \ldots, x_d) \mapsto \left(z + \sum_{i=1}^{d} x_i\right)^{-1},$$

where $d \geq 0$ is an integer. In the following lemma, we gather a few elementary properties of this transformation, which are immediate to check but will be of constant use throughout the paper.

Lemma 4.6. *For any integer $d \geq 0$ and $z \in \mathbb{H}_+$,*

(i) $\phi_{z,d}$ maps analytically $\mathbb{H}_+^d$ into $\mathbb{H}_+$

(ii) $|\phi_{z,d}|$ is uniformly bounded by $1/\Re(z)$ on $\mathbb{H}_+^d$.

From Lemma 4.6(i), it follows that the cavity ratio of a finite rooted graph belongs to $\mathcal{H}$, when viewed as a function of the temperature z. Lemma 4.6(ii) and Montel's theorem guarantee that the family of all those cavity ratios is tight in $\mathcal{H}$. This analytic tightness can also be found in [26]. Combined with the following uniqueness property at high temperature, it will quickly lead to the proof of Lemma 4.3.

The local recursion (16) also involves graph transformations of the form $(G, o) \mapsto (G - o, v)$, where $v \sim o$. Starting from a given $(G, o) \in \mathcal{G}_d^*$, we let $\mathrm{Succ}^*(\mathrm{G}, \mathrm{o}) \subseteq \mathcal{G}_{\mathrm{d}}^*$ denote the (denumerable) set of all rooted graphs that can be obtained by successively applying finitely many such transformations. Let

$$\mathbb{D} = \left\{z \in \mathbb{C}; \Re(z) > \sqrt{d}\right\} \subseteq \mathbb{H}_+.$$

Lemma 4.7 (Uniqueness at high temperature)**.** *Let $(G, o) \in \mathcal{G}_d^*$ and $z \in \mathbb{D}$. If*

$$R_z^1, R_z^2 \colon \mathrm{Succ}^*(\mathrm{G}, \mathrm{o}) \to \mathbb{H}_+$$

both satisfy the local recursion (16) then $R_z^1 = R_z^2$.

Proof. Set $\alpha = 2/\Re(z)$ and $\beta = \Re(z)^{-2}$. From (16) and Lemma 4.6(ii), the absolute difference $\Delta = |R_z^1 - R_z^2|$ must satisfy

$$\Delta(G, o) \leq \alpha \qquad \text{and} \qquad \Delta(G, o) \leq \beta \sum_{v \sim o} \Delta(G - o, v).$$

In turn, each $\Delta(G - o, v)$ appearing in the second upper-bound may be further expanded into $\beta \sum_{w \sim v, w \neq o} \Delta(G - o - v, w)$. Iterating this procedure k times, one obtains $\Delta(G, o) \leq \beta^k d^k \alpha$. Taking the infimum over all k yields $\Delta(G, o) = 0$, since $z \in \mathbb{D}$ means precisely $\beta d < 1$. $\square$

We may now prove Lemma 4.3.

Proof of Lemma 4.3. For clarity we divide the proof in three parts.

Step 1 : Analytic existence. Fix $(G,o) \in \mathcal{G}_d^*$, and consider an arbitrary collection of $\mathbb{H}_+ \to \mathbb{H}_+$ analytic functions $z \mapsto R_z^0(H,u)$, indexed by the elements $(H,u) \in \mathrm{Succ}^*(\mathrm{G},\mathrm{o})$. For every $n \geq 1$, define recursively

$$R_z^n(H,u) = \left(z + \sum_{v \sim u} R_z^{n-1}(H-u,v)\right)^{-1}, \tag{18}$$

for all $z \in \mathbb{H}_+$ and $(H,u) \in \mathrm{Succ}^*(\mathrm{G},\mathrm{o})$. By Lemma 4.6, each sequence $(z \mapsto R_z^n(H,u))\,, n \in \mathbb{N}$, is tight in $\mathcal{H}$. Consequently, their joint collection as (H,u) varies in the denumerable set $\mathrm{Succ}^*(\mathrm{G},\mathrm{o})$ is sequentially tight in the product space $\mathcal{H}^{\mathrm{Succ}^*(\mathrm{G},\mathrm{o})}$. Passing to the limit in (18), we see that any pre-limit $R_z \colon \mathrm{Succ}^*(\mathrm{G},\mathrm{o}) \to \mathbb{H}_+$ must automatically satisfy (16) for each $z \in \mathbb{H}_+$. By Lemma 4.7, this determines uniquely the value of $R_z(G,o)$ for z with sufficiently large real part, and hence everywhere in $\mathbb{H}_+$ by analycity. To sum up, we have just proved the following : for every $(G,o) \in \mathcal{G}_d^*$, the limit

$$R(G,o) := \lim_{n\to\infty} R^n(G,o) \tag{19}$$

exists in $\mathcal{H}$, satisfies the recursion (16), and does not depend upon the choice of the initial condition $R^0 \in \mathcal{H}^{\mathrm{Succ}^*(\mathrm{G},\mathrm{o})}$.

Step 2 : Pointwise uniqueness. Let us now show that any $S \colon \mathrm{Succ}^*(\mathrm{G},\mathrm{o}) \to \mathbb{H}_+$ satisfying the recursion (16) at a fixed value $z = z_0 \in \mathbb{H}_+$ must coincide with the $z = z_0$ specialization of the analytic solution constructed above. For each $(H,u) \in \mathrm{Succ}^*(\mathrm{G},\mathrm{o})$, the constant initial function $R_z^0(H,u) = S(H,u)$ is trivially analytic from $\mathbb{H}_+$ to $\mathbb{H}_+$, so the iteration (18) must converge to the analytic solution R_z. Since $R_{z_0}^n = S$ for any integer $n \geq 0$, we obtain $R_{z_0} = S$, as desired. The first two steps prove claims (i)-(ii) of the lemma.

Step 3 : Continuity. Finally, we prove claim (iii). Consider a sequence $(G_n,o)\,, n \geq 1$ in $\mathcal{G}_d^*$ converging locally to (G,o). Let us show that in $\mathcal{H}$,

$$R(G_n,o_n) \xrightarrow[n\to\infty]{} R(G,o). \tag{20}$$

It is routine that, up to rooted isomorphisms, $G, G_1, G_2, \ldots$ may be represented on a common vertex set, in such a way that $o = o_n$ and for each fixed $k \geq 1$, $(G_n,o)_k = (G,o)_k$ for all $n \geq n_k$. By construction, any simple path $v_1 \ldots v_k$ starting from the root in G is now also a simple path starting from the root in each $G_n, n \geq n_k$, so the $\mathcal{H}-$valued sequence $(R(G_n - \{v_1,\ldots,v_{k-1}\},v_k))\,, n \geq n_k$, is well defined, and tight (Lemma 4.6). Again, the denumerable collection of all sequences obtained by letting the simple path $v_1 \ldots v_k$ vary in (G,o) is sequentially tight for the product topology, and any pre-limit must by construction satisfy (16). By pointwise uniqueness, the convergence (20) must hold. $\square$

4.2.3 Convergence of the Boltzmann distribution : proof of Theorem 4.4 and Corollary 4.5

Proof of Theorem 4.4. Consider an infinite $(G, o) \in \mathcal{G}_d^*$, and let $(G_n, o_n)_{n\geq 1}$ be a sequence of finite rooted connected graphs converging locally to (G, o). As above, represent $G, G_1, G_2, \dots$ on a common vertex set, in such a way that $o_n = o$ and for each $k \geq 1$, $(G_n, o)_k = (G, o)_k$ for all $n \geq n_k$. Now fix an arbitrary finite matching m in G, and denote by $v_1, \dots, v_{2|m|}$ the vertices spanned by m, in any order. By construction, m is also a matching of G_n for large enough n. But the matchings of G_n that contain m are exactly the matchings of $G_n - \{v_1, \dots, v_{2|m|}\}$, and hence, by Lemma 4.1,

$$\mu_{G_n}^z(m \subseteq M) = \prod_{k=1}^{2|m|} R_z(G_n - \{v_1, \dots, v_{k-1}\}, v_k).$$

But $(G_n - \{v_1, \dots, v_{k-1}\}, v_k)$ converges locally to $(G - \{v_1, \dots, v_{k-1}\}, v_k)$, and by Lemma 4.3(iii) R_z is continuous. We get

$$\mu_{G_n}^z(m \subseteq M) \xrightarrow[n\to\infty]{} \prod_{k=1}^{2|m|} R_z(G - \{v_1, \dots, v_{k-1}\}, v_k).$$

as requested. □

Proof of Corollary 4.5. From recursion (16) and Lemma 4.6(ii), we have

$$|1 - xR_x(G, o)| = \left| \frac{\sum_{v\sim o} R_x(G - o, v)}{x + \sum_{v\sim o} R_x(G - o, v)} \right| \leq \frac{\deg_G(o)}{\Re(x)^2}.$$

On the other end, from Lemma 4.1, for any $z \in \mathbb{H}_+$,

$$\frac{1}{|V_n|} \log P_{G_n}(z) = \log z + \int_z^\infty \frac{1 - x\mathbb{E}_{U(G_n)} R_x(G, v)}{x} dx.$$

From what precedes, if $2|E_n| \leq c|V_n|$,

$$\left| \frac{1 - x\mathbb{E}_{U(G_n)} R_x(G, v)}{x} \right| \leq \frac{c}{\Re(x)^3}.$$

Analytic convergence of the free entropy follows from Theorem 4.4 and Montel's Theorem. □

4.3 The zero-temperature limit

4.3.1 Continuity of the matching number

Motivated by the asymptotic study of maximum matchings, we now let the temperature $z \to 0$ and study the matching number defined in (11).

Theorem 4.8 (matching number is estimable)**.** *Let $G_n = (V_n, E_n), n \in \mathbb{N}$, be a sequence of finite graphs such that $U(G_n)$ converges weakly to $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$. Then,*

$$\lim_{n\to\infty} \frac{\nu(G_n)}{|V_n|} = \gamma, \tag{21}$$

where γ depends only on ρ. Moreover, if $\deg_G(o) \leq d$ for some integer $d \geq 1$ then

$$\gamma = \frac{1 - \lim_{z\to 0} \mathbb{E}_\rho z R_z(G,o)}{2}, \tag{22}$$

where under $\mathbb{E}_\rho$, (G,o) has distribution ρ.

Statement (21) was first proved by Elek and Lippner [24], when the underlying graph is the lattice, it is a consequence of [26, Lemma 8.7]. We will follow the proof of [16] which is more constructive and we allow explicit computations in the next subsection. To some extent it is again possible to extend the definition (22) for γ to a larger class of $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$. Also, it can expressed in terms of the matching measure defined in (17) as $\gamma = (1 - \mathbb{E}_\rho \mu_{(G,o)}\{0\})/2$.

4.3.2 Proof of Theorem 4.8

We first use Theorem 4.4 to prove a version of Theorem 4.8 for graphs with bounded degree.

Lemma 4.9 (The zero temperature limit in graphs with bounded degree)**.** *For any $(G,o) \in \mathcal{G}_d^*$, the zero temperature limit*

$$S(G,o) = \lim_{z\to 0} \downarrow zR_z(G,o)$$

exists. Moreover, $S\colon \mathcal{G}_d^* \to [0,1]$ is the largest solution to the recursion

$$S(G,o) = \left(1 + \sum_{v\sim o}\left(\sum_{w\sim v} S(G-o-v,w)\right)^{-1}\right)^{-1}, \tag{23}$$

with the conventions $0^{-1} = \infty$, $\infty^{-1} = 0$. When G is finite, $S(G,o)$ is the probability that o is exposed in a uniform maximum matching.

Proof. Fix $(G,o) \in \mathcal{G}_d^*$. First, we claim that $z \mapsto zR_z(G,o)$ is non-decreasing on $\mathbb{R}_+$. Indeed, this is obvious if G is reduced to o, since in that case the $zR_z(G,o) = 1$. It then inductively extends to any finite graph (G,o), because iterating twice (16) gives

$$zR_z(G,o) = \left(1 + \sum_{v\sim o}\left(z^2 + \sum_{w\sim v} zR_z(G-o-v,w)\right)^{-1}\right)^{-1}. \tag{24}$$

For the infinite case, (G,o) is the local limit of the sequence of finite truncations $((G,o)_k)\,, k \geq 1$, so by continuity of the cavity ratio Lemma 4.3(iii), $R_z(G,o) = \lim_{k\to\infty} R_z(G,o)_k$ must be non-decreasing in z as well.

This guarantees the existence of the $[0,1]-$valued limit

$$S(G,o) = \lim_{z\to 0} \downarrow zR_z(G,o).$$

Moreover, taking the $z \to 0$ limit in (24) guarantees the recursive formula (23).

Finally, consider $T\colon \mathrm{Succ}^*(\mathrm{G,o}) \to [0,1]$ satisfying the recursion (23). Let us show by induction over integer $k \geq 0$ that for every $(H,u) \in \mathrm{Succ}^*(\mathrm{G,o})$ and $z > 0$,

$$T(H,u) \leq zR_z(H,u)_{2k}. \tag{25}$$

The statement is trivial when $k = 0$ ($zR_z(H,u)_0 = 1$), and is preserved from k to $k+1$ because

$$\begin{aligned} zR_z(H,u)_{2k+2} &= \left(1 + \sum_{v\sim u}\left(z^2 + \sum_{w\sim v} zR_z(H-u-v,w)_{2k}\right)^{-1}\right)^{-1} \\ &\geq \left(1 + \sum_{v\sim u}\left(\sum_{w\sim v} T(H-u-v,w)\right)^{-1}\right)^{-1} = T(H,u). \end{aligned}$$

Letting $k \to \infty$ and then $z \to 0$ in (25) yields $S \leq T$, which completes the proof ☐

This naturally raises the following question : may the zero temperature limit be interchanged with the infinite volume limit ? The answer is not straightforward : unlike recursion (16), the recursion (23) may admit several distinct solutions, and this translates as follows : in the limit of zero temperature, correlation decay breaks for the monomer-dimer model, in the precise sense that the functional $S\colon \mathcal{G}_d^* \to [0,1]$ is no longer continuous with respect to local convergence. For example, one can easily construct an infinite rooted tree (T,o) with bounded degree such that

$$\lim_{k\to\infty} \downarrow S(T,o)_{2k} \quad \neq \quad \lim_{k\to\infty} \uparrow S(T,o)_{2k+1}.$$

Despite this lack of correlation decay, the interchange of limits turns out to be valid "on average", i.e. when looking at a uniformly chosen vertex o. The key lemma is the next result.

Lemma 4.10 (Uniform logarithmic error)**.** *Let $G = (V,E)$ be a finite graph. For any $0 < z < 1$,*

$$\mathbb{E}_{U(G)} zR_z + \frac{|E|}{|V|}\frac{\log 2}{\log z} \leq \mathbb{E}_{U(G)} S \leq \mathbb{E}_{U(G)} zR_z.$$

Proof. From Lemma 4.9, $z \mapsto \mathbb{E}_{U(G)} zR_z$ is non-decreasing. It follows

$$\mathbb{E}_{U(G)} S \leq \mathbb{E}_{U(G)} zR_z \leq \frac{-1}{\log z}\int_z^1 s^{-1}\mathbb{E}_{U(G)} sR_s ds.$$

Use (14), we rewrite this as

$$\mathbb{E}_{U(G)} S \leq \mathbb{E}_{U(G)} zR_z \leq \frac{1}{|V|\log z}\log\frac{P_G(z)}{P_G(1)}.$$

Now, $P_G(1)$ is the total number of matchings and is thus clearly at most $2^{|E|}$, while $P_G(z)$ is at least $z^{|V|-2\nu(G)}$. Since $\mathbb{E}_{U(G)} S = 1 - 2\nu(G)/|V|$, these two bounds yield exactly the statement. ☐

We may now prove Theorem 4.8.

Proof of Theorem 4.8. The proof is in two steps.
Step 1: we assume that $G_n = (V_n, E_n)$ with $|E_n| = O(|V_n|)$ and ρ-a.s. $\deg_G(o) \le d$ for some $d \ge 1$. Since $U(G_n)$ converges weakly to ρ, by Lemma 4.3, for any $z > 0$,

$$\lim_{n\to\infty} \mathbb{E}_{U(G_n)} R_z = \mathbb{E}_\rho R_z.$$

Let $c > 0$ large enough such that $\sup_{n\ge1} |E_n|/|V_n| \le c$. Using Lemma 4.10 for G_n and letting $n \to \infty$ in , we see that for any $z < 1$,

$$\mathbb{E}_\rho z R_z + \frac{c \log 2}{\log z} \le \liminf_{n\to\infty} \mathbb{E}_{U(G_n)} S \le \limsup_{n\to\infty} \mathbb{E}_{U(G_n)} S \le \mathbb{E}_\rho z R_z.$$

Letting finally $z \to 0$, we obtain the statement of the theorem.
Step 2: truncation. We now establish Theorem 4.8 in full generality. To this end, we introduce the $d-$truncation G^d, $d \ge 0$, of a graph $G = (V, E)$, obtained from G by removing any edge incident to a vertex with degree larger than d. This transformation is clearly continuous on $\mathcal{G}^*$ endowed with the local topology. Moreover, if G is finite, its effect on the matching number can be easily controlled :

$$\nu(G^d) \le \nu(G) \le \nu(G^d) + |\{v \in V : \deg_G(v) > d\}|. \tag{26}$$

Now, consider a sequence of finite graphs $(G_n), n \ge 1$, such that $U(G_n)$ converges weakly to ρ. First, we fix $d \in \mathbb{N}$ and apply step 1 to the sequence $(G_n^d), n \ge 1$, to obtain :

$$\lim_{n\to\infty} \frac{\nu(G_n^d)}{|V_n|} = \frac{1 - \mathbb{E}_{\rho_d} S}{2},$$

where ρ_d is the d-truncation of ρ. Second, we may rewrite (26) as

$$\left| \frac{\nu(G_n^d)}{|V_n|} - \frac{\nu(G_n)}{|V_n|} \right| \le \mathbb{P}_{U(G_n)} \left(\deg_G(o) > d \right) = \frac{\left| \{v \in V_n : \deg_{G_n}(v) > d\} \right|}{|V_n|}.$$

Letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \left| \frac{1 - \mathbb{E}_{\rho_d} S}{2} - \frac{\nu(G_n)}{|V_n|} \right| \le \mathbb{P}_\rho \left(\deg_G(o) > d \right).$$

This last line is, by an elementary application of Cauchy criterion, enough to guarantee the convergence promised by Theorem 4.8 with $\gamma = \lim_{d\to\infty} (1 - \mathbb{E}_{\rho_d} S)/2$. □

4.4 Computation on Unimodular Galton-Watson trees

We now investigate the special case when $\rho = \mathrm{UGW}(P)$. Thanks to the recursive nature of the branching process and the unimodularity, the recursion defining γ simplifies into a recursive distributional equation (RDE), which can been explicitly solved.

Theorem 4.11. *With the notation of Theorem 4.8, if ρ is a* UGW *tree with degree distribution P and moment generating function $\varphi(t) = \sum_k P(k)t^k$, we have*

$$\gamma = \frac{1 - \max_{t\in[0,1]} M(t)}{2},$$

where

$$M(t) = t\varphi'(1-t) + \varphi(1-t) + \varphi\left(1 - \frac{\varphi'(1-t)}{\varphi'(1)}\right) - 1. \tag{27}$$

Differentiating the above expression, we see that any t achieving the maximum must satisfy

$$\varphi'(1)t = \varphi'\left(1 - \frac{\varphi'(1-t)}{\varphi'(1)}\right). \tag{28}$$

For Erdős-Rényi random graphs with connectivity c, the degree of the limiting UGW tree is Poisson with parameter c (i.e. $\varphi(t) = \exp(ct - c)$), so that (28) becomes $t = e^{-ce^{-ct}}$. Theorem 4.11 is thus consistent with a celebrated result by Karp and Sipser [27] on the matching number of Erdős-Rényi graphs. Similarly, for random graphs with a prescribed degree sequence with a log-concave assumption on the degree distribution, Theorem 4.11 covers a result by Bohmann and Frieze [11].

Given $z > 0$, $Q \in \mathcal{P}(\mathbb{N})$ and $\mu \in \mathcal{P}([0,1])$, we denote by $\Theta_{Q,z}(\mu)$ the law of the $[0,1]-$valued r.v.

$$Y = \frac{1}{z + \sum_{i=1}^N X_i},$$

where $N \sim Q$ and $X_1, X_2, \ldots \sim \mu$, all of them being independent. This defines an operator $\Theta_{\nu,z}$ on $\mathcal{P}([0,1])$. The corresponding fixed point equation $\mu = \Theta_{\nu,z}(\mu)$ belongs to the general class of RDE. Equivalently, it can be rewritten as

$$X \stackrel{d}{=} \frac{1}{z + \sum_{i=1}^N X_i},$$

where $X_1, X_2, \ldots$ are i.i.d. copies of the unknown random variable X. Note that the same RDE appears in the analysis of the spectrum and rank of adjacency matrices of random graphs [14, 15]. With this notation in hands, the infinite system of equations (16) defining R_z clearly leads to the following distributional characterization:

Lemma 4.12. *If (T,o) has distribution* UGW(P)*, then for any $z > 0$, $R_z(T,o)$ has distribution $\Theta_{P,z}(\mu_z)$, where ν_z is the unique solution of the RDE $\nu_z = \Theta_{\widehat{P},z}(\nu_z)$.*

The same program can be carried out in the zero temperature limit. Specifically, given $Q, Q' \in \mathcal{P}(\mathbb{N})$ and $\mu \in \mathcal{P}([0,1])$, we define $\Theta_{Q,Q'}(\nu)$ as the law of the $[0,1]-$valued r.v.

$$Y = \frac{1}{1 + \sum_{i=1}^N \left(\sum_{j=1}^{N_i'} X_{ij}\right)^{-1}}, \tag{29}$$

where $N \sim Q$, $N_i' \sim Q'$, and $X_{ij} \sim \nu$, all of them being independent. This defines an operator $\Theta_{Q,Q'}$ on $\mathcal{P}([0,1])$ whose fixed points will play a crucial role in our study. Indeed, Theorem 4.9 implies:

Lemma 4.13. *If (T,o) has distribution $\mathrm{UGW}(P)$, the random variable $S(T,o)$ has law $\Theta_{P,\widehat{P}}(\nu)$, where ν is the largest solution (for stochastic domination) to the RDE $\nu = \Theta_{\widehat{P},\widehat{P}}(\nu)$ or with $X \sim \nu$,*

$$X \stackrel{d}{=} \frac{1}{1 + \sum_{i=1}^{N} \left(\sum_{j=1}^{N_i} X_{ij} \right)^{-1}}, \tag{30}$$

where $N \sim \widehat{P}$, $N_i' \sim \widehat{P}$, and $X_{ij} \sim \nu$, all of them being independent,

Above, the stochastic domination on $\mathcal{P}([0,1])$ is the partial order $\nu_1 \leq \nu_2$ if for any continuous, increasing function $\varphi\colon [0,1] \to \mathbb{R}$,

$$\int \varphi d\nu_1 \leq \int \varphi d\nu_2.$$

Recall that by Theorem 4.8, the mean of $\Theta_{P,\widehat{P}}(\nu)$ gives precisely the asymptotic size of a maximum matching for any sequence of finite random graphs whose random weak limit is $\mathrm{UGW}(P)$. We will now solve this RDE. Combined with Theorem 4.8 and a simple continuity argument to remove the bounded degree assumption, this will prove Theorem 4.11.

The remaining of this section is dedicated to solving (30) when P has a finite second moment. We will assume that $P(0) + P(1) < 1$, otherwise $\widehat{P} = \delta_0$ and $\mu = \delta_1$ is clearly the only solution to (30). We recall that $\varphi(z)$ is the generating function of P. For any $x \in [0,1]$, we set $\overline{x} = \varphi'(1-x)/\varphi'(1)$ and then M defined in (27) can be rewritten as

$$M(x) = \varphi'(1)x\overline{x} + \varphi(1-x) + \varphi(1-\overline{x}) - 1.$$

Observe that $M'(x) = \varphi_*''(1-x)\left(\overline{\overline{x}} - x\right)$ and therefore, any $x \in [0,1]$ where M admits a local extremum must satisfy $x = \overline{\overline{x}}$. We will say that M admits an historical record at x if $x = \overline{\overline{x}}$ and $M(x) > M(y)$ for any $0 \leq y < x$. Since M is analytic, there are only finitely many such records in $[0,1]$. In fact, they are in one-to-one correspondence with the solutions to the RDE (30).

Lemma 4.14. *If $p_1 < \ldots < p_r$ are the locations of the historical records of M, then the RDE (30) admits exactly r solutions ; moreover, these solutions can be stochastically ordered, say $\nu_1 < \ldots < \nu_r$, and for any $i \in \{1, \ldots, r\}$,*

(i) $\nu_i(\{0\}^c) = p_i$;

(ii) $\Theta_{P,\widehat{P}}(\nu_i)$ *has mean* $M(p_i)$.

Theorem 4.11 follows immediately from Lemma 4.14 and Lemma 4.13. It thus remains to prove Lemma 4.14. The space $\mathcal{P}([0,1])$ is naturally equipped with the weak topology and the stochastic domination. The proof of Lemma 4.14 will be based on two lemmas, the first one being straightforward.

Lemma 4.15. *For any $Q, Q' \in \mathcal{P}(\mathbb{N}) \backslash \{\delta_0\}$, $\Theta_{Q,Q'}$ is continuous and strictly increasing on $\mathcal{P}([0,1])$.*

Lemma 4.16. *For any $\nu \in \mathcal{P}([0,1])$, letting $p = \nu(\{0\}^c)$, we have*

(i) $\Theta_{\widehat{P},\widehat{P}}(\nu)(\{0\}^c) = \overline{\overline{p}}$

(ii) if $\Theta_{\widehat{P},\widehat{P}}(\nu) \leq \nu$, *then the mean of* $\Theta_{P,\widehat{P}}(\nu)$ *is at least* $M(p)$.

(iii) if $\Theta_{\widehat{P},\widehat{P}}(\nu) \geq \nu$, *then the mean of* $\Theta_{P,\widehat{P}}(\nu)$ *is at most* $M(p)$;

In particular, if $\nu = \Theta_{\widehat{P},\widehat{P}}(\nu)$, *then* $p = \overline{\overline{p}}$ *and* $\Theta_{P,\widehat{P}}(\nu)$ *has mean* $M(p)$.

Proof. In equation (29) it is clear that $Y > 0$ if and only if for any $i \in \{1,\dots,N\}$, there exists $j \in \{1,\dots,N_i'\}$ such that $X_{ij} > 0$. Denoting by $\widehat{\varphi}$ the generating function of $\widehat{P}$, this rewrites:

$$\Theta_{\widehat{P},\widehat{P}}(\nu)(\{0\}^c) = \widehat{\varphi}\left(1 - \widehat{\varphi}\left(1 - \nu(\{0\}^c)\right)\right).$$

But from (2), it follows that $\widehat{\varphi}(x) = \varphi'(x)/\varphi'(1)$, i.e. $\widehat{\varphi}(1-x) = \overline{x}$, hence the first result.

Now let $X \sim \nu$, $N \sim P$, $\widehat{N} \sim \widehat{P}$, and let $S, S_1, \dots$ have the distribution of the sum of $\widehat{N}$ i.i.d. copies of X, all these variables being independent. Then, $\Theta_{P,\widehat{P}}(\nu)$ has mean

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{1+\sum_{i=1}^N {S_i}^{-1}}\right] &= \mathbb{E}\left[\left(1 - \frac{\sum_{i=1}^N {S_i}^{-1}}{1+\sum_{i=1}^N {S_i}^{-1}}\right)\mathbf{1}_{\{\forall i=1\dots N, S_i>0\}}\right] \\
&= \varphi(1-\overline{p}) - \varphi'(1)\mathbb{E}\left[\frac{S^{-1}}{S^{-1}+1+\sum_{i=1}^{\widehat{N}} {S_i}^{-1}}\mathbf{1}_{\{S>0,\forall i=1\dots\widehat{N}, S_i>0\}}\right] \\
&= \varphi(1-\overline{p}) - \varphi'(1)\mathbb{E}\left[\frac{Y}{Y+S}\mathbf{1}_{\{S>0\}}\right],
\end{aligned}$$

where the second and last lines follow from (2), see also (10), and $Y \sim \Theta_{\widehat{P},\widehat{P}}(\nu)$, respectively. Now, for any $s > 0$, $x \mapsto \frac{x}{x+s}$ is increasing and hence, depending on whether $\Theta_{\widehat{P},\widehat{P}}(\nu) \geq \nu$ or $\Theta_{\widehat{P},\widehat{P}}(\nu) \leq \nu$, $\Theta_{P,\widehat{P}}(\nu)$ has mean at most/least:

$$\varphi(1-\overline{p}) - \varphi'(1)\mathbb{E}\left[\frac{X}{X+S}\mathbf{1}_{\{S>0\}}\right] = \varphi(1-\overline{p}) - p\varphi'(1)\mathbb{E}\left[\frac{1}{1+N'}\mathbf{1}_{\{N'\geq 1\}}\right],$$

with $N' = \sum_{i=1}^{\widehat{N}} \mathbf{1}_{\{X_i>0\}}$ and we used the exchangeability of the vector $(X, X_1, \cdots, X_{\widehat{N}})$ given $\widehat{N}$. Finally, using the definition (2) and the well known identity $(n+1)\binom{n}{d} = (d+1)\binom{n+1}{d+1}$, we find

$$\begin{aligned}
p\varphi'(1)\mathbb{E}\left[\frac{1}{1+N'}\mathbf{1}_{\{N'\geq 1\}}\right] &= p\varphi'(1)\sum_{n\geq 1}\widehat{P}(n)\sum_{d=1}^{n}\binom{n}{d}\frac{p^d(1-p)^{n-d}}{d+1} \\
&= \sum_{n\geq 1} P(n+1)\big(1 - p(n+1)(1-p)^n - (1-p)^{n+1}\big) \\
&= \sum_{k\geq 0} P(k)\Big(1 - pk(1-p)^{k-1} - (1-p)^k\Big) \\
&= 1 - p\varphi'(1-p) - \varphi(1-p).
\end{aligned}$$

Since $\bar{p} = \varphi'(1-p)/\varphi'(1)$, we get finally, $\varphi(1-\overline{p}) - p\varphi'(1)\mathbb{E}\left[\frac{1}{1+N'}\mathbf{1}_{\{N'\geq 1\}}\right] = M(p)$ as requested. $\square$

We now have all the ingredients we need to prove Lemma 4.14.

Proof of Lemma 4.14. Let $p \in [0,1]$ such that $\overline{\overline{p}} = p$, and define $\nu_0 = \text{Bernoulli}(p)$. From Lemma 4.16 we know that $\Theta_{\widehat{P},\widehat{P}}(\nu_0)(\{0\}^c) = p$, and since Bernoulli(p) is the largest element of $\mathcal{P}([0,1])$ putting mass p on $\{0\}^c$, we have $\Theta_{\widehat{P},\widehat{P}}(\nu_0) \leq \nu_0$. Immediately, Lemma 4.15 guarantees that the limit

$$\nu_\infty = \lim_{k\to\infty} \searrow \Theta^k_{\widehat{P},\widehat{P}}(\nu_0)$$

exists in $\mathcal{P}([0,1])$ and is a fixed point of $\Theta_{\widehat{P},\widehat{P}}$. Moreover, by Fatou's Lemma, the number $p_\infty = \nu_\infty(\{0\}^c)$ must satisfy $p_\infty \leq p$. But then the mean of $\Theta_{P,\widehat{P}}(\nu_\infty)$ must be both

- equal to $M(p_\infty)$ by Lemma 4.16 with $\nu = \nu_\infty$;
- at least $M(p)$ since $\forall k \geq 0$, the mean of $\Theta_{P,\widehat{P}}\left(\Theta^k_{\widehat{P},\widehat{P}}(\mu_0)\right)$ is at least $M(p)$ (by Lemma 4.16 with $\nu = \Theta^k_{\widehat{P},\widehat{P}}(\nu_0)$).

We have just shown both $M(p) \leq M(p_\infty)$ and $p_\infty \leq p$. From this, we will now deduce the one-to-one correspondence between historical records of M and fixed points of $\Theta_{\widehat{P},\widehat{P}}$. We treat each inclusion separately.

First, if M admits an historical record at p, then clearly $p_\infty = p$, so ν_∞ is a fixed point satisfying $\nu_\infty(\{0\}^c) = p$.

Conversely, considering a fixed point ν with $\nu(\{0\}^c) = p$, we want to deduce that M admits an historical record at p. We first claim that ν is the above defined limit ν_∞. Indeed, $\nu \leq \text{Bernoulli}(p)$ implies $\nu \leq \nu_\infty$ ($\Theta_{\widehat{P},\widehat{P}}$ is increasing), and in particular $p \leq p_\infty$. Therefore, $p = p_\infty$ and $M(p) = M(p_\infty)$. In other words, the two ordered distributions $\Theta_{P,\widehat{P}}(\nu) \leq \Theta_{P,\widehat{P}}(\nu_\infty)$ share the same mean, hence are equal. This ensures $\nu = \nu_\infty$. Now, if $q < p$ is any historical record location, we know from what precedes that

$$\lambda_\infty = \lim_{k\to\infty} \searrow \Theta^k_{\widehat{P},\widehat{P}}(\text{Bernoulli}(q))$$

is a fixed point of $\Theta_{\widehat{P},\widehat{P}}$ satisfying $\lambda_\infty(\{0\}^c) = q$. But $q < p$, so $\text{Bernoulli}(q) < \text{Bernoulli}(p)$, hence $\lambda_\infty \leq \nu_\infty$. Moreover, this limit inequality is strict because $\lambda_\infty(\{0\}^c) = q < p = \nu_\infty(\{0\}^c)$. Consequently, $\Theta_{P,\widehat{P}}(\lambda_\infty) < \Theta_{P,\widehat{P}}(\nu_\infty)$ and taking expectations, $M(q) < M(p)$. Thus, M admits an historical record at p. ☐

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